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THE INSTABILITY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

YOUNHEE KO

ABSTRACT. We consider a system of functional differential equations $x'(t) = F(t, x_t)$ and obtain conditions on a Liapunov functional and a Liapunov function to ensure the instability of the zero solution.

1. Introduction

It is well known that Liapunov's direct method sometimes provides a useful tool in the study of instability of functional differential equations (FDEs). See, for example, [3] and [13,14]. However, most instability results are for the autonomous functional differential equations. One of goals of this paper is to provide an instability theorem for the nonautonomous functional differential equations with finite delay using the Liapunov's direct method.

One the other hand, an obstacle often is encountered when one tries to apply the Liapunov's direct method; namely, it frequently is difficult - if not impossible - to construct appropriate Liapunov functions or functionals in order to make use of known instability theorems. Another purpose of this paper is to provide an instability theorem that eliminates some of the obstacles imposed by this difficulty. In particular, we employ Liapunov-Razumikhin techniques and omega limit set properties in order to present an instability result for autonomous functional differential equations with infinite delay. Our result may be the first instability result which employs Liapunov-Razumikhin techniques and omega limit set properties for autonomous functional differential equations with infinite delay.

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Now we present the fundamental notation to which we will refer throughout this paper.

For $x \in R^n$ with $x = (x_1, x_2, \cdots, x_n)$, $|x|$ denotes the usual norm in R^n.

2. Instability for the Function Differential Equations with Finite Delay

The purpose of this section is to provide an instability theorem for the functional differential equations (including the nonautonomous functional differential equations) with finite delay using the Liapunov's direct method. Also example is given as an application for this theorem.

Now we present the fundamental notation and definitions to which we will refer throughout this section. For fixed $r \geq 0$, C denotes the space of continuous functions mapping $[-r, 0]$ into R^n, and for $\phi \in C$,

$$||\phi|| = \sup_{-r \leq s \leq 0} |\phi(s)| \, .$$

Also, C_H denotes the set of $\phi \in C$ with $||\phi|| < H$ and $0 < H \leq \infty$. If x is a continuous function of u defined for $-r \leq u < A$, with $A > 0$, and if t is a fixed number satisfying $0 \leq t < A$, then x_t denotes the restriction of x to $[t - r, t]$ so that x_t is an element of C denoted by

$$x_t(\theta) = x(t + \theta) \text{ for } -r \leq \theta \leq 0 \, .$$

We consider the system

$$x'(t) = F(t, x_t) \, , \tag{2.1}$$

where $F : R_+ \times C_H \to R^n$ is continuous and takes closed bounded sets into bounded sets; $0 < H \leq \infty$. We denote by $x(t_0, \phi)$ a solution of (2.1) with initial condition $\phi \in C$ where $x_{t_0}(t_0, \phi) = \phi$ and we denote by $x(t, t_0, \phi)$ the value of $x(t_0, \phi)$ at t. x' denotes the right-hand derivative. It is well known (Burton[3,4]) that for each $t_0 \in R_+ = [0, \infty)$ and each $\phi \in C_H$, there is at least one solution $x(t_0, \phi)$ defined on an interval $[t_0, t + \alpha)$ and, if there is an $H_1 < H$ with $|x(t, t_0, \phi)| \leq H_1$ for all t for which $x(t, t_0, \phi)$ is defined, then $\alpha = \infty$.

A Liapunov functional is a continuous function $V : R_+ \times C_H \to R_+$ which is locally Lipschitz with respect to ϕ. The derivative of a Liapunov functional $V(t,\phi)$ along a solution $x(t)$ of (2.1) may be defined in several equivalent ways. If V is differentiable, the natural derivative is obtained using the chain rule. Then $V'_{(2.1)}(t,\phi)$ denotes the derivative of functional V with respect to (2.1) defined by

$$V'_{(2.1)}(t,\phi) = \lim_{\delta\to 0+} \sup \frac{V(t+\delta, x_{t+\delta}(t,\phi)) - V(t,\phi)}{\delta} .$$

DEFINITION 2.1. A continuous function $W : R_+ = [0,\infty) \to R_+$ is called a *wedge* if $W(0) = 0$ and W is strictly increasing on R_+.

DEFINITION 2.2. Let $F(t,0) = 0$ for all $t \geq 0$.

(a) The zero solution of (2.1) is said to be stable if for each $\epsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[\phi \in C_\delta, t \geq t_0]$ imply $|x(t,t_0,\phi)| < \epsilon$.

(b) The zero solution of (2.1) is said to be unstable if there exist $\epsilon > 0$ and $t_0 > 0$ such that for any $\delta > 0$ there is an ϕ with $||\phi|| < \delta$ and a $t_1 > t_0$ such that $|x(t_1,t_0,\phi)| \geq \epsilon$.

Notice that stability requires all solutions starting near zero to stay near zero, but instability calls for the existence of some solutions starting near zero to move well away from zero.

THEOREM 2.1. *Let $H > 0$ and let $D, E, V : R_+ \times C_H \to R$ be continuous and locally Lipschitz in ϕ, and let $\eta : R_+ \to R_+$ be a function with $\int_0^\infty \eta(s)ds = \infty$. Suppose that there exist wedges W_1, W_2, W_3 and W_4 such that, for all $t \geq 0$ and $\phi \in C_H$,*

(i) $V(t,\phi) \leq W_1(D(t,\phi))$,

(ii) $V'_{(2.1)}(t,x_t) \geq \eta(t) W_2(E(t,x_t))$,

(iii) $D(t,x_t) \leq W_3(||x_t||)$ *and*

(iv) $D(t,x_t) \geq W_4(E(t,x_t)) \geq 0$.

If we can choose a sequence $\{t_n,\ \phi_n\} \in R_+ \times C_H$ such that $V(t_n,\phi_n) > 0$ and $\lim_{n\to\infty} ||\phi_n|| = 0$. Then the zero solution of (2.1) is unstable.

Proof. Suppose that $x = 0$ is stable. For $\epsilon > 0$ and $t_0 \geq 0$ there exists $\delta = \delta(t_0, \epsilon) > 0$ with $0 < \delta < \epsilon$ such that $||\phi|| < \delta$ implies $|x(t, t_0, \phi)| < \epsilon$ for any $t \geq t_0$. Now we may take the initial function ϕ with $\frac{\delta}{2} < |\phi(s)| < \delta$ for any $s \in [-r, 0]$ such that $V(t_0, \phi) > 0$. Thus

$$V(t, x_t) \leq W_1(D(t, x_t)) < W_1 \circ W_3(||x_t||) < W_1 \circ W_3(\epsilon)$$

for any $t \geq t_0 \geq 0$. That is, $V(t, x_t)$ is bounded above on $[0, \infty)$. But

$$\begin{aligned} V(t, x_t) &\geq V(t_0, \phi) + \int_{t_0}^{t} \eta(s) W_2(E(s, x_s)) ds \\ &\geq V(t_0, \phi) + \int_{t_0}^{t} \eta(s) W_2 \circ W_4^{-1}(D(s, x_s)) ds \\ &\geq V(t_0, \phi) + W_2 \circ W_4^{-1} \circ W_1^{-1}(V(t_0, \phi)) \int_{t_0}^{t} \eta(s) ds \to \infty \end{aligned}$$

as $t \to \infty$, which is a contradiction. Hence the proof is complete. □

REMARK 2.1. Some comments for Theorem 2.1 are in order.

(i) The results of Theorem 2.1 can be directly applied to the instability theory for ordinary differential equations. Thus we may consider Theorem 4.1.24 in [3] as the corollary of Theorem 2.1

(ii) Many instability theorems for FDEs with finite delay are dependent on the length of finite delay. So it is impossible to extend those theorems in a straightforward manner to FDEs with unbounded delay. But Theorem 2.1 is independent of the length of finite delay. So its results can be extended in a straightforward manner without much difficulty to FDEs with unbounded delay.

(iii) Theorem 2.1 has the weakest condition of the coefficient function η in order to insure that the zero solution is unstable.

EXAMPLE 2.1. Consider a scalar equation

$$x'(t) = a(t)x(t) + b(t) \int_{t-r}^{t} x(u) du \,. \tag{2.2}$$

where $a, b : R_+ \to R$ are continuous such that

$$\eta(t) = 2a(t) - \int_{t-r}^{t} |b(-s)|\, ds - r|b(t)| \geq 0$$

and $\int_0^\infty \eta(s) ds = \infty$. Then the zero solution of (2.2) is unstable.

Proof. Consider the Liapunov functional

$$V(t,x_t)=x^2(t)-\int_{-r}^{0}\int_{t+s}^{t}|b(u-s)|x^2(u)duds$$

Then we have

$$\begin{aligned}
&V'_{(2.2)}(t,x_t)\\
&=2x(t)x'(t)-\int_{-r}^{0}\frac{d}{dt}\left\{\int_{t+s}^{t}|b(u-s)|x^2(u)du\right\}ds\\
&=2x(t)\{a(t)x(t)+b(t)\int_{t-r}^{t}x(u)du\}-\int_{-r}^{0}|b(t-s)|x^2(t)ds\\
&\quad+\int_{-r}^{0}|b(t)|x^2(t+s)ds\\
&=2a(t)x^2(t)+2b(t)x(t)\int_{t-r}^{t}x(s)ds-x^2(t)\int_{-r}^{0}|b(t-s)|ds\\
&\quad+|b(t)|\int_{-r}^{0}x^2(t+s)ds\\
&\geq 2a(t)x^2(t)-x^2(t)\int_{t-r}^{t}|b(-s)|ds-2|b(t)||x(t)|\int_{t-r}^{t}|x(s)|ds\\
&\quad+|b(t)|\int_{t-r}^{t}x^2(s)ds\\
&\geq 2a(t)x^2(t)-x^2(t)\int_{t-r}^{t}|b(-s)|ds-|b(t)|\int_{t-r}^{t}x^2(t)ds\\
&=2a(t)x^2(t)-x^2(t)\int_{t-r}^{t}|b(-s)|ds-|b(t)|x^2(t)r\\
&=\{2a(t)-\int_{t-r}^{t}|b(-s)|ds-r|b(t)|\}x^2(t),
\end{aligned}$$

since

$$\begin{aligned}
&|b(t)|\int_{t-r}^{t}(|x(s)|-|x(t)|)^2\,ds\\
&=|b(t)|\int_{t-r}^{t}x^2(s)ds-2|b(t)||x(t)|\int_{t-r}^{t}|x(s)|ds+|b(t)|\int_{t-r}^{t}x^2(t)dt
\end{aligned}$$

$$= |b(t)| \int_{t-r}^{t} x^2(x)ds - 2|b(t)||x(t)| \int_{t-r}^{t} |x(s)|ds + r|b(t)|x^2(t) \geq 0,$$

that is,

$$|b(t)| \int_{t-r}^{t} x^2(s)ds - 2|b(t)||x(t)| \int_{t-r}^{t} |x(s)|ds \geq -r|b(t)|x^2(t).$$

Consider $W_1(t) = W_2(t) = t^2$, $W_3(t) = W_4(t) = t$ and $D(t, x_t) = E(t, x_t) = |x(t)|$. Then all conditions in Theorem 2.1 are satisfied. Hence, the proof is complete. □

REMARK 2.2. In Example 2.1 if $r = 0$, then (2.2) can be reduced to the ordinary differential equation $x'(t) = a(t)x(t)$. Now we can solve the equation $x'(t) = a(t)x(t)$ directly. Thus the general solution of $x'(t) = a(t)x(t)$ is given by

$$x(t) = x(t_0)e^{\int_{t_0}^{t} a(s)\,ds}$$

for any $t_0 \geq 0$. If $\int_0^\infty a(t)\,dt = \infty$, then we can check that the zero solution of $x'(t) = a(t)x(t)$ is unstable.

3. Instability for Autonomous Functional Differential Equations with Infinite Delay

The purpose of this section is to provide an instability theorem that eliminates some of the obstacles which we explained in Section 1. In particular, we employ Liapunov-Razumikhin techniques and omega limit set properties in order to present an instability theorem for autonomous functional differential equations with infinite delay. Examples are given to illustrate that this theorem often is straightforward to apply when applicable.

We present the standard notation for infinite delay functional differential equations and the basic definitions to which we will refer through this section. Also we provide underlying spaces that often arise in a natural way, and for which standard existence and uniqueness type results as well as fundamental properties of positive limit sets hold. And we provide the space which gives the information regarding precompactness of positive orbits.

Let B be a real vector space either of continuous functions that map $(-\infty, 0]$ into R^n or of measurable functions that map $(-\infty, 0]$ into R^n with elements ϕ and ψ in B identified when $\phi = \psi$ $a.e$ on $(-\infty, 0]$ and $\phi(0) = \psi(0)$. In either case, we assume B - endowed with norm $|\cdot|_B$ - is a Banach space.

If $x : (-\infty, A) \to R^n$, $-\infty < A \le \infty$, then, for any t in $(-\infty, A)$, define $x_t : (-\infty, 0] \to R^n$ by $x_t(s) = x(t+s)$, $s \le 0$. Then x_t is the translate of x on $(-\infty, t]$ to $(-\infty, 0]$, and x_0 is merely x restricted to $(-\infty, 0]$.

We consider autonomous FDEs with infinite delay of the form

$$x' = f(x_t), \tag{3.1}$$

where x' denotes the right-hand derivative with respect to t, and $f : D \to R^n$, $D \subseteq B$.

DEFINITION 3.1. A space B (as defined above) is said to be admissible [with respect to (3.1)] whenever there exist continuous functions $K, M : [0, \infty) \to [0, \infty)$ and a constant $J > 0$ such that the following conditions hold: If $x : (-\infty, A) \to R^n$ is continuous on $[a, A)$ with x_a in B for some $a < A$, then, for all t in $[a, A)$,

(B1) x_t is an element of B;
(B2) x_t is continuous in t with respect to $|\cdot|_B$;
(B3) $|x_t|_B \le K(t-a) \max_{a \le s \le t} |x(s)| + M(t-a)|x_a|_B$; and
(B4) $|\phi(0)| \le J|\phi|_B$ for all ϕ in B.

REMARK 3.1. The above conditions assure a Peano-type existence result: that is, if D is open and $f : D \to R^n$ is continuous, then any Cauchy problem

$$x' = f(x_t), \ x_0 = \phi, \ \phi \in D, \tag{3.2}$$

possesses a continuously differentiable solution that satisfies (3.1) for all t in some interval $[0, A)$, $0 < A < \infty$. Likewise, typical uniqueness, continuous-dependence, and continuation results can be obtained by employing (B1)-(B4). Proofs of results along these lines may be found in Hale and Kato [15], to name one source. A brief survey, which includes a substantial list of references related to [15], is given in Haddock [8]; whereas, an extensive survey of infinite delay equations is given in Corduneanu and Lakshmikantham [6].

Definition 3.2. A set $T \subseteq B$ is said to be *invariant* [with respect to (3.1)] if, for each $\phi \in T$, there exists a function $x : (-\infty, \infty) \to R^n$ such that $x_0 = \phi$ and, for all $t \in (-\infty, \infty)$,

i. $x_t \in T$ and
ii. $x'(t) = f(x_t)$.

Definition 3.3. For $\phi \in B$, the *positive limit set of the positive orbit* $\{x_t(\phi) : t \geq 0\}$ is the (possibly empty) set

$$\Omega(\phi) = \{\psi \in B : x_{t_n}(\phi) \to \psi \text{ as } n \to \infty \text{ for some sequence } \{t_n\} \uparrow \infty\}.$$

The following fundamental theorem is well known. Two references that provide a detailed account of (a slight variation of) the result are [17, Section 6.4] and [18, Section 4.4]. We note that a set S is precompact (in B) if its closure, $Cl(S)$, is compact (in B).

Theorem 3.1. *The positive limit set, $\Omega(\phi)$, of a precompact positive orbit $\{x_t(\phi) : t > 0\}$ is nonempty, compact, connected, and invariant, and*

$$x_t(\phi) \to \Omega(\phi) \text{ as } t \to \infty.$$

Example 3.1. Suppose

$$g : (-\infty, 0] \to [1, \infty) \tag{3.3}$$

is continuous and nonincreasing on $(-\infty, 0]$ with $g(0) = 1$. Furthermore, suppose that f satisfies the following conditions:

(i) $\dfrac{g(s+u)}{g(s)} \to 1$ uniformly on $(-\infty, 0]$ as $u \to 0_-$, (3.4)

(ii) $g(s) \to \infty$ as $s \to -\infty$. (3.5)

Let C_g denote the space of continuous functions that map $(-\infty, 0]$ into R^n such that

$$\sup_{s \leq 0} \frac{|\phi(s)|}{g(s)} < \infty.$$

We define a norm $||\cdot||_g$ on C_g by

$$||\phi||_g = \sup_{s \leq 0} \frac{|\phi(s)|}{g(s)} \tag{3.6}$$

Then C_g with this norm is a Banach space (cf. Corduneanu [5]).

Now we consider

$$C_g^* = \left\{ \phi \in C_g | \frac{\phi}{g} \text{ is uniformly continuous on } [0, \infty) \right\}.$$

Then the subspace C_g^* of C_g is admissible (cf. Haddock [12]).

REMARK 3.2. It is illustrated in Atkinson and Haddock [1] that (3.4) and (3.5) are not overly restrictive and often can be included in a natural way in studies involving functional differential equations with infinite delay.

The next result deals with obtaining sufficient conditions for precompactness of a bounded positive orbit $\{x_t : t \geq 0\}$ in C_g spaces. Application of such results occurs in dealing the main theorem (Theorem 3.3).

THEOREM 3.2. *Suppose $g : (-\infty, \infty) \to [1, \infty)$ satisfies (3.3), (3.4) and (3,5) with $g(s) \equiv 1$ on $[0, \infty)$. If $x : (-\infty, \infty) \to R^n$ such that $\phi = x_0 \in C_g^*$ and $x(\phi)(t)$ is bounded and uniformly continuous on $[0, \infty)$ and*

$$\frac{\phi(s)}{g(s)} \to 0 \text{ as } s \to -\infty,$$

then the set $\{x_t(\phi) : t \geq 0\}$ is precompact in C_g.

Proof. First we note that $x_t(\phi) \in C_g^*$ for any $t \geq 0$, since $\phi = x_0 \in C_g^*$ and $x(\phi)(t)$ is bounded and uniformly continuous on $[0, \infty)$. Thus the mapping $t \to x_t(\phi)$ is continuous with respect to $|| \cdot ||_g$ on $[0, \infty)$, since C_g^* is an admissible space. Since $\dfrac{x_0(s)}{g(s)} = \dfrac{\phi(s)}{g(s)} \to 0$ as $s \to -\infty$ and x is bounded on $[0, \infty)$, the condition (3.2) in [9] is satisfied (by (3.5)). Hence, $\{x_t : t \geq 0\}$ is precompact in C_g by Theorem 3.1 in [9]. □

For the remainder of this section we consider an autonomous system of functional differential equations with infinite delay

$$x' = f(x_t), \tag{3.7}$$

where

$$f : C_g^* \to R^n \tag{3.8}$$

is completely continuous, solutions depend continuously on initial values, and C_g^* with norm $||\cdot||_g$ is an admissible space.

By a Liapunov (or Razumikhin) function, we mean a locally Lipschitzian function $V : R^+ \to R$ such that (a) $V(0) = 0$ and (b) if $0 \neq x(t_0)$ is such that x is differentiable at t_0, then $\left(\frac{d}{dt}\right)V[x(t)]$ exists at $t = t_0$. In particular,

$$\begin{aligned}\frac{d}{dt}V[x(t)] &= \mathrm{grad}V[x(t)] \cdot x'(t)_{t=t_0}\\ &= \sum_{i=1}^{n} \frac{\partial V[x(t_0)]}{\partial x_i} f_i(x_{t_0}).\end{aligned}$$

For a Liapunov function V, we define the derivative of V with respect to (3.7) by

$$V'_{(3.7)}(\phi) = V'(\phi) = \lim_{h\to 0^+} \sup \frac{V[\phi(0) + hf(\phi)] - V[\phi(0)]}{h} \tag{3.9}$$

REMARK 3.3. Let $x(t)$ be a solution of (3.7) and let $V'_{(3.7)}(x_t) \geq 0$. Then $V[x(t)]$ is non-decreasing function of t which implies that $V[x(t)]$ is non-decreasing along a solution of (3.7) (cf. Yoshizawa [19]).

DEFINITION 3.4. The zero solution, $x = 0$, of (3.7) is stable in C_g^* if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $||\phi||_g < \delta$ with $\phi \in C_g^*$ implies that $|x(\phi)(t)| < \epsilon$ for all $t \geq 0$.

The zero solution is said to be unstable in C_g^* if it is not stable in C_g^*.

THEOREM 3.3. *Suppose that there exists a Liapunov function* $V : R^n \to R_+$ *such that* (i) $V(x) > 0$ *if* $x \neq 0$, *and* $V(0) = 0$ (ii) $V'_{(3.7)}(\phi) > 0$ *for all* $\phi \in \mathrm{dom}(V)$, *where* $\mathrm{dom}(V) = \{\phi \in C_g^* : V[\phi(0)] = \sup_{s\leq 0}\frac{V[\phi(s)]}{g(s)} > 0\}$. *Then the zero solution of* (3.7) *is unstable in* C_g^*.

Proof. Let $\epsilon > 0$ be given and consider any δ with $0 < \delta < \epsilon$. Let $\phi \in C_g^*$ be chosen so that $V[\phi(0)] = \sup_{s\le 0} \frac{V[\phi(s)]}{g(s)}$ and $\frac{\delta}{2} \le |\phi(s)| < \delta$ for all $s \in (-\infty, 0]$. Thus,

$$||\phi||_g = \sup_{s\le 0} \frac{|\phi(s)|}{g(s)} < \frac{\delta}{g(0)} = \delta$$

and

$$V'_{(3.7)}[\phi] > 0,$$

since

$$V[\phi(0)] = \sup_{s\le 0} \frac{V[\phi(s)]}{g(s)} > 0.$$

We claim that there exists $t^* > 0$ such that $|x(\phi)(t)| = \epsilon$. Suppose not. Then $x(\phi)(t)$ and $x'(t)$ are defined and bounded on $[0, \infty)$. Now we may assume that $g(t) \equiv 1$ on $[0, \infty)$. Therefore we note that all conditions in Theorem 3.1 in [9] are satisfied. That is, $\{x_t(\phi) : t \ge 0\}$ is C_g-precompact. Also note that $\{x_t(\phi) : t \ge 0\} \subset C_g^*$, since $\phi \in C_g^*$ and $x(\phi)(t)$ is bounded. Thus $\{x_t(\phi) : t \ge 0\}$ is C_g^*-precompact. By Theorem 3.1, the positive limit set $\Omega(\phi)$ is nonempty. Therefore, there exist $\psi \in \Omega(\phi)$ and a sequence $\{t_n\} \uparrow \infty$ such that

$$x_{t_n}(\phi) \to \psi \text{ as } n \to \infty.$$

Note that $x_{t_n}(\phi)(s) \to \psi(s)$ as $n \to \infty$ for all $s \in (-\infty, 0]$.

Now we note that $\{V(x(\phi)(t)) : t \ge 0\}$ is strictly increasing and bounded above from the fact that $V[(\phi(0)] = \sup_{s\le 0} \frac{V[\phi(s)]}{g(s)}$, and $x(\phi)(t)$ is bounded on $[0, \infty)$. Therefore, there exists a positive real number p such that

$$p = \lim_{t\to\infty} V(x(\phi)(t)).$$

Thus we have

$$V[\psi(0)] = \lim_{n\to\infty} V[x_{t_n}(\phi)(0)] = \lim_{n\to\infty} V[x(\phi)(t_n)] = p$$

and

$$V[\psi(s)] = \lim_{n\to\infty} V[x_{t_n}(\phi)(s)] = \lim_{n\to\infty} V[x(\phi)(t_n + s)] \le p$$

for $s \leq 0$. Therefore, we note that

$$V[\psi(0)] = \sup_{s\leq 0} \frac{V[\psi(s)]}{g(s)} > 0$$

and $V'[\psi] > 0$.

But $x_t(\psi) \in \Omega(\phi)$ for any $t \geq 0$, since $\Omega(\phi)$ is positively invariant. This implies that for any $t \geq 0$ there is a sequence $\langle s_n \rangle \uparrow$ as $n \to \infty$ such that $x_t(\psi) = \lim_{n\to\infty} x_{s_n}(\phi)$. Thus we have

$$V[x_t(\psi)(0)] = V(\lim_{n\to\infty} x_{s_n}(\phi)(0)) = \lim_{n\to\infty} V(x(\phi)(s_n)) \leq p,$$

which contradicts $V'[\psi] > 0$. Hence, the proof is complete. □

EXAMPLE 3.2. Consider a scalar equation

$$x'(t) = ax(t) + \int_{-\infty}^{t} b(t-s)x(s)ds. \tag{3.10}$$

Let g be a continuous function which satisfies the conditions (3.3), (3.4) and (3.5), and let $b : [0, \infty) \to R$ be a continuous function such that $\int_{-\infty}^{0} |b(-s)|g(s)ds < a$. then the zero solution of (3.10) is unstable in C_g^*.

Proof. Let $V(x(t)) = |x(t)|$. If $x(t) > 0$, then

$$\begin{aligned} V'_{(3.10)}[x_t] &= |x(t)|' \\ &= x'(t) \\ &= ax(t) + \int_{-\infty}^{t} b(t-s)x(s)ds \\ &= ax(t) + \int_{-\infty}^{0} b(-s)x_t(s)ds \\ &\geq ax(t) - \int_{-\infty}^{0} |b(-s)|g(s)\frac{|x_t(s)|}{g(s)}ds \\ &\geq a||x_t||_g - ||x_t||_g \int_{-\infty}^{0} |b(-s)|g(s)ds \\ &= (a - \int_{-\infty}^{0} |b(-s)|g(s)ds)||x_t||_g > 0 \end{aligned}$$

if $||x_t||_g = |x(t)| = x(t) > 0$, that is,

$$|x(t)| = V[x_t(0)] = \sup_{s\le 0}\frac{V[x_t(s)]}{g(s)} > 0.$$

If $x(t) < 0$, then

$$\begin{aligned}
V'_{(3.10)}[x_t] &= |x(t)|' \\
&= -x'(t) \\
&= -ax(t) - \int_{-\infty}^{t} b(t-s)x(s)ds \\
&= -ax(t) - \int_{-\infty}^{0} b(-s)x_t(s)ds \\
&\ge a|x(t)| - \int_{-\infty}^{0} |b(-s)|g(s)\frac{|x_t(s)|}{g(s)}ds \\
&\ge a||x_t||_g - ||x_t||_g\int_{-\infty}^{0} |b(-s)|g(s)ds \\
&= ||x_t||_g(a - \int_{-\infty}^{0} |b(-s)|g(s)ds) > 0
\end{aligned}$$

if $||x_t||_g = |x(t)| = -x(t) > 0$. From the above theorem, the zero solution of (3.10) is unstable in C_g^*. □

EXAMPLE 3.3. Consider a scalar equation

$$x'(t) = ax(t) + bx(t-r) + \int_{-\infty}^{t} c(t-s)x(s)ds, \tag{3.11}$$

where $r > 0$. Let g be a continuous function which satisfies the conditions (3.3), (3.4) and (3.5) with $g(s) = 1$ on $[-r, 0]$, and let $c : [0, \infty) \to R$ be a continuous function such that $|b| + \int_{-\infty}^{0} |c(-s)|g(s) < a$. then the zero solution of (3.11) is unstable in C_g^*.

Proof. Consider a Liapunov function $V(x(t)) = |x(t)|$. If $x(t) > 0$, then

$$\begin{aligned}
V'_{(3.11)}[x_t] &= |x(t)|' \\
&= x'(t) \\
&= ax(t) + bx(t-r) + \int_{-\infty}^{t} b(t-s)x(s)ds \\
&\geq a|x(t)| - |b|||x_t||_g - \int_{-\infty}^{0} |b(-s)|g(s)\frac{|x_t(s)|}{g(s)}ds \\
&\geq a||x_t||_g - |b|||x_t||_g - ||x_t||_g \int_{-\infty}^{0} |b(-s)|g(s)ds \\
&= (a - |b| - \int_{-\infty}^{0} |b(-s)|g(s)ds)||x_t||_g > 0,
\end{aligned}$$

if $||x_t||_g = |x(t)| = x(t) > 0$. If $x(t) < 0$, then we can prove the example by the similar argument. Hence the proof is complete. □

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Department of Mathematics Education
Cheju National University
Cheju City, Cheju-do 690-756, Korea
E-mail: yheeko@cheju.cheju.ac.kr