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SINGULAR INNER FUNCTIONS OF L^1-TYPE

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ABSTRACT. Let $\mathcal{M}$ be the maximal ideal space of the Banach algebra H^∞ of bounded analytic functions on the open unit disc Δ. For a positive singular measure μ on $\partial\Delta$, let $L^1_+(\mu)$ be the set of measures ν with $0 \le \nu \ll \mu$ and ψ_ν the associated singular inner functions. Let $\mathcal{R}(\mu)$ and $\mathcal{R}_0(\mu)$ be the union sets of $\{|\psi_\nu| < 1\}$ and $\{\psi_\nu = 0\}$ in $\mathcal{M} \setminus \Delta$, $\nu \in L^1_+(\mu)$, respectively. It is proved that if $S(\mu) = \partial\Delta$, where $S(\mu)$ is the closed support set of μ, then $\mathcal{R}(\mu) = \mathcal{R}_0(\mu) = \mathcal{M} \setminus (\Delta \cup M(L^\infty(\partial\Delta)))$ and $L^\infty(\partial\Delta)$ is generated by H^∞ and $\overline{\psi_\nu}, \nu \in L^1_+(\mu)$. It is proved that $d\theta(S(\mu)) = 0$ if and only if there exists a Blaschke product b with zeros $\{z_n\}_n$ such that $\mathcal{R}(\mu) \subset \{|b| < 1\}$ and $S(\mu)$ coincides with the set of cluster points of $\{z_n\}_n$. While, we prove that μ is a sum of finitely many point measures if and only if there exists another positive singular measure λ such that $\mathcal{R}(\mu) \subset \{|\psi_\lambda| < 1\}$ and $S(\lambda) = S(\mu)$. Also it is studied conditions on μ for which $\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$.

1. Introduction

Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disc Δ. We denote by $\mathcal{M} = M(H^\infty)$ the maximal ideal space of H^∞, the space of nonzero multiplicative linear functionals of H^∞ with the weak*-topology. Considering point evaluation, we may consider that $\Delta \subset \mathcal{M}$ and Δ is an open subset of $\mathcal{M}$. Carleson's corona theorem [2] says that Δ is dense in $\mathcal{M}$. Identifying a function in H^∞ with its Gelfand transform, we may consider that H^∞ is the closed subalgebra of $C(\mathcal{M})$, the space of continuous functions on $\mathcal{M}$. We also identify a function in H^∞ with its boundary function. Then we may consider

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that H^∞ is an (essentially) supremum norm closed subalgebra of L^∞, the usual Lebesgue space on the unit circle $\partial\Delta$. For a subset F of L^∞, we denote by $H^\infty[F]$ the closed subalgebra generated by H^∞ and F. We may consider that the maximal ideal space $M(L^\infty)$ of L^∞ as a subset of $\mathcal{M}$ and $M(L^\infty)$ as the Shilov boundary of H^∞. A function f in H^∞ is called inner if $|f| = 1$ on $M(L^\infty)$. For a function f in H^∞, we put

$$\{|f| < 1\} = \{x \in \mathcal{M}\backslash\Delta; |f(x)| < 1\} \text{ and } Z(f) = \{x \in \mathcal{M}\backslash\Delta; f(x) = 0\}.$$

We note that these sets are considered in $\mathcal{M} \setminus \Delta$. For a subset E of $\mathcal{M}$, we denote by $\overline{E}$ the weak*-closure of E in $\mathcal{M}$. On the other hand, for a subset F of Δ, we denote by clF the closure of F in the complex plane. See [10] for the study of the structure of $\mathcal{M}$.

For a sequence $\{z_n\}_n$ in Δ satisfying $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$, we can define a function

$$b(z) = \prod_{n=1}^{\infty} \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z}, \quad z \in \Delta.$$

Then b is an inner function and called a Blaschke product with zeros $\{z_n\}_n$. Put $S(b) = cl\{z_n\}_n \setminus \{z_n\}_n \subset \partial\Delta$. Then $S(b)$ is the set of points $e^{i\theta} \in \partial\Delta$ on which b can not be extended analytically. Since $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$, there is a sequence of positive integers $p = (p_1, p_2, \cdots)$ such that

$$\sum_{n=1}^{\infty} p_n(1 - |z_n|) < \infty \quad \text{and} \quad p_n \to \infty \quad \text{as } n \to \infty.$$

We denote by $\mathcal{P}(b)$ the set of sequences p as above. Then for $p = (p_1, p_2, \cdots) \in \mathcal{P}(b)$ we have an associated Blaschke product defined by

$$b^p(z) = \prod_{n=1}^{\infty} \left(\frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z} \right)^{p_n}, \quad z \in \Delta.$$

In [12], the first author called Blaschke products $b^p, p \in \mathcal{P}(b)$, weak infinite powers of b and studied them. Let

$$\mathcal{R}(b) = \bigcup \{\{|b^p| < 1\}; p \in \mathcal{P}(b)\} \quad \text{and} \quad \mathcal{R}_0(b) = \bigcup \{Z(b^p); p \in \mathcal{P}(b)\}.$$

It is easy to see that $\mathcal{R}(b) = \mathcal{R}_0(b)$. The first author proved the following theorems.

THEOREM A. *Let b be a Blaschke product. Then the following conditions are equivalent.*

(i) $S(b) = \partial\Delta$.

(ii) $\mathcal{R}(b) = \mathcal{M}\backslash(\Delta \cup M(L^\infty))$.
(iii) $L^\infty = H^\infty[\overline{b^p}; p \in \mathcal{P}(b)]$.

Let $d\theta$ be the arc length measure on $\partial\Delta$.

THEOREM B. *Let b be a Blaschke product. Then the following conditions are equivalent.*

(i) $d\theta(S(b)) = 0$.
(ii) *There is a Blaschke product B such that $S(B) = S(b)$ and $\mathcal{R}(b) \subset \{|B| < 1\}$.*
(iii) *There is a Blaschke product B such that $S(B) = S(b)$ and $\mathcal{R}(b) \subset Z(B)$.*
(iv) *There is a Blaschke product B such that $S(B) = S(b)$ and $H^\infty[\overline{b^p}; p \in \mathcal{P}(b)] \subset H^\infty[\overline{B}]$.*

In this paper, we investigate singular inner function's versions of the above two theorems. We denote by $M(\partial\Delta)$ the Banach space of bounded regular Borel measures on $\partial\Delta$ with the total variation norm. Since $M(\partial\Delta)$ is the dual space of $C(\partial\Delta)$, the space of continuous functions on $\partial\Delta$, we may consider the weak*-topology on $M(\partial\Delta)$. Let M_s^+ be the set of positive (nonzero) singular measures in $M(\partial\Delta)$ with respect to the Lebesgue measure on $\partial\Delta$.

Let $P_z(e^{i\theta})$ be the Poisson kernel for $z \in \Delta$, that is,

$$P_z(e^{i\theta}) = \operatorname{Re}\frac{e^{i\theta} + z}{e^{i\theta} - z}.$$

For each $\mu \in M_s^+$, let

$$\psi_\mu(z) = \exp\left(-\int_{\partial\Delta} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(e^{i\theta})\right), \quad z \in \Delta.$$

Then ψ_μ is inner and called a singular inner function. We note that

$$|\psi_\mu(z)| = \exp\left(-\int_{\partial\Delta} P_z(e^{i\theta}) d\mu(e^{i\theta})\right), \quad z \in \Delta.$$

We denote by $S(\mu)$ the closed support set of μ. Then $S(\mu)$ is the set of points $e^{i\theta} \in \partial\Delta$ on which ψ_μ can not be extended analytically, see [4, 9]. Let

$$L^1_+(\mu) = \{\nu \in M_s^+; 0 \le \nu \ll \mu\ , \nu \ne 0\}.$$

Then we have a family of singular inner functions $\{\psi_\nu; \nu \in L^1_+(\mu)\}$. We call these functions associated singular inner functions of L^1-type for the measure μ. We may expect that these functions play the same role as weak infinite powers of a Blaschke product. We put

$$\mathcal{R}(\mu)=\bigcup\{\{|\psi_\nu|<1\};\nu\in L^1_+(\mu)\} \text{ and } \mathcal{R}_0(\mu)=\bigcup\{Z(\psi_\nu);\nu\in L^1_+(\mu)\}.$$

In section 2, we prove a singular inner function's version of Theorem A. In section 3, we study a singular inner function's version of Theorem B, and prove that $d\theta(S(\mu))=0$ if and only if there is a Blaschke product B such that $S(B)=S(\mu)$ and $\mathcal{R}(\mu)\subset Z(B)$. Also, we prove that for $\mu\in M_s^+$, if there exists $\lambda\in M_s^+$ such that $S(\lambda)=S(\mu)$ and $\mathcal{R}(\mu)\subset\{|\psi_\lambda|<1\}$, then $S(\mu)$ is a finite set, and there are no $\lambda\in M_s^+$ such that $S(\lambda)=S(\mu)$ and $\mathcal{R}(\mu)\subset Z(\psi_\lambda)$.

In the study of singular inner functions of L^1-type, we will find some difference of properties between Blaschke products and singular inner functions. We have $\mathcal{R}(b)=\mathcal{R}_0(b)$ for every Blaschke product b, but $\mathcal{R}(\mu)\neq\mathcal{R}_0(\mu)$ for some $\mu\in M_s^+$. For $e^{i\theta}\in\partial\Delta$, we denote by $\delta_{e^{i\theta}}$ the unit point mass at $e^{i\theta}\in\partial\Delta$. Then we have $\mathcal{R}_0(\delta_{e^{i\theta}})\subsetneqq\mathcal{R}(\delta_{e^{i\theta}})$. In section 4, we study conditions on $\mu\in M_s^+$ for which $\mathcal{R}(\mu)=\mathcal{R}_0(\mu)$. In section 5, we study especially on discrete measures.

We denote by $M_{s,c}^+$ and $M_{s,d}^+$ the sets of continuous and discrete measures in M_s^+, respectively.

2. Singular inner functions of L^1–type

First, we prove the following theorem.

THEOREM 2.1. *Let $\mu\in M_s^+$ such that $S(\mu)=\partial\Delta$. Then for every Blaschke product b, there exists $\nu\in L^1_+(\mu)$ such that $\{|b|<1\}\subset Z(\psi_\nu)$.*

We note that in [5] Gorkin proved that for a Blaschke product b there exists a discrete measure $\mu\in M_s^+$ such that $\{|b|<1\}\subset Z(\psi_\mu)$. To prove our theorem, we need some facts. For each $z\in\Delta$ with $|z|\neq 0$, we have $P_z(e^{i\theta})\le(1+|z|)/(1-|z|)$ for every $e^{i\theta}\in\partial\Delta$ and $P_z(z/|z|)=(1+|z|)/(1-|z|)$. Hence there exists an open subarc J of $\partial\Delta$ such that $z/|z|\in J$ and $P_z>1/(1-|z|)$ on J.

LEMMA 2.1. *Let $\{z_n\}_n$ be a sequence in Δ such that $\sum_{n=1}^\infty(1-|z_n|)<\infty$ and $|z_n|\neq 0$ for every n. Let $\{p_n\}_n$ be a sequence of positive numbers such that $\sum_{n=1}^\infty p_n(1-|z_n|)<\infty$ and $p_n\to\infty$ as $n\to\infty$. Let J_n be an open subarc of $\partial\Delta$ such that $z_n/|z_n|\in J_n$ and $P_{z_n}>1/(1-|z_n|)$ on*

J_n. Let $\mu \in M_s^+$ such that $\mu = \sum_{n=1}^\infty \mu_n, \mu_n \in M_s^+, S(\mu_n) \subset clJ_n$, and $\|\mu_n\| \geq p_n(1-|z_n|)$ for every n. Then $\psi_\mu(z_n) \to 0$ as $n \to \infty$.

Proof. By our assumption, we have

$$\begin{aligned}\int_{\partial\Delta} P_{z_n}(e^{i\theta})d\mu(e^{i\theta}) &\geq \int_{J_n} P_{z_n}(e^{i\theta})d\mu_n(e^{i\theta}) \\ &\geq \|\mu_n\|/(1-|z_n|) \\ &\geq p_n.\end{aligned}$$

Since $p_n \to \infty$ as $n \to \infty$,

$$|\psi_\mu(z_n)| = \exp\left(-\int_{\partial\Delta} P_{z_n}(e^{i\theta})d\mu(e^{i\theta})\right) \to 0 \quad \text{as } n \to \infty. \qquad \square$$

A Blaschke product b and its zeros $\{z_n\}_n$ are called interpolating if for every bounded sequence of complex numbers $\{a_n\}_n$, there exists a function $f \in H^\infty$ such that $f(z_n) = a_n$ for every n. It is known that if b is an interpolating Blaschke product, $Z(b) = \overline{\{z_n\}_n} \setminus \{z_n\}_n$, see [9, p. 205]. In the study of H^∞, interpolating Blaschke products play an important role, see [4, 11].

By [1, 7], we have the following, see also [5].

LEMMA 2.2. *Let b be an interpolating Blaschke product and $\mu \in M_s^+$. If $Z(b) \subset Z(\psi_\mu)$, then $\{|b| < 1\} \subset Z(\psi_\mu)$.*

Proof of Theorem 2.1. It is known that there is an interpolating Blaschke product q such that $\{|q| < 1\} = \{|b| < 1\}$, see [15] or [5]. Hence we may assume that b is interpolating with zeros $\{z_n\}_n$ and $|z_n| \neq 0$ for every n. Since $\sum_{n=1}^\infty(1-|z_n|) < \infty$, there is a sequence of positive numbers $\{p_n\}_n$ such that $\sum_{n=1}^\infty p_n(1-|z_n|) < \infty$ and $p_n \to \infty$ as $n \to \infty$. For each n, take an open subarc J_n of $\partial\Delta$ such that $z_n/|z_n| \in J_n$ and $P_{z_n} > 1/(1-|z_n|)$ on J_n. Since $S(\mu) = \partial\Delta$, $\|\mu_{|J_n}\| \neq 0$, where $\mu_{|J_n}$ is the restriction measure of μ on J_n. Put

$$\mu_n = \frac{p_n(1-|z_n|)\mu_{|J_n}}{\|\mu_{|J_n}\|} \quad \text{and} \quad \nu = \sum_{n=1}^\infty \mu_n.$$

Then

$$\|\nu\| \leq \sum_{n=1}^\infty \|\mu_n\| = \sum_{n=1}^\infty p_n(1-|z_n|) < \infty.$$

Since $\mu_n \in L^1_+(\mu)$, we have $\nu \in L^1_+(\mu)$. Also it is clear that $S(\mu_n) \subset cl J_n$. Then by Lemma 2.1, $\psi_\nu(z_n) \to 0$ as $n \to \infty$, so that $Z(b) \subset Z(\psi_\nu)$. Hence by Lemma 2.2, we obtain our assertion. □

As applications of Theorem 2.1, we have the following corollaries, see [12]. Here we note that by [8] if b is a Blaschke product and $\mu \in M_s^+$ such that $\{|b| < 1\} \subset Z(\psi_\mu)$, then $\psi_\mu \bar{b} \in H^\infty + C(\partial\Delta)$.

COROLLARY 2.1. *Let $\mu \in M_s^+$ such that $S(\mu) = \partial\Delta$. Then for $f \in L^\infty$, there exists $\nu \in L^1_+(\mu)$ such that $\psi_\nu f \in H^\infty + C(\partial\Delta)$.*

COROLLARY 2.2. *Let $\mu \in M_s^+$ such that $S(\mu) = \partial\Delta$. Then for $f \in L^\infty$ and $\varepsilon > 0$, there exists $\nu \in L^1_+(\mu)$ and $h \in H^\infty$ such that $\|\psi_\nu f - h\| < \varepsilon$.*

Proof. Use Corollary 2.1 and a fact that if $g \in C(\partial\Delta)$ and $\mu \in M_s^+$, then $\|g\psi_\mu^n + H^\infty\| \to 0$ as $n \to \infty$. □

The following is the singular inner function's version of Theorem A.

THEOREM 2.2. *Let $\mu \in M_s^+$. Then the following assertions are equivalent.*

(i) $S(\mu) = \partial\Delta$.
(ii) $\mathcal{R}_0(\mu) = \mathcal{M}\backslash(\Delta \cup M(L^\infty))$.
(iii) $\mathcal{R}(\mu) = \mathcal{M}\backslash(\Delta \cup M(L^\infty))$.
(iv) $L^\infty = H^\infty[\overline{\psi_\nu}; \nu \in L^1_+(\mu)]$.

Proof. (i) ⇒ (ii) Suppose that $S(\mu) = \partial\Delta$. For each Blaschke product b, by Theorem 2.1 there exists $\nu \in L^1_+(\mu)$ such that $\{|b| < 1\} \subset Z(\psi_\nu)$. By Newman's Theorem [14],

$$\bigcup \{\{|b| < 1\}; b \text{ is a Blaschke product}\} = \mathcal{M} \setminus (\Delta \cup M(L^\infty)),$$

hence (ii) holds.

(ii) ⇒ (iii) follows from $\mathcal{R}_0(\mu) \subset \mathcal{R}(\mu) \subset \mathcal{M} \setminus (\Delta \cup M(L^\infty))$.

(iii) ⇒ (i) Suppose that $S(\mu) \neq \partial\Delta$. Then ψ_μ can be extended analytically at each point in $\partial\Delta \setminus S(\mu)$ and $|\psi_\mu| = 1$ on $\partial\Delta \setminus S(\mu)$. Hence it is not difficult to see that $\mathcal{R}(\mu) \neq \mathcal{M} \setminus (\Delta \cup M(L^\infty))$.

(iii) ⇔ (iv) This follows from the Chang and Marshall Theorem [3, 13]. □

Next, we study $\mathcal{R}(\mu)$ when $S(\mu) \neq \partial\Delta$. For $-1 < R < 1$ and $e^{i\theta} \in \partial\Delta$, let

$$\Delta_R(e^{i\theta}) = \{z \in \Delta; |z - (1+R)e^{i\theta}/2| < (1-R)/2\}.$$

LEMMA 2.3. *Let $-1 < R < 1$ and $e^{i\theta} \in \partial\Delta$. Then*

$$\Delta_R(e^{i\theta}) = \left\{ z \in \Delta; |\psi_{\delta_{e^{i\theta}}}(z)| < e^{-\frac{1+R}{1-R}} \right\}.$$

Proof. This follows from that

$$|\psi_{\delta_{e^{i\theta}}}(z)| < e^{-\frac{1+R}{1-R}} \quad \text{if and only if} \quad \frac{1-|z|^2}{|e^{i\theta}-z|^2} > \frac{1+R}{1-R}. \qquad \square$$

The following theorem is the singular inner function's version of [12, Theorem 4.1].

THEOREM 2.3. *Let $\mu \in M_s^+$ and $e^{i\theta_0} \in S(\mu)$. Then there exists $\nu \in L^1_+(\mu)$ such that $\psi_\nu(re^{i\theta_0}) \to 0$ as $r \to 1, 0 < r < 1$.*

Proof. We may assume that $e^{i\theta_0} = 1$. Let $\{R_n\}_n$ be a sequence of increasing positive numbers such that $\sum_{n=1}^\infty (1-R_n) < \infty$ and $0 < R_n < 1$. Then there exists another sequence of positive numbers $\{p_n\}_n$ such that

$$(2.1) \qquad \sum_{n=1}^\infty p_n(1-R_n) < \infty \quad \text{and} \quad p_n \to \infty \quad \text{as } n \to \infty.$$

We have $(R_n, R_{n+1}] \subset \Delta_{R_n}(1)$. Hence by Lemma 2.3,

$$(2.2) \qquad |\psi_{\delta_1}| < e^{-\frac{1+R_n}{1-R_n}} \quad \text{on } (R_n, R_{n+1}].$$

For each positive integer j, let $J_j = \{e^{i\theta} \in \partial\Delta; |\theta| \leq 1/j\}$. Since $1 = e^{i\theta_0} \in S(\mu)$, $\|\mu_{|J_j}\| \neq 0$. Put

$$(2.3) \qquad \mu_j = \mu_{|J_j}/\|\mu_{|J_j}\|.$$

Then μ_j converges to δ_1 as $j \to \infty$ in the weak*-topology of $M(\partial\Delta)$, so that $\psi_{\mu_j}(z) \to \psi_{\delta_1}(z)$ uniformly on compact subsets of Δ. Hence for each positive integer n, by (2.2) there exists a positive integer j_n such that

$$(2.4) \qquad |\psi_{\mu_{j_n}}| < e^{-\frac{1+R_n}{1-R_n}+1} \quad \text{on } (R_n, R_{n+1}].$$

Let

$$\nu = \sum_{k=1}^\infty p_k(1-R_k)\mu_{j_k}.$$

Then by (2.1) and (2.3), $\nu \in L^1_+(\mu)$, and for every $r \in (R_n, R_{n+1}]$ we have

$$\begin{aligned} |\psi_\nu(r)| &= \prod_{k=1}^{\infty} |\psi_{\mu_{j_k}}(r)|^{p_k(1-R_k)} \\ &\le |\psi_{\mu_{j_n}}(r)|^{p_n(1-R_n)} \\ &< e^{\left(-\frac{1+R_n}{1-R_n}+1\right)p_n(1-R_n)} \qquad \text{by (2.4).} \end{aligned}$$

Hence

$$\sup_{R_n < r \le R_{n+1}} |\psi_\nu(r)| \le e^{-p_n(1+R_n)+p_n(1-R_n)}.$$

By (2.1), $p_n(1+R_n) \to \infty$ and $p_n(1-R_n) \to 0$ as $n \to \infty$. Hence

$$\sup_{R_n < r \le R_{n+1}} |\psi_\nu(r)| \to 0 \quad \text{as } n \to \infty.$$

Thus we get our assertion. □

COROLLARY 2.3. *Let $\mu_1, \mu_2 \in M_s^+$. Then the following conditions are equivalent.*

(i) $S(\mu_1) \cap S(\mu_2) = \emptyset$.
(ii) $\mathcal{R}(\mu_1) \cap \mathcal{R}(\mu_2) = \emptyset$.
(iii) $\mathcal{R}_0(\mu_1) \cap \mathcal{R}_0(\mu_2) = \emptyset$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. To prove (iii) $\Rightarrow$ (i), suppose that $S(\mu_1) \cap S(\mu_2) \neq \emptyset$. Take $e^{i\theta_0} \in S(\mu_1) \cap S(\mu_2)$. Then, by Theorem 2.3, there exist $\nu \in L^1_+(\mu_1)$ and $\sigma \in L^1_+(\mu_2)$ such that $\psi_\nu(re^{i\theta_0}) \to 0$ and $\psi_\sigma(re^{i\theta_0}) \to 0$ as $r \to 1, 0 < r < 1$. Thus both sets $\mathcal{R}_0(\mu_1)$ and $\mathcal{R}_0(\mu_2)$ contain $\overline{\{re^{i\theta_0}; 0 < r < 1\}} \setminus \{re^{i\theta_0}; 0 < r < 1\}$. Hence we have $\mathcal{R}_0(\mu_1) \cap \mathcal{R}_0(\mu_2) \neq \emptyset$. □

COROLLARY 2.4. *Let $\mu \in M_s^+$. If $d\theta(S(\mu)) > 0$, then there are no nonzero $f \in H^\infty$ such that $\mathcal{R}_0(\mu) \subset Z(f)$.*

Proof. Suppose that $d\theta(S(\mu)) > 0$ and $\mathcal{R}_0(\mu) \subset Z(f)$ for some $f \in H^\infty$. By Theorem 2.3, $f(re^{i\theta}) \to 0$ as $r \to 1, 0 < r < 1$, for every $e^{i\theta} \in S(\mu)$. Since $d\theta(S(\mu)) > 0$, we have $f = 0$. □

3. Associated domains

In [12], the first author defined an associated domain of a Blaschke product b with zeros $\{z_n\}_n$ as follows; $\Omega(b) = \cup \{z \in \Delta; \rho(z_n, z) < |z_n|\}$, where $\rho(z_n, z) = |z - z_n|/|1 - \overline{z}_n z|$. In this section, we define an associated domain $\Omega(\mu)$ of $\mu \in M_s^+$, and using this properties we study $\mathcal{R}(\mu)$.

Let $\mu \in M_s^+$. Suppose that $S(\mu) \neq \partial\Delta$. Then there is a sequence (may be finite) of disjoint open subarcs $\{J_k\}_k$ of $\partial\Delta$ such that

$$\partial\Delta \setminus S(\mu) = \bigcup_{k=1}^{\infty} J_k. \tag{3.1}$$

Put $J_k = \{e^{i\theta}; s_k < \theta < t_k\}$. For each positive integer k, take two sequences $\{s_{k,n}\}_n$ and $\{t_{k,n}\}_n$ such that $s_k < s_{k,n} < s_{k,1} < t_{k,1} < t_{k,n} < t_k$, $s_{k,n} \to s_k$ and $t_{k,n} \to t_k$ as $n \to \infty$. Let $\{R_n\}_n$ be a sequence of decreasing numbers such that

$$-1 < R_n < 0 \quad \text{and} \quad \lim_{n\to\infty} R_n = -1. \tag{3.2}$$

Put

$$\Omega_n = \bigcup \{\Delta_{R_n}(e^{i\theta}); e^{i\theta} \in S(\mu)\}. \tag{3.3}$$

Then Ω_n is a simply connected domain, $\Omega_n \subset \Omega_{n+1}$, and $\partial\Omega_n \cap \partial\Delta = S(\mu)$. By (3.2) and (3.3), $\cup_{n=1}^{\infty}\Omega_n = \Delta$. For each pair of integers k and n such that $1 \leq k \leq n$, let

$$E_{k,n} = \{z \in \Delta; s_{k,n} \leq \arg z \leq t_{k,n}, z \notin \Omega_n\}. \tag{3.4}$$

Then $E_{k,n}$ is a closed subset of Δ, and for each fixed k, the sequence $\{E_{k,n}\}_n$ converges to J_k as $n \to \infty$. Let

$$E = \bigcup_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_{k,n}. \tag{3.5}$$

Then E is a closed subset of Δ. Let

$$\Omega(\mu) = \Delta \setminus E. \tag{3.6}$$

When $S(\mu) = \partial\Delta$, we put $\Omega(\mu) = \Delta$. By our construction, $\Omega(\mu)$ is a simply connected domain and

$$\partial\Omega(\mu) \cap \partial\Delta = S(\mu). \tag{3.7}$$

We call $\Omega(\mu)$ the associated domain of μ. Since there are infinitely many choices of sequences $\{E_{k,n}\}_{k,n}$, $\Omega(\mu)$ is not determined uniquely. In any way, for each μ we assign $\Omega(\mu)$ one of such domains.

The following theorem is the singular inner function's version of [12, Theorem 4.4].

THEOREM 3.1. *Let $\mu \in M_s^+$. Then $\mathcal{R}(\mu) \subset \overline{\Omega(\mu)}$.*

To prove Theorem 3.1, we need the following lemma.

LEMMA 3.1. *Let $\mu \in M_s^+$ and $z_0 \in \Delta, z_0 \neq 0$. Let $e^{it_0} \in S(\mu)$ be the nearest point from z_0. Then $|\psi_\mu(z_0)| \geq |\psi_{\delta_{e^{it_0}}}(z_0)|^{\|\mu\|}$.*

Proof. By simple calculation, we have

$$\begin{aligned} |\psi_\mu(z_0)| &= \exp\left(-\int_{\partial\Delta} P_{z_0}(e^{i\theta})d\mu(e^{i\theta})\right) \\ &\geq \exp\left(-\|\mu\| \int_{\partial\Delta} P_{z_0}(e^{i\theta})d\delta_{e^{it_0}}\right) \\ &= |\psi_{\delta_{e^{it_0}}}(z_0)|^{\|\mu\|}. \end{aligned}$$

□

Proof of Theorem 3.1. When $S(\mu) = \partial\Delta$, $\overline{\Omega(\mu)} = \overline{\Delta} = \mathcal{M}$, so that we may assume that $S(\mu) \neq \partial\Delta$. To describe $\Omega(\mu)$, we use (3.1) - (3.7).

To prove our assertion, suppose not. Then there exist $\nu \in L_+^1(\mu)$ and a sequence $\{z_j\}_j$ in $\Delta \setminus \Omega(\mu)$ such that $|z_j| \to 1$ and

$$\limsup_{j\to\infty} |\psi_\nu(z_j)| < 1. \tag{3.8}$$

For each j, by (3.5) and (3.6), $z_j \in E_{k_j,n_j}$ for some $k_j \leq n_j$. Then by (3.4), $z_j \notin \Omega_{n_j}$. Hence by (3.3), $z_j \notin \Delta_{R_{n_j}}(e^{i\theta})$ for every $e^{i\theta} \in S(\mu)$. By Lemma 2.3,

$$|\psi_{\delta_{e^{i\theta}}}(z_j)| \geq e^{-\frac{1+R_{n_j}}{1-R_{n_j}}} \quad \text{for every } e^{i\theta} \in S(\mu).$$

Since $S(\nu) \subset S(\mu)$, by Lemma 3.1 we have

$$|\psi_\nu(z_j)| \geq e^{-\frac{1+R_{n_j}}{1-R_{n_j}}\|\nu\|} \quad \text{for every } j. \tag{3.9}$$

By considering a subsequence of $\{z_j\}_j$, we may assume that $n_j \to \infty$ as $j \to \infty$ or $n_j = n_0$ for every j. When $n_j \to \infty$ as $j \to \infty$, by (3.2) and (3.9) we have $|\psi_\nu(z_j)| \to 1$. This contradicts (3.8). When $n_j = n_0$ for every j, we have $1 \leq k_j \leq n_0$ for every j. Hence we may assume that $k_j = k_0$ for every j. Then $z_j \in E_{k_0,n_0}$ for every j. Since $|z_j| \to 1$, by (3.4) we have $cl\{z_j\}_j \setminus \{z_j\}_j \subset J_{k_0}$. Since $S(\nu) \subset S(\mu)$, $S(\nu) \cap J_{k_0} = \emptyset$. Hence $|\psi_\nu(z_j)| \to 1$ as $j \to \infty$. This also contradicts (3.8). □

By (3.7), $\Omega(\mu)$ is a simply connected domain and $\partial\Omega(\mu)\cap\partial\Delta = S(\mu)$. In [12], the first author proved that for a Blaschke product b, $d\theta(S(b)) = 0$ if and only if there exists a Blaschke product B such that $S(B) = S(b)$ and $\overline{\Omega(b)}\setminus\Delta \subset \{|B| < 1\}$. In the same way as the proof of [12, Theorem 4.6] and using Theorem 3.1, we can prove the following theorem which is the singular inner function's version of Theorem B.

THEOREM 3.2. *Let $\mu \in M_s^+$. Then the following conditions are equivalent.*

(i) $d\theta(S(\mu)) = 0$.
(ii) *There exists a Blaschke product b such that $S(b) = S(\mu)$ and $\mathcal{R}(\mu) \subset Z(b)$.*
(iii) *There exists a Blaschke product b such that $S(b) = S(\mu)$ and $\mathcal{R}(\mu) \subset \{|b| < 1\}$.*
(iv) *There exists a Blaschke product b such that $S(b) = S(\mu)$ and $H^\infty[\overline{\psi_\nu}; \nu \in L^1_+(\mu)] \subset H^\infty[\overline{b}]$.*

Here, we have a question for which $\mu \in M_s^+$ there exists $\lambda \in M_s^+$ such that $S(\lambda) = S(\mu)$ and $\mathcal{R}(\mu) \subset \{|\psi_\lambda| < 1\}$. We note that $\mathcal{R}(\delta_{e^{i\theta}}) = \{|\psi_{\delta_{e^{i\theta}}}| < 1\}$. Hence if $S(\mu)$ is a finite set, then $\mathcal{R}_0(\mu) = Z(\psi_\mu) \subsetneqq \{|\psi_\mu| < 1\} = \mathcal{R}(\mu)$. Now we have the following theorem.

THEOREM 3.3. *Let $\mu \in M_s^+$. Then the following conditions are equivalent.*

(i) *$S(\mu)$ is a finite set.*
(ii) $\mathcal{R}(\mu) = \{|\psi_\mu| < 1\}$.
(iii) *There exists $\lambda \in M_s^+$ such that $S(\lambda) = S(\mu)$ and $\mathcal{R}(\mu) \subset \{|\psi_\lambda| < 1\}$.*
(iv) *There exists $\lambda \in M_s^+$ such that $S(\lambda) = S(\mu)$ and $\mathcal{R}_0(\mu) \subset \{|\psi_\lambda| < 1\}$.*
(v) *There exists $\lambda \in M_s^+$ such that $S(\lambda) = S(\mu)$ and $\mathcal{R}_0(\mu) \subset Z(\psi_\lambda)$.*

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (v) are already mentioned.

(ii) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (iii) Suppose that there exists $\lambda \in M_s^+$ such that $S(\lambda) = S(\mu)$ and $\mathcal{R}_0(\mu) \subset \{|\psi_\lambda| < 1\}$. To prove (iii), let $x \in \mathcal{R}(\mu)$. Then there exists $\nu \in L^1_+(\mu)$ such that $|\psi_\nu(x)| < 1$, so that by [1, p. 92] there exists $y \in \mathcal{M}$ such that $\operatorname{supp}\mu_y \subset \operatorname{supp}\mu_x$ and $\psi_\nu(y) = 0$, where μ_x is the representing measure on $M(L^\infty)$ for x, that is,

$$f(x) = \int_{M(L^\infty)} f\, d\mu_x \qquad \text{for } f \in H^\infty$$

and $\operatorname{supp}\mu_x$ is the closed support set of μ_x. Then $y \in \mathcal{R}_0(\mu)$, so that by our assumption we have $|\psi_\lambda(y)| < 1$. Since $\operatorname{supp}\mu_y \subset \operatorname{supp}\mu_x$, we have $|\psi_\lambda(x)| < 1$. Thus we get (iii).

(iii) $\Rightarrow$ (i) Let $\lambda \in M_s^+$ such that

$$S(\lambda) = S(\mu) \quad \text{and} \quad \mathcal{R}(\mu) \subset \{|\psi_\lambda| < 1\}. \tag{3.10}$$

By Frostman's theorem (see [4]), there exists a Blaschke product B such that $S(B) = S(\lambda)$ and $\{|B| < 1\} = \{|\psi_\lambda| < 1\}$. Then by (3.10), $\mathcal{R}(\mu) \subset \{|B| < 1\}$, so that by Theorem 3.2 we have $d\theta(S(\mu)) = 0$. If $\partial\Delta \setminus S(\mu)$ consists of finitely many disjoint open subarcs of $\partial\Delta$, $S(\mu)$ is a finite set.

So we shall show that $\partial\Delta \setminus S(\mu)$ consists of finitely many disjoint open subarcs. To prove this, suppose not. Then $\partial\Delta \setminus S(\mu) = \cup_{n=1}^\infty J_n$, where J_n is an open subarc of $\partial\Delta$, say $J_n = \{e^{i\theta}; s_n < \theta < t_n\}$, such that

$$J_n \cap S(\mu) = \emptyset \quad \text{and} \quad e^{is_n}, e^{it_n} \in S(\mu). \tag{3.11}$$

For the sake of simplicity, we assume that

$$0 < s_{n+1} < t_{n+1} < s_n < t_n \quad \text{and} \quad \lim_{n\to\infty} t_n = 0.$$

Put

$$\lambda_n = \lambda_{|\{e^{i\theta}; t_{n+1} \le \theta \le s_n\}}. \tag{3.12}$$

Then by (3.10) and (3.11), we have $\|\lambda_n\| \neq 0$. We also have $\sum_{n=1}^\infty \|\lambda_n\| \le \|\lambda\| < \infty$. Hence there exists a sequence of positive numbers $\{p_n\}_n$ such that

$$\sum_{n=1}^\infty p_n \|\lambda_n\| < \infty \quad \text{and} \quad p_n \to \infty \quad \text{as } n \to \infty. \tag{3.13}$$

Here we can take $-1 < R_n < 1$ such that

$$\frac{1 + R_n}{1 - R_n} \|\lambda_n\| = \frac{1}{p_n}. \tag{3.14}$$

Put $\lambda_n' = \lambda - \lambda_n$. Then by (3.12) we have $e^{is_n} \notin S(\lambda_n')$, so that for each fixed n, we have $|\psi_{\lambda_n'}(z)| \to 1$ as $z \to e^{is_n}$, $z \in \Delta$. Hence there exists a sequence $\{z_n\}_n$ such that

$$z_n \in \partial\Delta_{R_n}(e^{is_n}), \quad s_n < \arg z_n < (s_n + t_n)/2, \tag{3.15}$$

and

$$|\psi_{\lambda'_n}(z_n)| \to 1. \tag{3.16}$$

By (3.15) and Lemma 2.3, we have

$$|\psi_{\delta_{e^{is_n}}}(z_n)| = e^{-\frac{1+R_n}{1-R_n}}. \tag{3.17}$$

Hence

$$\begin{aligned} |\psi_\lambda(z_n)| &= |\psi_{\lambda_n}(z_n)||\psi_{\lambda'_n}(z_n)| \\ &\geq |\psi_{\delta_{e^{is_n}}}(z_n)|^{\|\lambda_n\|}|\psi_{\lambda'_n}(z_n)| \qquad \text{by (3.15) and Lemma 3.1} \\ &= e^{-\frac{1+R_n}{1-R_n}\|\lambda_n\|}|\psi_{\lambda'_n}(z_n)| \qquad \text{by (3.17)} \\ &= e^{-\frac{1}{p_n}}|\psi_{\lambda'_n}(z_n)| \qquad \text{by (3.14)}. \end{aligned}$$

Thus by (3.13) and (3.16),

$$|\psi_\lambda(z_n)| \to 1 \quad \text{as } n \to \infty. \tag{3.18}$$

For each $c < s_n$, let $\mu_{n,c} = \mu_{|\{e^{i\theta};c\leq\theta\leq s_n\}}$. By (3.11), $\|\mu_{n,c}\| \neq 0$. Then $\mu_{n,c}/\|\mu_{n,c}\| \to \delta_{e^{is_n}}$ as $c \to s_n$ in the weak*-topology of $M(\partial\Delta)$, so that there exists $c_n < s_n$ such that

$$|\psi_{\mu_n}(z_n)| \leq |\psi_{\delta_{e^{is_n}}}(z_n)|^{1/2}, \tag{3.19}$$

where $\mu_n = \mu_{n,c_n}/\|\mu_{n,c_n}\|$. Let

$$\nu = \sum_{n=1}^{\infty} p_n\|\lambda_n\|\mu_n. \tag{3.20}$$

Then by (3.13), $\|\nu\| \leq \sum_{n=1}^{\infty} p_n\|\lambda_n\| < \infty$. Hence $\nu \in L^1_+(\mu)$ and

$$\begin{aligned} |\psi_\nu(z_n)| &\leq |\psi_{\mu_n}(z_n)|^{p_n\|\lambda_n\|} \qquad \text{by (3.20)} \\ &\leq |\psi_{\delta_{e^{is_n}}}(z_n)|^{p_n\|\lambda_n\|/2} \qquad \text{by (3.19)} \\ &= e^{-\frac{1+R_n}{1-R_n}\frac{p_n\|\lambda_n\|}{2}} \qquad \text{by (3.17)} \\ &= e^{-1/2} \qquad \text{by (3.14)}. \end{aligned}$$

Therefore $\overline{\{z_n\}_n} \setminus \{z_n\}_n \subset \{|\psi_\nu| < 1\} \subset \mathcal{R}(\mu)$. On the other hand, by (3.18) we have $\overline{\{z_n\}_n} \setminus \{z_n\}_n \subset \{|\psi_\lambda| = 1\}$. Thus $\mathcal{R}(\mu) \not\subset \{|\psi_\lambda| < 1\}$. This contradicts (3.10). □

COROLLARY 3.1. *Let $\mu \in M_s^+$. Then there are no measures $\lambda \in M_s^+$ such that $S(\lambda) = S(\mu)$ and $\mathcal{R}(\mu) \subset Z(\psi_\lambda)$.*

Proof. Suppose that $\mathcal{R}(\mu) \subset Z(\psi_\lambda)$ for some $\lambda \in M_s^+$ with $S(\lambda) = S(\mu)$. By Theorem 3.3, $S(\lambda)$ is a finite set, so that $\mathcal{R}(\mu) = \{|\psi_\mu| < 1\} = \{|\psi_\lambda| < 1\} \supsetneqq Z(\psi_\lambda)$. This is a contradiction. □

4. Zero sets of singular inner functions of L^1-type

In Theorem 2.2, we proved that $\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$ for $\mu \in M_s^+$ with $S(\mu) = \partial\Delta$. In this section, we study measures $\mu \in M_s^+$ satisfying $\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$. For $\zeta \in \partial\Delta$, let $\mathcal{M}_\zeta = \{x \in \mathcal{M}; z(x) = \zeta\}$, where z is the coordinate function on Δ. First, we prove

PROPOSITION 4.1. *Suppose that $\mu \perp \delta_\zeta$, where $\mu \in M_s^+$ and $\zeta \in \partial\Delta$. Then there exists $\nu \in L^1_+(\mu)$ such that $\mathcal{M}_\zeta \cap \{|\psi_\mu| < 1\} \subset Z(\psi_\nu)$.*

Proof. When $\zeta \notin S(\mu)$, then $\mathcal{M}_\zeta \cap \{|\psi_\mu| < 1\} = \emptyset$, so our assertion is clear. Hence we assume that $\zeta \in S(\mu)$. Take a sequence of decreasing open subarcs $\{J_n\}_n$ of $\partial\Delta$ such that $\cap_{n=1}^\infty J_n = \{\zeta\}$. Put $\mu_n = \mu_{|J_n}$. Then $\|\mu_n\| \neq 0$. Since $\mu \perp \delta_\zeta$, $\|\mu_n\| \to 0$ as $n \to \infty$. Moreover we may assume that $\sum_{n=1}^\infty \|\mu_n\| < \infty$. Then there exists a sequence of positive numbers $\{p_n\}_n$ such that $\sum_{n=1}^\infty p_n\|\mu_n\| < \infty$ and $p_n \to \infty$ as $n \to \infty$. We put $\nu = \sum_{n=1}^\infty p_n\mu_n$. Since $\mu_n \in L^1_+(\mu)$ and $\|\nu\| \le \sum_{n=1}^\infty p_n\|\mu_n\| < \infty$, we have $\nu \in L^1_+(\mu)$. We note that $|\psi_{\mu_n}| = |\psi_\mu|$ on $\mathcal{M}_\zeta$. Then we have $|\psi_\nu| \le |\psi_{\mu_n}|^{p_n} = |\psi_\mu|^{p_n}$ on $\mathcal{M}_\zeta$. Since $p_n \to \infty$ as $n \to \infty$, we obtain our assertion. □

PROPOSITION 4.2. *Let $\mu \in M_{s,c}^+$. Then $\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$.*

Proof. It is trivial that

$$\mathcal{R}_0(\mu) \subset \mathcal{R}(\mu) \subset \bigcup\{\mathcal{M}_\zeta; \zeta \in S(\mu)\}. \tag{4.1}$$

To prove our assertion, let $\nu \in L^1_+(\mu)$ and $\zeta \in S(\mu)$. By Proposition 4.1, there exists $\sigma \in L^1_+(\nu)$ such that

$$\mathcal{M}_\zeta \cap \{|\psi_\nu| < 1\} \subset Z(\psi_\sigma). \tag{4.2}$$

Since $\sigma \in L^1_+(\nu) \subset L^1_+(\mu)$, by (4.2) we have $\mathcal{M}_\zeta \cap \{|\psi_\nu| < 1\} \subset \mathcal{R}_0(\mu)$. Thus we get $\mathcal{M}_\zeta \cap \mathcal{R}(\mu) \subset \mathcal{R}_0(\mu)$ for $\zeta \in S(\mu)$. Hence by (4.1), $\mathcal{R}(\mu) \subset \mathcal{R}_0(\mu)$. □

Next, we study $\mu \in M_s^+$ satisfying $\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$.

THEOREM 4.1. *Let $\mu \in M_s^+$ and $\mu = \mu_c + \mu_d$, where $\mu_c \in M_{s,c}^+$ and $\mu_d \in M_{s,d}^+$. Let $\mu_d = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}}$, where $a_n > 0$ for every n. Then the following assertions are equivalent.*

(i) $\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$.

(ii) *For every n, there exists $\nu_n \in L_+^1(\mu)$ such that $\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\} \subset Z(\psi_{\nu_n})$.*

(iii) *For every n, there exists $\lambda_n \in M_s^+$ such that $S(\lambda_n) \subset S(\mu)$ and $\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\} \subset Z(\psi_{\lambda_n})$.*

To prove our theorem, we need some lemmas.

LEMMA 4.1. *Let $\mu = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}} \in M_{s,d}^+$, where $a_n > 0$ for every n. Then*

$$\mathcal{R}(\mu) = \mathcal{R}_0(\mu) \cup \bigcup_{n=1}^{\infty} \{|\psi_{\delta_{e^{i\theta_n}}}| < 1\}.$$

Proof. We know

$$\mathcal{R}_0(\mu) \cup \bigcup_{n=1}^{\infty} \{|\psi_{\delta_{e^{i\theta_n}}}| < 1\} \subset \mathcal{R}(\mu).$$

To prove our assertion, suppose that $\mathcal{R}(\mu) \neq \mathcal{R}_0(\mu)$. Let $x \in \mathcal{R}(\mu) \setminus \mathcal{R}_0(\mu)$. Then there exists $\nu \in L_+^1(\mu)$ such that

$$0 < |\psi_\nu(x)| < 1. \tag{4.3}$$

It is sufficient to prove that $|\psi_{\delta_{e^{i\theta_n}}}(x)| < 1$ for some n. To prove this, suppose not. Then

$$|\psi_{\delta_{e^{i\theta_n}}}(x)| = 1 \quad \text{for every } n. \tag{4.4}$$

Let $\nu = \sum_{n=1}^{\infty} b_n \delta_{e^{i\theta_n}}$, where $b_n \geq 0$ for every n. By (4.3) and (4.4), $b_n > 0$ for infinitely many n. Without loss of generality, we may assume that $b_n > 0$ for every n. Since $\sum_{n=1}^{\infty} b_n < \infty$, there exists a sequence of increasing positive numbers $\{p_n\}_n$ such that $\sum_{n=1}^{\infty} p_n b_n < \infty$ and $p_n \to \infty$ as $n \to \infty$. Put $\sigma = \sum_{n=1}^{\infty} p_n b_n \delta_{e^{i\theta_n}}$. Then $\sigma \in L_+^1(\mu)$. For each positive integer k, put

$$\nu_k = \sum_{n=k}^{\infty} b_n \delta_{e^{i\theta_n}}.$$

Then $\sigma \geq p_k \nu_k$ for every k. Hence by (4.4),

$$|\psi_\sigma(x)| \leq |\psi_{\nu_k}(x)|^{p_k} = |\psi_\nu(x)|^{p_k}$$

Since $p_k \to \infty$ as $k \to \infty$, by (4.3) we have $x \in Z(\psi_\sigma)$. Hence $x \in \mathcal{R}_0(\mu)$, this is a contradiction. □

LEMMA 4.2. *Let $\mu \in M_s^+$ and $e^{i\theta} \in \partial\Delta$ such that $\{|\psi_{\delta_{e^{i\theta}}}| < 1\} \subset Z(\psi_\mu)$. Put $\mu_1 = \mu - \mu(\{e^{i\theta}\})\delta_{e^{i\theta}}$. Then $\{|\psi_{\delta_{e^{i\theta}}}| < 1\} \subset Z(\psi_{\mu_1})$.*

Proof. For $0 < r < 1$, let

$$b(z) = \frac{\psi_{\delta_{e^{i\theta}}}(z) - r}{1 - r\psi_{\delta_{e^{i\theta}}}(z)}, \quad z \in \Delta.$$

Then b is an interpolating Blaschke product, see [6]. Let $\{w_n\}_n$ be the zeros of b. Then $\psi_{\delta_{e^{i\theta}}}(w_n) = r$ for every n. By our assumption, we have $\psi_{\mu_1} = 0$ on $\overline{\{w_n\}_n} \setminus \{w_n\}_n = Z(b)$. Hence by Lemma 2.2, $\{|b| < 1\} \subset Z(\psi_{\mu_1})$. Since $\{|b| < 1\} = \{|\psi_{\delta_{e^{i\theta}}}| < 1\}$, we get our assertion. □

LEMMA 4.3. *Let $\mu \in M_s^+$, $-1 < R < 1$, and $J = \{e^{i\theta}; \theta_0 < \theta < \theta_1\}$. Suppose that $J \cap S(\mu) = \emptyset$. If $\mu \perp \delta_{e^{i\theta_0}}$, then $|\psi_\mu(z)| \to 1$ as $|z| \to 1$, $z \in \partial\Delta_R(e^{i\theta_0})$ and $\theta_0 < \arg z < \theta_1$.*

Proof. We may assume that $e^{i\theta_0} = 1$. It is not difficult to see that our assertion holds when $1 = e^{i\theta_0} \notin S(\mu)$. So we assume $1 \in S(\mu)$. Let $J_n = \{e^{i\theta}; -1/n < \theta < 0\}$. Since $1 \in S(\mu)$, $J \cap S(\mu) = \emptyset$, and $\mu \perp \delta_1$, we have $\|\mu_{|J_n}\| \neq 0$. Put $\mu_n = \mu_{|J_n}$ and $\mu'_n = \mu_{|\partial\Delta \setminus J_n}$. Then $\|\mu_n\| \to 0$ as $n \to \infty$ and $\mu = \mu_n + \mu'_n$ for every n.

To prove our assertion, let $\varepsilon > 0$ arbitrary. Then there exists a positive integer k such that

$$e^{-\frac{1+R}{1-R}\|\mu_k\|} > 1 - \varepsilon. \tag{4.5}$$

Since $1 \notin S(\mu'_k)$, $|\psi_{\mu'_k}(z)| \to 1$ as $|z| \to 1, z \in \partial\Delta_R(1)$. Hence there exists $r_0, 0 < r_0 < \theta_1/2$, such that

$$|\psi_{\mu'_k}(z)| > 1 - \varepsilon \quad \text{for } z \in \partial\Delta_R(1), 0 < \arg z < r_0. \tag{4.6}$$

For every $z \in \partial\Delta_R(1)$ with $0 < \arg z < r_0$, we have

$$\begin{aligned} |\psi_\mu(z)| &= |\psi_{\mu_k}(z)||\psi_{\mu'_k}(z)| \\ &\geq |\psi_{\delta_1}(z)|^{\|\mu_k\|}|\psi_{\mu'_k}(z)| \qquad \text{by Lemma 3.1} \\ &= e^{-\frac{1-R}{1-R}\|\mu_k\|}|\psi_{\mu'_k}(z)| \qquad \text{by Lemma 2.3} \\ &> (1-\varepsilon)^2 \qquad \text{by (4.5) and (4.6).} \end{aligned}$$

This shows our assertion. □

Proof of Theorem 4.1. (i) $\Rightarrow$ (ii) It is sufficient to prove (ii) for $n = 1$. For $0 < r < 1$, let

$$b(z) = \frac{\psi_{\delta_{e^{i\theta_1}}}(z) - r}{1 - r\psi_{\delta_{e^{i\theta_1}}}(z)}, \quad z \in \Delta.$$

Then by [6] b is an interpolating Blaschke product and

$$\psi_{\delta_{e^{i\theta_1}}} = r \quad \text{on } Z(b). \tag{4.7}$$

Since $\{|\psi_{\delta_{e^{i\theta_1}}}| < 1\} \subset \mathcal{R}(\mu)$, $Z(b) \subset \mathcal{R}(\mu)$. By condition (i), for each $x \in Z(b)$ there exists $\sigma_x \in L^1_+(\mu)$ such that $\psi_{\sigma_x}(x) = 0$. By (4.7), we may assume that $\delta_{e^{i\theta_1}} \perp \sigma_x$. Since $Z(b)$ is compact, there exist $\sigma_1, \dots, \sigma_k \in L^1_+(\mu)$ such that $Z(b) \subset \cup_{j=1}^k \{|\psi_{\sigma_j}| < 1\}$ and $\delta_{e^{i\theta_1}} \perp \sigma_j$ for every j. Put $\lambda = \sum_{j=1}^k \sigma_j$. Then $\lambda \in L^1_+(\mu)$, $\delta_{e^{i\theta_1}} \perp \lambda$, and

$$Z(b) \subset \{|\psi_\lambda| < 1\}. \tag{4.8}$$

By Proposition 4.1, there exists $\nu \in L^1_+(\lambda)$ such that $\mathcal{M}_{e^{i\theta_1}} \cap \{|\psi_\lambda| < 1\} \subset Z(\psi_\nu)$. Since $\lambda \in L^1_+(\mu)$, $\nu \in L^1_+(\mu)$. Since $Z(b) \subset \mathcal{M}_{e^{i\theta_1}}$, by (4.8) we have $Z(b) \subset Z(\psi_\nu)$. Hence by Lemma 2.2, $\{|b| < 1\} \subset Z(\psi_\nu)$. Since $\{|b| < 1\} = \{|\psi_{\delta_{e^{i\theta_1}}}| < 1\}$, condition (ii) holds.

(ii) $\Rightarrow$ (i) It is easy to see that $\mathcal{R}(\mu) = \mathcal{R}(\mu_c) \cup \mathcal{R}(\mu_d)$ and $\mathcal{R}_0(\mu) = \mathcal{R}_0(\mu_c) \cup \mathcal{R}_0(\mu_d)$. Hence

$$\begin{aligned}
\mathcal{R}(\mu) &= \mathcal{R}_0(\mu_c) \cup \mathcal{R}(\mu_d) && \text{by Proposition 4.2} \\
&= \mathcal{R}_0(\mu_c) \cup \left(\mathcal{R}_0(\mu_d) \cup \bigcup_{n=1}^\infty \{|\psi_{\delta_{e^{i\theta_n}}}| < 1\} \right) && \text{by Lemma 4.1} \\
&\subset \mathcal{R}_0(\mu) && \text{by condition (ii).}
\end{aligned}$$

Thus we get condition (i).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii) It is sufficient to prove for the case $n = 1$. We may assume that $e^{i\theta_1} = 1$. Let

$$\Gamma_+ = \partial\Delta_0(1) \cap \{\operatorname{Im} z > 0\} \quad \text{and} \quad \Gamma_- = \partial\Delta_0(1) \cap \{\operatorname{Im} z < 0\}.$$

We shall show the existence of $\tau, \tau' \in L^1_+(\mu)$ such that

$$\psi_\tau(z) \to 0 \quad \text{as } |z| \to 1, z \in \Gamma_+ \tag{4.9}$$

and

$$\psi_{\tau'}(z) \to 0 \quad \text{as } |z| \to 1, z \in \Gamma_-. \tag{4.10}$$

We prove only (4.9). In the same way, we can prove (4.8). By condition (iii), there exists $\lambda \in M_s^+$ such that

$$\{|\psi_{\delta_1}| < 1\} \subset Z(\psi_\lambda) \quad \text{and} \quad S(\lambda) \subset S(\mu). \tag{4.11}$$

By Lemma 4.2, we may assume that $\lambda \perp \delta_1$. Since $|\psi_{\delta_1}(z)| \to e^{-1}$ as $|z| \to 1, z \in \Gamma_+$, by (4.11) we have

$$\psi_\lambda(z) \to 0 \quad \text{as } |z| \to 1,\ z \in \Gamma_+. \tag{4.12}$$

Then by Lemma 4.3, we may further assume that

$$S(\lambda) \subset J = \{e^{i\theta}; 0 \leq \theta < r_1\} \quad \text{for some } 0 < r_1 < 1$$

and $\lambda(\{e^{i\theta}; 0 < \theta < \varepsilon\}) > 0$ for every $0 < \varepsilon < r_1$. Then we can take a sequence of decreasing numbers $\{r_n\}_n, r_n > 0$, such that $\lambda(\{e^{ir_n}\}) = \mu(\{e^{ir_n}\}) = 0$ and $\lambda(J_n) \neq 0$ for every n, where $J_n = \{e^{i\theta}; r_{n+1} < \theta < r_n\}$. Put

$$\lambda_n = \lambda_{|J_n}. \tag{4.13}$$

Then we have

$$\lambda = \sum_{n=1}^{\infty} \lambda_n, \quad \|\lambda\| = \sum_{n=1}^{\infty} \|\lambda_n\|, \quad \text{and} \quad \|\lambda_n\| \neq 0. \tag{4.14}$$

Let $\xi(t), 0 < t < 1$, be a one to one continuous map onto Γ_+ such that $\xi(t) \to 1$ as $t \to 0$. Then by (4.12),

$$\psi_\lambda(\xi(t)) \to 0 \quad \text{as } t \to 0. \tag{4.15}$$

Since $1 \notin S(\lambda_1)$, we also have

$$\psi_{\lambda-\lambda_1}(\xi(t)) \to 0 \quad \text{as } t \to 0. \tag{4.16}$$

Let $\{\varepsilon_n\}_n$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ as $n \to \infty$. By (4.15), there exists c_1, $0 < c_1 < 1$, such that

$$|\psi_\lambda(\xi(t))| < \varepsilon_1 \quad \text{for } 0 < t \leq c_1. \tag{4.17}$$

By (4.11) and (4.13), we have $S(\lambda_1) \subset S(\mu_{|J_1})$. Then it is not difficult to see the existence of a sequence $\{\mu_{1,k}\}_k$ in $L^1_+(\mu_{|J_1})$ such that $\|\mu_{1,k}\| \leq \|\lambda_1\|$ and $\mu_{1,k} \to \lambda_1$ in the weak*-topology of $M(\partial\Delta)$. We note that $|\psi_{\mu_{1,k}}| \to |\psi_{\lambda_1}|$ uniformly on compact subsets of Δ. Put $\mu_1 = \mu_{1,k}$ for sufficiently large k, and then put

$$\tau_1 = \mu_1 + \sum_{j=2}^{\infty} \lambda_j. \tag{4.18}$$

By (4.14), (4.16), and (4.17), we may assume moreover that $|\psi_{\tau_1}(\xi(t))| < \varepsilon_1$ for $0 < t \leq c_1$. Since $1 \notin S(\lambda_1) \cup S(\mu_1)$, $|\psi_{\lambda_1}(\xi(t))| \to 1$ and $|\psi_{\mu_1}(\xi(t))| \to 1$ as $t \to 0$. Hence by (4.16), $\psi_{\tau_1}(\xi(t)) \to 0$ as $t \to 0$. Then take c_2 such that $0 < c_2 < c_1$ and $|\psi_{\tau_1}(\xi(t))| < \varepsilon_2$ for $0 < t \leq c_2$.

In the same way, there exists $\mu_2 \in L^1_+(\mu_{|J_2})$ such that $\|\mu_2\| \leq \|\lambda_2\|$ and $|\psi_{\tau_2}(\xi(t))| < \varepsilon_k$ for $0 < t \leq c_k, k = 1, 2, 3$, where $\tau_2 = \mu_1 + \mu_2 + \sum_{j=3}^{\infty} \lambda_j$. We note that $\psi_{\tau_2}(\xi(t)) \to 0$ as $t \to 0$. Repeat this process. Then we get sequences of measures $\{\mu_n\}_n$ and $\{\tau_n\}_n$, and a sequence of decreasing numbers $\{c_n\}_n$ such that

$$c_n \to 0 \quad \text{as } n \to \infty, \tag{4.19}$$

$$\mu_n \in L^1_+(\mu_{|J_n}), \tag{4.20}$$

$$\|\mu_n\| \leq \|\lambda_n\|, \tag{4.21}$$

$$\tau_n = \sum_{j=1}^{n} \mu_j + \sum_{j=n+1}^{\infty} \lambda_j, \tag{4.22}$$

and

$$|\psi_{\tau_n}(\xi(t))| < \varepsilon_k \quad \text{for } 0 < t \leq c_k,\ k = 1, 2, \dots, n+1. \tag{4.23}$$

Let

$$\tau = \sum_{j=1}^{\infty} \mu_j. \tag{4.24}$$

Then by (4.14), (4.20) and (4.21), $\|\tau\| < \infty$ and $\tau \in L^1_+(\mu)$. Also by (4.22) and (4.24), $\|\tau - \tau_n\| \to 0$ as $n \to \infty$. Hence by (4.23), $|\psi_\tau(\xi(t))| \leq \varepsilon_k$, $0 < t \leq c_k$, for every k. By (4.19), we get (4.9).

Now we prove the existence of $\nu \in L^1_+(\mu)$ such that

$$\{|\psi_{\delta_1}| < 1\} \subset Z(\psi_\nu). \tag{4.25}$$

Let

$$b(z) = \frac{\psi_{\delta_1}(z) - e^{-1}}{1 - e^{-1}\psi_{\delta_1}(z)}, \quad z \in \Delta.$$

Then by [6], b is an interpolating Blaschke product with zeros $\{z_n\}_n$ such that $z_n \in \partial\Delta_0(1)$ and $z_n \to 1$. Let

$$\nu = \tau + \tau' \in L^1_+(\mu).$$

Then by (4.9) and (4.10), $\psi_\nu(z_n) \to 0$ as $n \to \infty$. Hence $Z(b) \subset Z(\psi_\nu)$ and by Lemma 2.2, $\{|b| < 1\} \subset Z(\psi_\nu)$. Since $\{|b| < 1\} = \{|\psi_{\delta_1}| < 1\}$, we get (4.25).

□

The following follows from Theorem 4.1.

COROLLARY 4.1. *Let $\mu \in M_s^+$ and $\mu = \mu_c + \mu_d$, where $\mu_c \in M_{s,c}^+$ and $\mu_d \in M_{s,d}^+$. Suppose that $\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$. Then*

(i) *$S(\mu)$ does not contain any isolated points.*
(ii) *If $\nu \in L_+^1(\mu)$ and $S(\nu) = S(\mu)$, then $\mathcal{R}(\nu) = \mathcal{R}_0(\nu)$.*
(iii) *If $\lambda \in M_{s,c}^+$ and $S(\lambda) = S(\mu)$, then $\mathcal{R}(\lambda + \mu_d) = \mathcal{R}_0(\lambda + \mu_d)$.*

We note that for $\mu \in M_s^+$ with $S(\mu) \neq \partial\Delta$, there exists $\lambda \in M_s^+$ such that $S(\lambda) = S(\mu)$ and $\mathcal{R}(\lambda) \neq \mathcal{R}_0(\lambda)$. For, there is an open subarc $J = \{e^{i\theta}; \theta_0 < \theta < \theta_1\}$ such that $J \cap S(\mu) = \emptyset$ and $e^{i\theta_0} \in S(\mu)$. Put $\lambda = \mu + \delta_{e^{i\theta_0}}$. Then $S(\lambda) = S(\mu)$. By Lemma 4.3, we have $\{|\psi_{\delta_{e^{i\theta_0}}}| < 1\} \not\subset \mathcal{R}_0(\lambda)$. Hence by Theorem 4.1, $\mathcal{R}(\lambda) \neq \mathcal{R}_0(\lambda)$

From Theorem 4.1, we have the following problem.

PROBLEM 4.1. Characterize closed subsets E of $\partial\Delta$ with $1 \in E$ satisfying the following condition; there exists $\mu \in M_s^+$ such that $S(\mu) \subset E$ and $\{|\psi_{\delta_1}| < 1\} \subset Z(\psi_\mu)$.

5. Discrete measures

When μ is a discrete measure, we have other equivalent conditions on μ such that $\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$.

THEOREM 5.1. *Let $\mu = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}} \in M_{s,d}^+$, where $a_n > 0$ for every n. Then the following conditions are equivalent.*

(i) *$\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$.*
(ii) *For every n, there exists $\nu_n \in L_+^1(\mu)$ such that $\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\} \subset Z(\psi_{\nu_n})$.*
(iii) *For every n, there exists $\lambda_n \in M_s^+$ such that $S(\lambda_n) \subset S(\mu)$ and $\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\} \subset Z(\psi_{\lambda_n})$.*
(iv) *There exists $\nu \in L_+^1(\mu)$ such that $\{|\psi_\mu| < 1\} \subset Z(\psi_\nu)$.*
(v) *There exists $\lambda \in M_s^+$ such that $S(\lambda) = S(\mu)$ and $\{|\psi_\mu| < 1\} \subset Z(\psi_\lambda)$.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow from Theorem 4.1.

(iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) are trivial.

(ii) $\Rightarrow$ (iv) By condition (ii), there exists $\nu_n \in L^1_+(\mu)$ such that

$$\{|\psi_{\delta_{e^{i\theta_n}}}| < 1\} \subset Z(\psi_{\nu_n}) \quad \text{for every } n. \tag{5.1}$$

Since $Z(\psi_{\nu_n}) = Z(\psi_{c\nu_n})$ for $c > 0$, we may assume that $\sum_{n=1}^{\infty} \|\nu_n\| < \infty$. Let

$$\lambda = \sum_{n=1}^{\infty} \nu_n. \tag{5.2}$$

Then $\lambda \in L^1_+(\mu)$, so that we can write λ as

$$\lambda = \sum_{n=1}^{\infty} b_n \delta_{e^{i\theta_n}} \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n| < \infty. \tag{5.3}$$

Hence there exists a sequence of increasing positive integers $\{p_n\}_n$ such that

$$\sum_{n=1}^{\infty} p_n(a_n + b_n) < \infty \quad \text{and} \quad p_n \to \infty \quad \text{as } n \to \infty. \tag{5.4}$$

Let

$$\nu = \sum_{n=1}^{\infty} p_n(a_n + b_n)\delta_{e^{i\theta_n}}. \tag{5.5}$$

Then by (5.4), $\nu \in L^1_+(\mu)$. To prove condition (iv), let $x \in \mathcal{M} \setminus \Delta$ such that

$$|\psi_\mu(x)| < 1. \tag{5.6}$$

We shall prove that

$$x \in Z(\psi_\nu). \tag{5.7}$$

By (5.2), (5.3), and (5.5), we have $\nu \geq \sum_{n=1}^{\infty} \nu_n \geq \nu_n$, so that $|\psi_\nu| \leq |\psi_{\nu_n}|$ on $\mathcal{M} \setminus \Delta$ for every n. If $|\psi_{\delta_{e^{i\theta_n}}}(x)| < 1$ for some n, then by (5.1) we obtain (5.7).

Next, suppose that

$$|\psi_{\delta_{e^{i\theta_n}}}(x)| = 1 \quad \text{for every } n. \tag{5.8}$$

Put

$$\nu'_n = \sum_{j=n}^{\infty} p_j(a_j + b_j)\delta_{e^{i\theta_j}} \quad \text{and} \quad \mu'_n = \sum_{j=n}^{\infty} a_j \delta_{e^{i\theta_j}}.$$

Then $\nu'_n \geq p_n \sum_{j=n}^{\infty} a_j \delta_{e^{i\theta_j}} = p_n \mu'_n$. Hence

$$|\psi_{\nu'_n}| \leq |\psi_{\mu'_n}|^{p_n} \quad \text{for every } n. \tag{5.9}$$

Therefore

$$\begin{aligned} |\psi_\nu(x)| &= |\psi_{\nu'_n}(x)| \prod_{j=1}^{n-1} |\psi_{\delta_{e^{i\theta_j}}}(x)|^{p_j(a_j+b_j)} \\ &= |\psi_{\nu'_n}(x)| \qquad \text{by (5.8)} \\ &\leq |\psi_{\mu'_n}(x)|^{p_n} \qquad \text{by (5.9)} \\ &= |\psi_\mu(x)|^{p_n} \qquad \text{by (5.8)}. \end{aligned}$$

Since $p_n \to \infty$, by (5.6) we have $\psi_\nu(x) = 0$. Thus we get (5.7). □

We show the existence of positive discrete measures which satisfy conditions in Theorem 5.1.

EXAMPLE 5.1. Let $\{e^{i\theta_n}\}_n$ be a dense subset of $\partial\Delta$ and $\{a_n\}_n$ a sequence of positive numbers such that $\Sigma_{n=1}^{\infty} a_n < \infty$. Let $\mu = \sum_{n=1}^{\infty} a_n \delta_{e^{i\theta_n}}$. Then $S(\mu) = \partial\Delta$. By Theorem 2.1, $\mathcal{R}(\mu) = \mathcal{R}_0(\mu)$.

EXAMPLE 5.2. By Gorkin [5], for every $\nu \in M_s^+$ there exists $\lambda \in M_{s,d}^+$ such that $\{|\psi_\nu| < 1\} \subset Z(\psi_\lambda)$, where we can take λ such as $\|\lambda\|$ is sufficiently small. We use this fact inductively.

Let $\nu_0 \in M_{s,d}^+$ with $\|\nu_0\| \leq 1$. Then there exists $\nu_1 \in M_{s,d}^+$ such that $\{|\psi_{\nu_0}| < 1\} \subset Z(\psi_{\nu_1})$ and $\|\nu_1\| \leq 1/2$. Then there exists $\nu_2 \in M_{s,d}^+$ such that

$$\{|\psi_{\nu_1}| < 1\} \subset Z(\psi_{\nu_2}) \quad \text{and} \quad \|\nu_2\| \leq (1/2)^2.$$

By induction, we can get a sequence $\{\nu_k\}_k$ in $M_{s,d}^+$ such that

$$\{|\psi_{\nu_{k-1}}| < 1\} \subset Z(\psi_{\nu_k}) \tag{5.10}$$

and $\|\nu_k\| \leq (1/2)^k$. Let

$$\mu = \sum_{k=0}^{\infty} \nu_k.$$

Then $\mu \in M_{s,d}^+$ and $\nu_k \in L_+^1(\mu)$ for every k. Suppose that $\mu(\{e^{i\theta_0}\}) > 0$. Then there exists a positive integer k such that $\nu_k(\{e^{i\theta_0}\}) \neq 0$. Then $\{|\psi_{\delta_{e^{i\theta_0}}}| < 1\} \subset \{|\psi_{\nu_k}| < 1\}$. Hence by (5.10), we have $\{|\psi_{\delta_{e^{i\theta_0}}}| < 1\} \subset Z(\psi_{\nu_{k-1}})$. Therefore μ satisfies condition (ii) of Theorem 5.1.

Start from $\nu_0 = \delta_1$. Then by the above construction of μ, we can prove the existence of $\mu \in M^+_{s,d}$ satisfying conditions in Theorem 5.1 and $d\theta(S(\mu)) = 0$.

From Proposition 4.2 and Theorem 5.1, we have the following problem on continuous measures.

PROBLEM 5.1. Let $\mu \in M^+_{s,c}$. Does there exist $\nu \in L^1_+(\mu)$ such that $\{|\psi_\mu| < 1\} \subset Z(\psi_\nu)$?

The following is a partial answer.

PROPOSITION 5.1. *Let $\mu \in M^+_{s,c}$. Suppose that $S(\mu) = \{e^{i\theta}; \theta_0 \leq \theta \leq \theta_1\}, \theta_0 < \theta_1$. Then there exists $\nu \in L^1_+(\mu)$ such that $\{|\psi_\mu| < 1\} \subset Z(\psi_\nu)$.*

Proof. When $S(\mu) = \partial\Delta$, by Theorem 2.1 we have our assertion. So we assume that $S(\mu) \neq \partial\Delta$. Let $J = \{e^{i\theta}; \theta_0 \leq \theta \leq \theta_1\}$ and $\Omega_0 = \{re^{i\theta}; 0 < r < 1, \theta_0 \leq \theta \leq \theta_1\}$. As mentioned before, there exists an interpolating Blaschke product b with zeros $\{z_k\}_k$ such that $z_k \neq 0$ for every k and

$$\{|\psi_\mu| < 1\} = \{|b| < 1\}. \tag{5.11}$$

Then

$$cl\{z_k\}_k \setminus \{z_k\}_k = J \tag{5.12}$$

and there exists a positive number r_0 such that

$$|\psi_\mu(z_k)| \leq r_0 < 1 \quad \text{for every } k. \tag{5.13}$$

Put

$$\{\zeta_n\}_n = \{z_k; z_k \in \Omega_0\} \quad \text{and} \quad \{\xi_n\}_n = \{z_k; z_k \notin \Omega_0\}. \tag{5.14}$$

In the same way as the proof of Theorem 2.1, there exists $\nu_1 \in L^1_+(\mu)$ such that

$$\psi_{\nu_1}(\zeta_n) \to 0 \quad \text{as } n \to \infty. \tag{5.15}$$

Next, we prove the existence of $\nu_2 \in L^1_+(\mu)$ such that

$$\psi_{\nu_2}(\xi_n) \to 0 \quad \text{as } n \to \infty. \tag{5.16}$$

Take a sequence of strictly increasing closed subarcs $\{J_n\}_n$ of J such that $cl(\cup_{n=1}^\infty J_n) = J$. Let $J_0 = \emptyset$. Put $I_n = J_n \setminus J_{n-1}$ and $\mu_n = \mu_{|I_n}$. Then

$\|\mu_n\| > 0$,

$$\mu = \sum_{n=1}^{\infty} \mu_n, \tag{5.17}$$

and there exists a sequence of increasing positive numbers $\{p_n\}_n$ such that

$$\sum_{n=1}^{\infty} p_n\|\mu_n\| < \infty \quad \text{and} \quad p_n \to \infty \quad \text{as } n \to \infty. \tag{5.18}$$

Let

$$\nu_2 = \sum_{n=1}^{\infty} p_n\mu_n. \tag{5.19}$$

Then $\nu_2 \in L^1_+(\mu)$. By (5.12) and (5.14), $cl\{\xi_n\}_n \setminus \{\xi_n\}_n \subset \{e^{i\theta_0}, e^{i\theta_1}\}$. Hence we have

$$|\psi_{\mu_k}(\xi_n)| \to 1 \quad \text{as } n \to \infty \text{ for every } k. \tag{5.20}$$

By (5.18) and (5.19), we have $\nu_2 \geq p_k \sum_{n=k}^{\infty} \mu_n$. Hence by (5.17) and (5.20), $|\psi_{\nu_2}| \leq |\psi_\mu|^{p_k}$ on $\overline{\{\xi_n\}_n} \setminus \{\xi_n\}_n$. By (5.13), $|\psi_\mu| \leq r_0$ on $\overline{\{\xi_n\}_n} \setminus \{\xi_n\}_n$. Therefore by (5.18), $\psi_{\nu_2} = 0$ on $\overline{\{\xi_n\}_n} \setminus \{\xi_n\}_n$. Thus we get (5.16).

Put $\nu = \nu_1 + \nu_2$. Then $\nu \in L^1_+(\mu)$, and by (5.14), (5.15), and (5.16) we have that $\psi_\nu(z_n) \to 0$ as $n \to \infty$. Hence by Lemma 2.2, $\{|b| < 1\} \subset Z(\psi_\nu)$. Thus by (5.11), we have our assertion. □

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