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BOUNDARY POINTWISE ERROR ESTIMATE FOR FINITE ELEMENT METHOD

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ABSTRACT. This paper is devoted to the pointwise error estimate up to boundary for the standard finite element solution of Poisson equation with Dirichlet boundary condition. Our new approach uses the discrete maximum principle for the discrete harmonic solution. Once the mesh in our domain satisfies the $\beta-$ condition defined by us, the discrete harmonic solution with Dirichlet boundary condition has the discrete maximum principle and the pointwise error should be bounded by L_1- errors newly obtained.

1. Introduction

In the field of finite element methods the error estimates for Poisson type equations have been extensively studied for several decades by many authors because of its basic importance and applications. For example, many problems physically interesting are concerned with Laplacian operator as principal part. If all other terms except for Laplacian are moved to right-hand side, then we can get the equation of Poisson type. Thus the errors in the regime of function spaces considered are interesting. Especially, the pointwise error is natural and important thing for many engineers and mathematicians. However it has been known that its error analysis is hard one even for good mathematicians.

The key idea of our method for the pointwise error up to boundary is to use the discrete maximum principle for discrete harmonic by which we

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mean the finite element solution for Laplace equation in term of piecewise linear interpolation polynomials. Recently we have obtained the sufficient condition for triangulation under which the discrete harmonic function satisfies maximum principle, i.e., its maximum can be obtained on the boundary. We called this the $\beta-$ condition for the triangulation. See [1].

In this paper we consider the finite element solution for the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

where Ω is a bounded open set in $\mathbf{R}^N$. As usual triangulation in finite element method, our domain Ω is divided by quasi-uniform polyhedrons T_i. Let $\mathcal{T}_h\,(0 < h < 1)$ be the class of T_i's and $\Omega_h = \cup_{T_i \in \mathcal{T}_h} T_i$. By quasi-uniformity, we mean that, for any $T_i \in \mathcal{T}_h$, there exist constants $c > 0$ and $C > 0$ such that

$$\operatorname{diam}(T_i) \leq C\,h \quad \text{and} \quad \sup\{r > 0 \,|\, B_r \subset T_i\} \geq c\,h.$$

Let us consider the finite element space $\mathcal{S}^h$ which is composed of piecewise linear and continuous functions. Thus we have $\mathcal{S}^h \subset W^{1,\infty}(\Omega_h)$. From a usual Hilbert space theory we can find a solution $u_h \in \mathcal{S}_0^h$ satisfying

$$\int_{\Omega_h} \nabla u_h \cdot \nabla \chi \, dx = \int_{\Omega_h} f \chi \, dx$$

for all $\chi \in \mathcal{S}_0^h$ where $\chi \in \mathcal{S}_0^h$ has supp $\chi \subset \Omega_h$. We assume that any harmonic function $u \in W^{1,\infty}(\bar{\Omega}_h)$ with piecewise linear boundary data satisfies the localized estimate

$$||u - P_1 u||_{L^2(B_R)} \leq c R^{\frac{N}{2}} \log\left(\frac{1}{h}\right) ||u||_{L^\infty(B_{2R})} \tag{1.1}$$

for some large constant c and $B_R \subset \Omega_h$, where P_1 is the H^1 projection. For the detail of the assumption, we refer (10.8) of [8]. In fact Schatz and Wahlbin ([8]) proved (1.1) with suitable cut-off function.

Our main concern here is the L^∞ error estimate of the solution to discrete Poisson equation globally.

Let us define $B(x_0, r) = \{x \in \mathbf{R}^N \,|\, |x - x_0| < r\}$. L^p is the usual Lebesque space and $W^{k,p}(\Omega)$ is the Sobolev space equipped with the usual norm

$$\|u\|_{k,p} = \left(\sum_{|\alpha| \le k} \int_{\Omega} |D^{\alpha} u|^p \, dx \right)^{\frac{1}{p}},$$

and $W^{k,2}(\Omega) = H^k(\Omega)$ as usual notation. It is assumed that an extendibility condition for our domain Ω_h is satisfied, i.e., there is a sufficiently large domain $\overline{\Omega}_h(\overline{\mathcal{T}}_h)$ with $\overline{\Omega}_h(\overline{\mathcal{T}}_h) \supset \Omega_h(\mathcal{T}_h)$ such that $\overline{\mathcal{T}}_h \supset \mathcal{T}_h$ and

$$\overline{\Omega}_h = \bigcup_{\overline{T}_i \in \overline{\mathcal{T}}_h} \overline{T}_i, \quad \Omega_h = \bigcup_{T_i \in \mathcal{T}_h} T_i.$$

Moreover, at each point $x_0 \in \partial\Omega_h$, $B(x_0, R_0) \subset \overline{\Omega}_h$ for some fixed $R_0 > 0$ independent of x_0. Now we state our main theorem.

THEOREM 1. *Suppose Ω_h is extendible and its triangulation $\overline{\mathcal{T}_h}$ satisfies $\beta-$ condition. Then*

$$\begin{aligned} \|u - u_h\|_{L^\infty\left(B(x_0, \frac{R}{2}) \cap \Omega\right)} &\le C \left(\log \tfrac{R}{h}\right) \min_{\chi \in S^h} \|u - \chi\|_{L^\infty(B(x_0,R) \cap \Omega)} \\ &\quad + C\, d^{-\frac{N}{2} - s} \|u - u_h\|_{H^{-s}(B(x_0,R) \cap \Omega)} \end{aligned}$$

where $d = dist\left(\partial(B(x_0, \frac{R}{2}) \cap \Omega) \backslash \partial\Omega, \partial(B(x_0, R) \cap \Omega) \backslash \partial\Omega\right)$.

This theorem is achieved by a discrete maximum principle which is known for the triangulation satisfying $\beta-$condition. Here an important ingredient is a representation formula for the discrete harmonic functions.

2. Discrete Representation Formula

In this section we solve a discrete boundary value problem by weight. The weight corresponds to the Poisson kernel for the continuous case. In order to expand our new theory several topological definitions are needed for discrete cases. For readers' convenience let $\overline{\mathcal{T}}_h$ be an extended triangulation of $\mathcal{T}_h$ in Ω, the set of all nodes in Ω_h(or $\overline{\Omega}_h$) denoted by Φ(or $\overline{\Phi}$) and the set of all vertices in a polyhedron $T \in \mathcal{T}_h$(or $\overline{\mathcal{T}}_h$) by $\Sigma(T)$.

DEFINITION (Path). Let $V_a \in \Phi$ and $V_b \in \Phi$ be nodes of our triangulation $\mathcal{T}_h$. We define the discrete path $P(V_a, V_b)$ by the following ordered set of nodes:

$$P(V_a, V_b) = \{V_i \in \overline{\Phi} \mid i = 0, 1, \cdots, k, V_0 = V_a, V_k = V_b\}$$

where the successive nodes V_i and V_{i+1} are contained in an edge of the same polyhedron. The length of $P(V_a, V_b)$ is defined by

$$\|P(V_a, V_b)\| = k.$$

DEFINITION (Discrete Distance). Let us define the discrete distance between V_a and V_b by

$$D(V_a, V_b) = \begin{cases} \min\limits_P \|P(V_a, V_b)\|, & \text{if } \; V_a \neq V_b \\ 0, & \text{otherwise} \end{cases}$$

where the minimum is taken among all possible path P between V_a and V_b.

REMARK. The distance $D(V_a, V_b)$ as defined above is in fact a metric on Φ.

DEFINITION (Discrete ball). The discrete ball $B_{dsc}(V_0, L)$ is the set of node V satisfying

$$B_{dsc}(V_0, L) = \{V \in \overline{\Phi} \mid D(V_0, V) < L\}.$$

Furthermore, we define the continuous domain $\overline{B}_{dsc}(V_0, L)$ by

$$\overline{B}_{dsc}(V_0, L) = \bigcup\{T \in \mathcal{T}_h \mid \Sigma(T) \cap B_{dsc}(V_0, L) \neq \emptyset \\ \text{or } \; D(V_0, V) = L \; \text{ for all } \; V \in \Sigma(T)\}.$$

DEFINITION (h-convex). The discrete ball $\overline{B}_{dsc}(V_0, L)$ is h-convex if there is a hyperplane $\mathcal{P}(V_j)$ for each node $V_j \in \partial\overline{B}_{dsc}(V_0, L)$ such that

$$\mathcal{P}(V_j) \cap \overline{B}_{dsc}(V_0, L) = \emptyset \quad \text{and} \quad \text{dist}(V_j, \mathcal{P}(V_j)) \leq C\,h$$

for some C independent of L.

LEMMA 1. *Let a polyhedron $T \in \mathcal{T}_h$ be given. $T \subset \overline{B}_{dsc}(V_c, L)$ if and only if $D(V_c, V) \leq L$ for all $V \in \Sigma(T)$.*

Proof. Assume $T \subset \overline{B}_{dsc}(V_c, L)$. If there is a vertex $V_* \in \Sigma(T)$ but $D(V_c, V_*) > L$, then trivially $V_* \notin B_{dsc}(V_c, L+1)$. This means $D(V_c, V) \geq L$ for all $V \in \Sigma(T)$ since $D(V_*, V) \leq 1$ when $V \in \Sigma(T)$. Thus we have $T \notin \overline{B}_{dsc}(V_c, L)$, which is contradiction to our assumption. So, the sufficiency has been proven.

Now, assume $D(V_c, V) \leq L$ for all $V \in \Sigma(T)$. If $T \subset \overline{B}_{dsc}(V_c, L)$, the we have done. If $T \notin \overline{B}_{dsc}(V_c, L)$, then from the definition of discrete ball, $\Sigma(T) \cap B_{dsc}(V_c, L) = \emptyset$. This implies that $D(V_c, V) \geq L$ for all $V \in \Sigma(T)$. From the hypothesis of $D(V_c, V) \leq L$, we conclude that $D(V_c, V) = L$ for all $V \in \Sigma(T)$. Therefore, $T \in \overline{B}_{dsc}(V_c, L)$. □

LEMMA 2. *Let $V_c \in \Phi$. Then the following characterization is true.*

$$\partial\overline{B}_{dsc}(V_c, L) \cap \overline{\Phi} = \{V \in \overline{\Phi} \mid D(V_c, V) = L \text{ and } \exists T \ni V \text{ such that } T \notin \overline{B}_{dsc}(V_c, L)\}$$

Proof. Assume that V_* is a node which satisfies $D(V_c, V_*) = L$ and there exists $T \in \overline{\mathcal{T}}_h$ containing V_* as vertex such that $T \notin \overline{B}_{dsc}(V_c, L)$. First we note that there exist at least one polyhedron $T \ni V_*$ such that $T \in \overline{B}_{dsc}(V_c, L)$ since $D(V_c, V_1) = L - 1$ for some vertex V_1 of T_1 where T_1 has V_* as vertex and so we have $T_1 \in \overline{B}_{dsc}(V_c, L)$ from the definition of $\overline{B}_{dsc}(V_c, L)$. Therefore V_* is a boundary point of $\overline{B}_{dsc}(V_c, L)$.

To prove the converse, let $V_* \in \partial\overline{B}_{dsc}(V_c, L) \cap \overline{\Phi}$. Then there must be two polyhedron $T_{in} \in \overline{B}_{dsc}(V_c, L)$ and $T_{out} \notin \overline{B}_{dsc}(V_c, L)$,in which V_* is contained as vertex, such that there are two vertices $V_{in} \in \Sigma(T_{in})$ with $D(V_c, V_{in}) \leq L$ and $V_{out} \in \Sigma(T_{out})$ with $D(V_c, V_{out}) > L$, respectively. If $D(V_c, V_*) < L$, then from the fact that $D(V_c, V_{out}) > L$, we have $D(V_*, V_{out}) > 1$. It is impossible. Thus we have $D(V_c, V_*) \geq L$. This implies that $D(V_c, V_*) = L$ since $T_{in} \subset \overline{B}_{dsc}(V_c, L)$. □

COROLLARY 1. *Assume that Ω_h is extendible. Let $V_c \in \Phi$ be a node in Ω_h and $h\,L < R_0$. Then $\overline{B}_{dsc}(V_c, L)$ is simply connected polygonal domain. If the triangulation of Ω_h can be extendible to quasi-uniform one of $\overline{\Omega}_h$, then $\partial\overline{B}_{dec}(V_c, L)$ is uniform Lipschitz domain.*

Proof. Our hypothesis $h\,L < R_0$ guarantees that $\partial\overline{B}_{rel}(V_c, L) \cap \partial\Omega_h = \emptyset$. From Lemma 1 and 2, our corollary can be obtained. □

Let u be the harmonic function in a domain $\Omega \subset \mathbf{R}^{\mathbf{N}}$ and $B(x, \rho) = \{y \in \mathbf{R}^{\mathbf{N}} \mid \|x - y\| < \rho\}$ be a ball. Then harmonic function u satisfies the well known mean value property of the following form

$$u(x) = \frac{1}{|\partial B(x, \rho)|} \int_{\partial B(x,\rho)} u(y)\, d\Gamma$$

whenever $B(x, \rho) \subset \Omega$. For the discrete harmonic function, we can obtain the mean value property as analogous form to continuous case, which is described below.

DEFINITION (Proper triangulation). Let $\mathcal{T}_h$ be a triangulation of a domain Ω. If ,for any discrete 1-ball $\overline{B}_{dsc}(V,1)$, its all interior node is $\{V\}$, then the triangulation $\mathcal{T}_h$ is called a proper triangulation of Ω.

REMARK. If $\mathcal{T}_h$ is a proper triangulation of Ω, then any discrete 1-ball must be written as follows

$$\overline{B}_{dsc}(V,1) = \bigcup\{T \in \mathcal{T}_h \,|\, \Sigma(T) \cap V \neq \emptyset\}$$

whenever V is the node of $\mathcal{T}_h$.

Let us define the index set of nodes on the boundary of $\overline{B}_{dsc}(V,L)$ by

$$\Psi(V,L) = \{1,2,\cdots,n_L^V\}$$

and its corresponding set of nodes by

$$\Theta(V,L) = \{V_k^L \,|\, k \in \Psi(V,L)\}$$

and u_h a discrete harmonic in $\overline{B}_{dsc}(V_0,L)$. Then we obtain the following representation formula for u_h with respect to the boundary node values

$$u_h = \sum_{k=1}^{n_L^{V_0}} u_h(V_k^L)\,\omega_k^L \tag{2.2}$$

where each ω_k^L is the discrete harmonic function in $\overline{B}_{dsc}(V_0,L)$ such that its boundary node values are defined by

$$\omega_k^L(V_i^L) = \delta_{i\,k}, \quad i = 1,2,\cdots,n_L^{V_0} \tag{2.3}$$

where $\delta_{i\,k} = 1$ if $i = k$, otherwise, 0.

Let us assume that our triangulation $\mathcal{T}_h$ is assumed to be quasi-uniform such that

$$h_T = \min\{r \,|\, B(x,r) \supset T\}, \qquad \rho_T = \max\{r \,|\, B(x,r) \subset T\}$$
$$h = \max\{h_T \,|\, T \in \mathcal{T}_h\}, \qquad \rho = \min\{\rho_T \,|\, T \in \mathcal{T}_h\}.$$

From the theorem of discrete maximum principle by H. J. Choe and D. W. Kim [1] the following estimate can be obtained under the proper triangulation

$$0 < \frac{c}{n_1^{V_0}} < \omega_k^1(V_0) < \frac{C}{n_1^{V_0}} \leq 1$$

where c is constant independent of mesh size h and node point V_0. If we let for some γ with $0 < \gamma < 1$

$$\rho = \gamma\, h,$$

then the following inequalities hold

$$\frac{1}{2^N}\gamma^N < \frac{1}{n_1^{V_0}} < \frac{1}{2^N}\left(\frac{1}{\gamma}\right)^N$$

where N is the space dimension. Therefore we can obtain the following inequality

$$0 < \delta < \omega_k^1(V_0), \quad \text{for all } k \in \Psi(V_0, 1) \tag{2.4}$$

for some constants δ independent of h and V_0.

The following Lemmas will be strongly used to prove our main theorem. They can be obtained from our discrete ball theory combined with the maximum principle of discrete harmonic function.

LEMMA 3. *Let $\mathcal{T}_h$ be a proper quasi-uniform triangulation of $\Omega \subset \mathbf{R}^N$. Suppose $V_0 \in \Phi$ and $\overline{B}_{dsc}(V_0, L) \subset \Omega_h$ and $u_h \in \mathcal{S}^h(\Omega_h)$ satisfies the following equations*

$$\begin{aligned} \int_{\overline{B}_{dsc}(V_0,L)} \nabla u_h \cdot \nabla v \, dx &= 0, \quad \text{for all } v \in \mathcal{S}_0^h(\overline{B}_{dsc}(V_0, L)) \\ u_h|_{\partial \overline{B}_{dsc}(V_0,L)} &= u_{0\,h} \in \mathcal{S}^h(\overline{B}_{dsc}(V_0, L)). \end{aligned}$$

Then we have the consequences

$$u_h(V_0) = \sum_{k=1}^{n_L^{V_0}} \beta_{0\,k}^L \, u_{0\,h}(V_k^L), \quad \sum_{k=1}^{n_L^{V_0}} \beta_{0\,k}^L = 1, \tag{2.5}$$

where the coefficients are satisfy

$$\frac{\delta^L}{L^{N-1}} < \beta_{0\,k}^L. \tag{2.6}$$

Proof. The equations (2.5) is obvious from the representation formula (2.2) for the discrete harmonic function. Furthermore, it must be true that

$$\beta_{0\,k}^L = \omega_k^L(V_0) \quad \text{for all } k \in \Psi(V_0, L).$$

Thus it will be proved by induction for the argument L that ,for any integer $L \geq 1$,

$$0 < \frac{\delta^L}{L^{N-1}} < \omega_k^L(V_0) \quad \text{for all } k \in \Psi(V_0, L). \tag{2.7}$$

From the estimate (2.4), our claim (2.7) is true for $L=1$. Assume that it is true for less than or equal to $L-1$. The representation formula (2.2) has β_{0k}^L written as follows

$$\omega_k^L(V_0) = \sum_{j=1}^{n_{L-1}^{V_0}} \omega_j^{L-1}(V_0)\,\omega_k^L(V_j^{L-1}) \quad \text{for all } k \in \Psi(V_0, L).$$

If the nodes in $\Theta(V_0, L-1)$ are split into the following

$$\Theta(V_0, L-1) \supset \bigcup_{l=1}^{L} S_l$$

where $S_l = \Theta(V_k^L, l) \cap \Theta(V_0, L-1)$. The maximum principle for the discrete harmonic function ω_k^L yields the result of

$$0 < \delta^l < \omega_k^L(V_j^{L-1}) \quad \text{for all } V_j^{L-1} \in S_l \tag{2.8}$$

Since our assumption implies that

$$\frac{\delta^{L-1}}{(L-1)^{N-1}} < \omega_j^{L-1}(V_0),$$

we can obtain the following inequalities

$$\begin{aligned}
\omega_k^L(V_0) &> \frac{\delta^{L-1}}{(L-1)^{N-1}} \sum_{j=1}^{n_{L-1}^{V_0}} \omega_k^L(V_j^{L-1}) \\
&\geq \frac{\delta^{L-1}}{(L-1)^{N-1}} \sum_{l=1}^{L} \sum_{V \in S_l} \omega_k^L(V) \\
&\geq \frac{\delta^{L-1}}{(L-1)^{N-1}} \sum_{l=1}^{L} \delta^l \\
&\geq \frac{\delta^L}{L^{N-1}}
\end{aligned}$$

The third inequality in the above is obtainable from that of (2.8). □

Now we want to obtain an upper bound for β_{0j}^L.

LEMMA 4. *Let all discrete ball centered at V_0 be h-convex. If $\overline{B}_{dsc}(V_0, L)$ denotes arbitrary but fixed discrete ball contained in Ω_h, then the following is obtained*

$$\beta_{0j}^L \leq C\,\frac{1+\log L}{L^{N-1}}$$

for some C independent of L.

Proof. Let V_j be a node in $\partial\overline{B}_{dsc}(V_0, L)$. Then since we are assuming our discrete ball $\overline{B}_{dsc}(V_0, L)$ is h-convex, there is a hyperplane $\mathcal{P}(V_j)$ such that

$$\text{(2.9)} \qquad \operatorname{dist}(V_j, \mathcal{P}(V_j)) \leq C_1\, h \quad \text{and} \quad \mathcal{P}(V_j) \cap \overline{B}_{dsc}(V_0, L) = \emptyset.$$

Let $X_0 \in \mathcal{P}(V_j)$ such that

$$\operatorname{dist}(V_j, X_0) = \operatorname{dist}(V_j, \mathcal{P}(V_j)).$$

In general we may assume

$$X_0 = 0,$$
$$\mathcal{P}(V_j) = \{X = (x', x_N) \in \mathbf{R}^N \,|\, x_N = 0\} \quad \text{and}$$
$$\overline{B}_{dsc}(V_0, L) \subset \{x_N \geq 0\}.$$

For given c_0, $\lambda > 0$, we solve the problem

$$\Delta v = 0 \quad \text{in} \quad \{x_N > 0\}$$
$$v(x', 0) = c_0\, \chi_{B(0,\lambda h)\cap\{x_N=0\}}.$$

Then using the Poisson kernel for upper half plane in $\mathbf{R}^N$ we have the following solution for the problem:

$$\text{(2.10)} \qquad v(x', x_N) = \frac{c_0}{c_N}\int_{B(0,\lambda h)} \frac{x_N}{\sqrt{|x'-y'|^2 + x_N^2}^{\,N}}\, dy'$$

where $c_N = \frac{\Gamma(\frac{N}{2})}{\pi^{\frac{N}{2}}}$. If we let $T_j \in \mathcal{T}_h$ be any polyhedron in $\overline{B}_{dsc}(V_0, L)$ containing V_j as a vertex, then for sufficiently large c_0 and λ depending only on C_1 which is the constant in expression (2.9)

$$v(x', x_N) \geq 1$$

for all $x = (x', x_N) \in T_j$.

Now we let $\bar{v}$ be the solution to

$$\Delta \bar{v} = 0 \quad \text{in} \quad \overline{B}_{dsc}(V_0, L)$$
$$\bar{v} = \omega_j^L \quad \text{on} \quad \partial\overline{B}_{dsc}(V_0, L)$$

where ω_j^L is the discrete harmonic function in $\mathcal{S}_h(\overline{B}_{dsc}(V_0, L))$ with its boundary values defined in (2.3). Then since $\bar{v} \leq v$ on $\partial\overline{B}_{dsc}(V_0, L)$ we have

$$\bar{v} \leq v \quad \text{in} \quad \overline{B}_{dsc}(V_0, L)$$

from the comparison principle.

From the quasi-uniform condition if we set $R_L = L\,h$, it can be assumed that

$$a_1\, R_L \leq \operatorname{dist}(V_0, \partial\overline{B}_{dsc}(V_0, L)) \leq a_2\, R_L.$$

From the Harnack principle (2.10), we have

$$\begin{aligned} \bar{v}(V_0) \leq v(V_0) \ &\leq \ \frac{c_0}{c_N} \int_{B(0,\lambda\, h)\cap\{x_N=0\}} \frac{a_2}{a_1} \frac{1}{R_L^{N-1}}\, dy' \\ &\leq \ C \left(\frac{h}{R_L}\right)^{N-1} \end{aligned} \tag{2.11}$$

where C is independent of L.

Let $\bar{v}_h$ be the H^1 projection of $\bar{v}$. From the comparison principle

$$\bar{v}_h(V_0) \geq \omega_j^L(V_0) \tag{2.12}$$

since $\bar{v}_h$ is discrete harmonic in $\overline{B}_{dsc}(V_0, L)$. Therefore from (2.11) and (2.12), the following estimates are obtained

$$\begin{aligned} \omega_j^L(V_0) \ &\leq \ \bar{v}_h(V_0) \leq \bar{v}(V_0) + (\bar{v}_h(V_0) - \bar{v}(V_0))) \\ &\leq \ C \left(\frac{h}{R_L}\right)^{N-1} + |\bar{v}(V_0) - \bar{v}_h(V_0)|\,. \end{aligned}$$

On the other hand, by the assumption (1.1) we obtain

$$\begin{aligned} \|\bar{v} - \bar{v}_h\|_{L^2(B_{\frac{R_L}{8}})} \ &\leq \ C\, h^{\frac{N}{2}} \log\left(\frac{R_L}{h}\right) \|\bar{v}\|_{L^\infty(\frac{R_L}{4})} \\ &\leq \ C \left(\frac{h}{R_L}\right)^{N-1} = C\, \frac{1}{L^{N-1}} \end{aligned}$$

and

$$\begin{aligned} |\bar{v}(V_0) - \bar{v}_h(V_0)| \ &\leq \ C \log\left(\frac{R_L}{h}\right) \inf_{\chi \in \mathcal{S}_0^h(\overline{B}_{dsc}(V_0, \frac{L}{4}))} \|\bar{v} - \chi\|_{L^\infty(B_{\frac{R_L}{4}})} \\ &\quad + \ R_L^{-\frac{N}{2}} \|\bar{v} - \bar{v}_h\|_{L^2(B_{\frac{R_L}{4}})} \end{aligned}$$

Therefore we obtain

$$|\bar{v}(V_0) - \bar{v}_h(V_0)| \le C\,(1+\log L)L^{-(N-1)}$$

and

$$\omega_j^L(V_0) \le C\frac{1+\log L}{L^{N-1}}.$$

□

From the bound of coefficients β_{0j}^L we can develop an L^1-estimate of discrete harmonic functions.

THEOREM 2. *Suppose u_h is a nonnegative discrete harmonic functions in $\overline{B}_{dsc}(V_0, L) \subset \Omega_h$. Then*

$$\sup_{\overline{B}_{dsc}(V_0,\frac{L}{2})} u_h \le C\left(1+\log L\right)\frac{1}{|\overline{B}_{dsc}(V_0,L)|}\int_{\overline{B}_{dsc}(V_0,L)} u_h\,dx$$

for some C independent of L.

Proof. Changing the radius of balls, we can assume

$$u_h(V_0^*) = \sup_{\overline{B}_{dsc}(V_0,L)} u_h.$$

Here we note $\sup_{\overline{B}_{dsc}(V_0,L)} u_h$ is always achieved on the node. Since $\beta_{j,0}^L \le C\frac{1+\log L}{L^{N-1}}$, we obtain

$$\begin{aligned} u_n(v_0) &= \sum_{V_j^L \in \Psi(V_0,L)} \beta_{0,j}^L u_n(V_j^L) \\ &\le C\frac{1+\log L}{L^{N-1}} \sum_{V_j^L \in \Psi(V_0,L)} \beta_{0,j}^L u_n(V_j^L) \end{aligned}$$

and

$$h^N\frac{L^{N-1}}{1+\log L}u_n(V_0) \le C \sum_{T\in\overline{B}_{dsc}(V_0,L)\setminus\overline{B}_{dsc}(V_0,L-1)} \int_T u_h\,dx.$$

Therefore,

$$\sum_{L=1}^{L_0} h^N\frac{L^{N-1}}{1+\log L}u_n(V_0) \le C\sum_{L=1}^{L_0} \sum_{T\in\overline{B}_{dsc}(V_0,L)\setminus\overline{B}_{dsc}(V_0,L-1)} \int_T u_h\,dx$$

and

$$\frac{1}{1+\log L_0}|\overline{B}_{dsc}(V_0,L_0)|u_h(V_0) \le C\int_{\overline{B}_{dsc}(V_0,L_0)} u_h\,dx.$$

□

Now we want to develop boundary maximum principle and L^1-theory for discrete harmonic functions. Let $V_0 \in \Omega_h$ be a node near boundary. From our extendibility assumption of the domain Ω_h, we can find a large discrete domain $\bar{\Omega}_h(\bar{\mathcal{T}}_h)$ with $\bar{\Omega}_h(\bar{\mathcal{T}}_h) \supset \Omega_h(\mathcal{T}_h)$ such that $\bar{\mathcal{T}}_h \supset \mathcal{T}_h$. Moreover, we assume that the reference discrete ball $B_{dsc}(V_0, L) \subset \bar{\Omega}_h$ satisfies our h-convexity condition.

THEOREM 3. *Assume that u_h is a nonnegative discrete harmonic function satisfying*

$$\begin{cases} \Delta_h u_h = 0 & \text{in } \Omega_h \cap B_{dsc}(V_0, L) \\ u_h = 0 & \text{on } \partial\Omega_h \cap B_{dsc}(V_0, L) \end{cases}$$

in the discrete sense. Then

$$u_h(V_0) \le C \frac{1 + \log L}{|\Omega_h \cap \partial \bar{B}_{dsc}(V_0, L)|} \int_{\Omega_h \cap \partial \bar{B}_{dsc}(V_0,L)} u_h \, dx.$$

Proof. We find out that

$$u_h(V_0) = \sum_{V_k^L \in \partial B_{dsc}(V_0,L)} w_k^L u_h(V_k^L)$$

since $u_h = 0$ on $\partial\Omega_h \cap B_{dsc}(V_0, L)$. We let

$$\begin{cases} \Delta_h v_h = 0 & \text{in } B_{dsc}(V_0, L) \\ v_h(V_j^L) = \delta_{jk} & \text{for } V_j^L \in \partial B_{dsc}(V_0, L). \end{cases}$$

Then from the estimate of Lemma 3,

$$v_h(V_0) \le C \frac{1 + \log L}{L^{N-1}}.$$

Therefore from the discrete maximum principle we have

$$w_k^L \le v_h(V_0) \le C \frac{1 + \log L}{L^{N-1}},$$

and

$$u_h(V_0) \le C \frac{1 + \log L}{L^{N-1}} \sum_{V_j^L \in \Omega_h \cap \partial B_{dsc}(V_0,L)} u_h(V_j^L).$$

Following the same argument of Theorem 2 we prove

$$u_h(V_0) \le C \frac{1 + \log L}{|\Omega_h \cap \partial \bar{B}_{dsc}(V_0, L)|} \int_{\Omega_h \cap \partial \bar{B}_{dsc}(V_0,L)} u_h \, dx.$$

□

The above estimate is the weak boundary maximum principle. With standard modification we prove the following Theorem.

THEOREM 4 (Boundary Pointwise Estimate). *Suppose that* $-\Delta u = f$ *in* Ω_h *with* $u = 0$ *on* $\partial\Omega_h$ *and* $-\Delta_h u_h = f$ *in* Ω_h *with* $u_h = 0$ *on* $\partial\Omega_h$. *Then the pointwise error estimate up to boundary is*

$$\begin{aligned}
&\|u - u_h\|_{L^\infty(B(V_0,R))} \le C\|u - P_1 u\|_{L^\infty(B(V_0,R))} \\
&+ C\left(\log\frac{R}{h}\right)\frac{1}{|B(V_0,2R)\cap\Omega_h|}\int_{B(V_0,2R)\cap\Omega_h}|u - P_1 u|\,dx \\
&+ C\left(\log\frac{R}{h}\right)\frac{1}{|B(V_0,2R)\cap\Omega_h|}\int_{B(V_0,2R)\cap\Omega_h}|u - u_h|\,dx
\end{aligned}$$

where $P_1 u$ *is the* H_0^1 *projection of* u *in* Ω_h.

Proof. From the quasi-uniform mesh $\mathcal{T}_h$, there is some discrete length L such that $B(V_0,R) \subset \overline{B}_{dsc}(V_0,L) \subset B(V_0,2R)$. From the fact that $P_1 u - u_h$ is a discrete harmonic, the pointwise error estimate satisfies the following inequality, using Theorem 3,

$$\begin{aligned}
\|u - u_h\|_{L^\infty(B(V_0,R))} &\le \|u - P_1 u\|_{L^\infty(B(V_0,R))} + C\|P_1 u - u_h\|_{L^\infty(\overline{B}_{dsc}(V_0,L))} \\
&\le \|u - P_1 u\|_{L^\infty(B(V_0,R))} \\
&\quad + C\log L\,\frac{1}{|\overline{B}_{dsc}(V_0,L)\cap\Omega_h|}\int_{\overline{B}_{dsc}(V_0,L)}|P_1 u - u_h|\,dx.
\end{aligned}$$

Since $Lh < CR$ for some constant $C > 0$ independent of h, we have proven our main theorem. □

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