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STATIONARY SOLUTIONS FOR ITERATED FUNCTION SYSTEMS CONTROLLED BY STATIONARY PROCESSES

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Abstract. We consider a class of discrete parameter processes on a locally compact Banach space S arising from successive compositions of strictly stationary random maps with state space $C(S,S)$, where $C(S,S)$ is the collection of continuous functions on S into itself. Sufficient conditions for stationary solutions are found. Existence of pth moments and convergence of empirical distributions for trajectories are proved.

1. Introduction

Let S be a locally compact Banach space with a norm $|\cdot|$ and let $C(S,S)$ be the set of all continuous functions from S into itself. Endow $C(S,S)$ with the compact open topology and $C(S,S)$ is a complete separable metric space.

Let $(\Omega,\mathcal{F},P)$ be a probability space on which is defined a sequence of stationary random maps $\Gamma_0,\Gamma_1,\Gamma_2,\cdots$ taking values on $C(S,S)$.

In this paper we consider the following sequence of process $\{X_n : n \geq 0\}$ on S,

$$X_0, \quad X_{n+1} = \Gamma_n(X_n), \ (n \geq 0) \tag{1.1}$$

where X_0 is arbitrarily prescribable S-valued random variable independent of $\{\Gamma_n \ : \ n \geq 0\}$.

Stationarity of model (1.1) is of importance in statistical inference and sufficient conditions for stationarity for various models have been

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found in many papers (see, e.g. [2], [3], and [6]-[15]). If we assume that $\{\Gamma_n : n \geq 0\}$ are independent and identically distributed, then $\{X_n\}$ given by (1.1) is a Markov chain. For Markov chains, stationary condition can be found by looking for conditions for ergodicity and in this case, Markov property can be used to investigate the problems. In [11] and references therein, sufficient conditions for ergodicity of φ−irreducible Markov chains are obtained, by using, so called Tweedie's criterion. Ergodicity of Markov chains without φ−irreducibility assumption is studied in, for example, [3], [9] and [15]. Processes given by more general sequences, such as, stationary process, finite Markov chain, finite semi-Markov chain, regenerative process are investigated in [8], [2], [13], and [12], respectively. The process $\{X_n\}$ generated by a stationary process $\{\Gamma_n\}$ is considered in [8] and it is proved that negative Lyapunov exponent with some additional assumptions is sufficient for stationarity of $\{X_n\}$. Time reversal idea and Kingman's subadditive ergodic theorem are used for the proof in [8].

Our main objective in this paper is to find out conditions that ensure the existence of a unique stationary solution of (1.1) via the L^p contraction. Also finiteness of pth moment of the limiting distribution and convergence of the empirical distribution of the trajectories are obtained.

Since $C(S,S)$ is a complete separable metric space, without loss of generality we may assume that $\{\Gamma_n : n \geq 0\}$ extends backward in time to $\{\Gamma_n : -\infty < n < \infty\}$. For solving our problems, it is more convenient to deal with the equation

$$X_{n+1} = \Gamma_n(X_n), \qquad n \in \mathbf{Z} \tag{1.2}$$

where $\mathbf{Z}$ denotes the set of all integers.

For S−valued random variable $X : \Omega \to S$, define

$$\|X\|_p = \begin{cases} (E|X|^p)^{1/p} & \text{if } 1 \leq p < \infty \\ E|X|^p & \text{if } 0 < p < 1 \\ \text{ess. sup. } |X| & \text{if } p = \infty \ . \end{cases}$$

We write $X \in L^p$ if $\|X\|_p < \infty$.

Note that for S−valued random variables X and Y,

$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p, \qquad \forall p \in (0, \infty]. \tag{1.3}$$

We make the following assumption.

Assumption A. There exist $p \in (0, \infty]$, $x_0 \in S$ and some $\delta > 0$ such that for every compact set $C \subset S$,

$$\|\mathrm{diam}(\Gamma_n \cdots \Gamma_0 C)\|_p \to 0 \quad \text{as} \quad n \to \infty \tag{1.4}$$

and

$$\sup_{n\geq 0} \|\Gamma_n \cdots \Gamma_1 \Gamma_0 x_0\|_{p+\delta} < \infty. \tag{1.5}$$

Here we write γx for the value of the map $\gamma \in C(S,S)$ at x, and $\gamma_n \cdots \gamma_1 \gamma_0$ for the compositions of the maps $\gamma_0, \gamma_1, \cdots, \gamma_n$. Also write for $B \subset S$, $\mathrm{diam}(\gamma B) = \sup\{|\gamma x - \gamma y| : x, y \in B\}$.

2. Main Results

For any $x \in S$, $-\infty < k < \infty$, $n \geq 1$, we define

$$X_{k,n}(x) = \Gamma_{k-1}\Gamma_{k-2} \cdots \Gamma_{k-n} x. \tag{2.1}$$

Our first main result is:

THEOREM 1. *Suppose that $\{\Gamma_n\}$ is a stationary process in $C(S,S)$ satisfying the Assumption A. Then there exists a process $\{Y_k : -\infty < k < \infty\}$ such that*
(1) *for any $x \in S$, $X_{k,n}(x)$ converges in L^p to Y_k as $n \to \infty$ and the distribution of Y_k is independent of x,*
(2) *if we take $X_0 = Y_0$, then $\{Y_k : k \geq 0\}$ is a unique stationary solution of the equation (1.1), and*
(3) *for any $x \in S$, $\Gamma_n \Gamma_{n-1} \cdots \Gamma_0 x$ converges in distribution to Y_0 as $n \to \infty$.*

To prove the theorem 1, we start with the following lemma:

LEMMA 1. *Let the Assumption A hold. Then for any $k \in \mathbf{Z}$, $\{X_{k,n}(x_0) : n \geq 1\}$ is Cauchy in L^p.*

Proof. For $\alpha > 0$, we define $h_\alpha : S \to S$ by

$$h_\alpha(x) = \begin{cases} x & \text{if } |x| < \alpha \\ 0 & \text{if } |x| \geq \alpha. \end{cases}$$

Then

$$(2.2)\ \|X_{k,n}(x_0) - X_{k,n+m}(x_0)\|_p \\ \leq \|X_{k,n}(x_0) - h_\alpha(X_{k,n}(x_0))\|_p + \|h_\alpha(X_{k,n}(x_0)) - h_\alpha(X_{k,n+m}(x_0))\|_p \\ + \|h_\alpha(X_{k,n+m}(x_0)) - X_{k,n+m}(x_0))\|_p.$$

Let three terms on the righthand side of (2.2) be denoted by I_1, I_2 and I_3 respectively.

We can easily show that

$$(2.3)\quad E|X_{k,n}(x_0) - h_\alpha(X_{k,n}(x_0))|^p = E\left[|X_{k,n}(x_0)|^p I_{\{|X_{k,n}(x_0)|\geq\alpha\}}\right] \leq \frac{1}{\alpha^\delta} E|X_{k,n}(x_0)|^{p+\delta}$$

and

$$(2.4)\quad E|h_\alpha(X_{k,n}(x_0)) - h_\alpha(X_{k,n+m}(x_0))|^p \\ \leq E\left[|X_{k,n}(x_0) - X_{k,n+m}(x_0)|^p I_{\{|X_{k,n}(x_0)|<\alpha, |X_{k,n+m}(x_0)|<\alpha\}}\right] \\ + \alpha^p P(|X_{k,n}(x_0)| \geq \alpha) + \alpha^p P(|X_{k,n+m}(x_0)| \geq \alpha).$$

From (1.5) and stationarity of $\{\Gamma_n\}$, we have for any $k \in \mathbf{Z}$,

$$(2.5)\qquad \frac{1}{\alpha^\delta} E|X_{k,n}(x_0)|^{p+\delta} \longrightarrow 0$$

and

$$(2.6)\qquad \alpha^p P(|X_{k,n}(x_0)| \geq \alpha) \longrightarrow 0$$

as $\alpha \to \infty$ uniformly in $n,\ n \geq 1$.

Now let $\epsilon > 0$ be given. Then from (2.3)-(2.6), we may choose $\alpha > 0$ sufficiently large that for any $k \in \mathbf{Z}$,

$$(2.7)\quad I_1 + I_3 + \alpha^p P(|X_{k,n}(x_0)| \geq \alpha) + \alpha^p P(|X_{k,n+m}(x_0)| \geq \alpha) < \frac{\epsilon}{2},$$

uniformly in m and n $(n, m \geq 1)$.

For α satisfying (2.7) above, we take $M \gg 2\alpha$ so that for any $k \in \mathbf{Z}$,

$$(2.8)\quad (2\alpha)^p P(|\Gamma_{k-n-1} \cdots \Gamma_{k-n-m} x_0| > M) \leq (2\alpha)^p \frac{\sup_n \|\Gamma_n \cdots \Gamma_0 x_0\|_{p+\delta}}{M^{p+\delta}} < \frac{\epsilon}{4},$$

for all positive integers m and n.

Take $B_M = \{x \in S \; : \; |x| \leq M\}$. Then we may assume, without loss of generality, that $x_0 \in B_M$ and hence

$$(2.9)\qquad \begin{aligned} &E[|X_{k,n}(x_0) - X_{k,n+m}(x_0)|^p I_{\{|X_{k,n}(x_0)|<\alpha, |X_{k,n+m}(x_0)|<\alpha\}}] \\ &\leq E[|X_{k,n}(x_0) - X_{k,n+m}(x_0)|^p I_{\{|\Gamma_{k-n-1}\cdots\Gamma_{k-n-m}x_0|\leq M\}}] \\ &\quad + E[|X_{k,n}(x_0) - X_{k,n+m}(x_0)|^p I_{\{|X_{k,n}(x_0)|<\alpha, |X_{k,n+m}(x_0)|<\alpha\}} \\ &\qquad \cdot I_{\{|\Gamma_{k-n-1}\cdots\Gamma_{k-n-m}x_0|>M\}}] \\ &\leq E|\mathrm{diam}(\Gamma_{k-1}\cdots\Gamma_{k-n}B_M)|^p \\ &\quad + (2\alpha)^p P(|\Gamma_{k-n-1}\cdots\Gamma_{k-n-m}x_0| > M). \end{aligned}$$

Moreover, by (1.4), there exist $n_0 = n_0(\epsilon)$ such that for any n, $n > n_0$

$$(2.10)\qquad \|\mathrm{diam}(\Gamma_{k-1}\cdots\Gamma_{k-n}B_M)\|_p < \frac{\epsilon}{4}, \quad \forall k \in \mathbf{Z}.$$

When $0 < p < 1$, combining (2.2)-(2.10), we have for each k, if $n > n_0$,

$$(2.11)\qquad \|X_{k,n}(x_0) - X_{k,n+m}(x_0)\|_p < \epsilon, \qquad m = 1, 2, 3, \cdots.$$

For the case $1 \leq p < \infty$, in the same manner as above, there are $\alpha > 0$ and $M \gg 2\alpha$ such that for any k in $\mathbf{Z}$,

$$(2.12)\quad I_1 + I_3 + \alpha\{P(|X_{k,n}(x_0)| \geq \alpha)\}^{\frac{1}{p}} + \alpha\{P(|X_{k,n+m}(x_0)| \geq \alpha)\}^{\frac{1}{p}} < \frac{\epsilon}{2},$$

and

$$(2.13)\qquad 2\alpha\{P(|\Gamma_{k-n-1}\cdots\Gamma_{k-n-m}x_0| > M)\}^{\frac{1}{p}} \leq \frac{\epsilon}{4},$$

for all positive integers m and n, and we obtain (2.11), by using (2.10), (2.12) and (2.13).

Now we assume that $p = \infty$. Then by (1.5) there exists K_0 such that $\sup_n \|\Gamma_n \cdots \Gamma_0 x_0\|_\infty \leq K_0 < \infty$. If we take $K = \max\{|x_0|, K_0\}$, then by (1.4), for $k \in \mathbf{Z}$,

$$(2.14)\quad \|X_{k,n}(x_0) - X_{k,n+m}(x_0)\|_\infty \leq \|\mathrm{diam}(\Gamma_{k-1}\cdots\Gamma_{k-n}B_K)\|_\infty \to 0$$

as $n \to \infty$, uniformly in m.

Hence for any p, $0 < p \leq \infty$, $\{X_{k,n}(x_0)\}$ is Cauchy in L^p.

We may now prove the theorem 1.

Proof of theorem 1. (1) By completeness of L^p space, $0 < p \leq \infty$

and above lemma , for each k, $\{X_{k,n}(x_0)\}$ converges in L^p to some random variable in L^p, say Y_k. But for any $x \in S$,

$$\|X_{k,n}(x) - Y_k\|_p \leq \|X_{k,n}(x) - X_{k,n}(x_0)\|_p + \|X_{k,n}(x_0) - Y_k\|_p \to 0$$

as $n \to \infty$, by assumption(1.4) and lemma 1. Hence L^p -limit of $X_{k,n}(x)$ is independent of x.

(2) Let k be fixed. Since $X_{k,n}(x_0) \to Y_k$ in L^p, there exists a subsequence $\{X_{k,n_j}(x_0) : j \geq 0\}$ such that $X_{k,n_j}(x_0) \to Y_k$ a.s. as $j \to \infty$. Since $\{\Gamma_k(X_{k,n_j}(x_0)) : j \geq 0\}$ is a subsequence of $\{X_{k+1,n}(x_0) : n \geq 1\}$, by uniqueness of L^p-limit, $\Gamma_k(X_{k,n_j}(x_0))$ and $X_{k+1,n}(x_0)$ have the same limit, Y_{k+1}, i.e.,

$$\Gamma_k(X_{k,n_j}(x_0)) \to Y_{k+1} \text{ in } L^p \text{ as } j \to \infty. \tag{2.15}$$

On the other hand, each $\gamma \in C(S, S)$, $\gamma(X_{k,n_j}) \to \gamma(Y_k)$ a.s. and hence

$$\Gamma_k(X_{k,n_j}(x_0)) \to \Gamma_k(Y_k) \quad a.s. \tag{2.16}$$

From (2.15) and (2.16) and uniqueness of the limit, $Y_{k+1} = \Gamma_k(Y_k)$ a.s., which says that Y_k is a solution of (1.1).

Stationarity of $\{Y_k\}$ follows from that of $\{\Gamma_k\}$.

(3) For $\forall x \in S$, $\Gamma_n \cdots \Gamma_0 x \stackrel{d}{=} \Gamma_{-1}\Gamma_{-2} \cdots \Gamma_{-n-1}x \to Y_0$ in L^p as $n \to \infty$, from which $\Gamma_n \cdots \Gamma_0 x \to Y_0$ in distribution as $n \to \infty$.

THEOREM 2. *Suppose that the Assumption A holds. In addition, assume for any compact set $C \subset S$, diam$(\Gamma_n \cdots \Gamma_0 C) \to 0$ a.s. as $n \to \infty$. Then for every $x \in S$,*

$$\frac{1}{n}\sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 x) \to E(f(Y_0)|\mathcal{I}) \quad a.s \tag{2.17}$$

as $n \to \infty$ for every bounded continuous real valued function f on S, where $\mathcal{I}$ is an invariant σ - field of $\{\Gamma_n\}$.

Proof. Since $E|f(Y_0)| < \infty$, by ergodic theorem,

$$\frac{1}{n}\sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 Y_0) \to E(f(Y_0)|\mathcal{I}) \quad a.s. \tag{2.18}$$

as $n \to \infty$. For any given $x \in S$, choose $r > 0$ such that $|x| < r$ and then define $\Omega_r = \{\omega \in \Omega : |Y_0(\omega)| \leq r\}$. Define Y_0^r by $Y_0^r = Y_0$ on Ω_r and 0 on

$\Omega - \Omega_r$. Clearly, on Ω_r, $\frac{1}{n}\sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 Y_0^r) = \frac{1}{n}\sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 Y_0)$. Moreover,

$$(2.19) \qquad |\Gamma_n \cdots \Gamma_0 Y_0^r - \Gamma_n \cdots \Gamma_0 x| \leq \operatorname{diam}(\Gamma_n \cdots \Gamma_0 B_r) \to 0 \quad a.s.$$

as $n \to \infty$.

Therefore on Ω_r, for every bounded uniformly continuous function f,

$$(2.20) \qquad \frac{1}{n}\sum_{i=0}^{n-1} f(\Gamma_i \cdots \Gamma_0 x) \to E(f(Y_0)|\mathcal{I}) \qquad a.s.$$

But from the fact that every continuous function defined on a compact set is uniformly continuous and the assumption $\operatorname{diam}(\Gamma_n \cdots \Gamma_0 B_r) \to 0$ as $n \to \infty$, (2.20) holds for every bounded continuous function, which together with $P(\Omega_r) \to 1$ implies the conclusion.

For $\gamma \in C(S,S)$, define a generalized norm

$$l(\gamma) = \sup_{x \neq y} \frac{|\gamma(x) - \gamma(y)|}{|x - y|}.$$

Let $Lip(S,S) = \{\gamma \in C(S,S) \mid l(\gamma) < \infty\}$. Then a function $\gamma \in Lip(S,S)$ is called a Lipschitzian map on S to S.

Remark. Suppose that $\{\Gamma_n\}$ is a sequence of independent and identically distributed random elements taking values on $Lip(S,S)$. Then $\{X_n\}$ obtained recursively by (1.1) is a Markov chain and using Markov property, it is proved that assumptions $E\log^+ l(\Gamma_1) < 0$ and $E\log^+ |x_0 - \Gamma_1(x_0)| < \infty$ for some x_0 in S are sufficient for ergodicity (stationarity) of X_n (see, e.g., [8],[9],[12]).

For following corollaries, we assume that $\{\Gamma_n : n \geq 0\}$ is a sequence of stationary processes in $Lip(S,S)$.

Corollary 1. *Suppose that $\{\Gamma_n\}$ is a stationary process in $Lip(S,S)$. If there exists $p \in (0,\infty)$ such that $\sup_{n\geq 0} \|\Gamma_n \cdots \Gamma_0 x_0\|_p < \infty$ for some x_0 and $\|l(\Gamma_n \cdots \Gamma_0)\|_p \to 0$ as $n \to \infty$, then the conclusions of theorem 1 hold with $p', 0 < p' < p$.*

Proof. For given $0 < p < \infty$, choose $p' > 0$ and $\delta > 0$, such that $p = p' + \delta$. Note that $|\gamma x - \gamma y| \leq l(\gamma)|x - y|$, $\forall \gamma \in Lip(S,S)$ and

$$(E|l(\Gamma_n \cdots \Gamma_0)|^{p'})^{\frac{1}{p'}} \leq E(|l(\Gamma_n \cdots \Gamma_0)|^p)^{\frac{1}{p}}, \quad 0 < p' < p.$$

Since for every compact set $C \subset S$,

$$\|\text{diam}(\Gamma_n \cdots \Gamma_0 C)\|_{p'} = \|l(\Gamma_n \cdots \Gamma_0)\text{diam}C\|_{p'} \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

(1.4) and (1.5) hold with $p = p'$ and hence the conclusion follows.

COROLLARY 2. *Suppose that for a stationary process* $\{\Gamma_n\}$ *in* $Lip(S,S)$, $\sum_{n=0}^{\infty} \|l(\Gamma_n \cdots \Gamma_0)\|_p < \infty$, *for some* $p \in (0,\infty)$. *If* $\|\Gamma_0 x_0\|_r < \infty$ *for some* $x_0 \in S$ *and* $r > p$, *then the conclusions in* theorem 1 *hold with* $p = p', 0 < p' < p$.

Proof. For given p, $0 < p < \infty$, let $p' > 0$ and $\delta > 0$ be such that $p = p' + \delta$. For any compact set $C \subset S$, we have

$$\|\text{diam}(\Gamma_n \cdots \Gamma_0 C)\|_{p'} \le \|\text{diam}C \cdot l(\Gamma_n \cdots \Gamma_0)\|_{p'} \to 0, \tag{2.21}$$

as $n \longrightarrow \infty$. Apply the Hölder's inequality to get

$$\begin{aligned} \|\Gamma_0 \cdots \Gamma_{-n} x_0 \quad - \quad & \Gamma_0 \cdots \Gamma_{-n+1} x_0\|_{p'+\frac{\delta}{2}} \\ \le \quad & \|l(\Gamma_0 \cdots \Gamma_{-n+1}) \cdot |\Gamma_{-n} x_0 - x_0|\|_{p'+\frac{\delta}{2}} \\ \le \quad & \|l(\Gamma_0 \cdots \Gamma_{-n+1})\|_{p'+\delta} \cdot \|\Gamma_{-n} x_0 - x_0\|_r \end{aligned} \tag{2.22}$$

where $\|\Gamma_{-n} x_0 - x_0\|_r \le K = \|\Gamma_{-n} x_0\|_r + \|x_0\|_r < \infty$ with $r = \frac{p(2p-\delta)}{\delta}$.

Now from stationarity of $\{\Gamma_n\}$ and (2.22), we have

$$\begin{aligned} \|\Gamma_n \cdots \Gamma_0 x_0\|_{p'+\frac{\delta}{2}} &= \|\Gamma_0 \cdots \Gamma_{-n} x_0\|_{p'+\frac{\delta}{2}} \\ &\le \|\Gamma_0 \cdots \Gamma_{-n} x_0 - \Gamma_0 \cdots \Gamma_{-n+1}\|_{p'+\frac{\delta}{2}} \\ &\quad + |\Gamma_0 \cdots \Gamma_{-n+1} x_0 - \Gamma_0 \cdots \Gamma_{-n+2} x_0\|_{p'+\frac{\delta}{2}} \\ &\quad + \cdots + \|\Gamma_0 \Gamma_{-1} x_0 - \Gamma_0 x_0\|_{p'+\frac{\delta}{2}} + \|\Gamma_0 x_0\|_{p'+\frac{\delta}{2}} \\ &\le K \left(\sum_{k=0}^{n-1} \|l(\Gamma_k \cdots \Gamma_0)\|_{p'+\delta} + 1 \right), \end{aligned}$$

and hence

$$\sup_n \|\Gamma_n \cdots \Gamma_0 x_0\|_{p'+\frac{\delta}{2}} < \infty. \tag{2.23}$$

If $r > p$, then r can be written by $r = \frac{p(2p-\delta)}{\delta}$ for some $\delta > 0$, $0 < \delta < p$, and hence we can find $p', 0 < p' < p$ satisfying $p = p' + \delta$. Therefore by (2.21) and (2.23), proof is completed.

REMARK. The assumptions in corollary 2 can be weakened as follows: there exists a positive integer m_0 such that $\sum_{n=0}^{\infty} \|l(\Gamma_{nm_0-1} \cdots \Gamma_0)\|_p < \infty$ for some $p \in (0,\infty)$ and $\sup_{0 \le n < m_0} \|\Gamma_n \cdots \Gamma_0 x_0\|_r < \infty$ for some

$x_0 \in S$, and $r > p$. The proof follows essentially the same line of corollary 2.

REMARK. Let $x \mapsto A_n x + b_n$ be random stationary affine maps of S into itself. Here A_n denotes a random linear operator on S, b_n a random vector in S. Consider the sequence $X_{n+1} = A_n X_n + b_n$. Then $\|\Gamma_n \cdots \Gamma_1\|_L = \|A_n \cdots A_1\|$, where $\|\cdot\|$ denotes the operator norm. Hence if $\sum_{n=1}^{\infty} \|\|A_n \cdots A_1\|\|_p < \infty$ for some $p > 0$ and $E|b_0|^r < \infty, \forall r > p$, then there exists a stationary solution $\{Y_n\}$ such that X_n converges to Y_0 in $L^{p'}$, $0 < p' < p$.

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