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ARITHMETIC OF THE MODULAR FUNCTION j_4

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ABSTRACT. Since the modular curve $X(4) = \Gamma(4)\backslash\mathfrak{H}^*$ has genus 0, we have a field isomorphism $K(X(4)) \approx \mathbb{C}(j_4)$ where $j_4(z) = \theta_3(\frac{z}{2})/\theta_4(\frac{z}{2})$ is a quotient of Jacobi theta series ([9]). We derive recursion formulas for the Fourier coefficients of j_4 and $N(j_4)$ (=the normalized generator), respectively. And we apply these modular functions to Thompson series and the construction of class fields.

1. Introduction

Let $\mathfrak{H}$ be the complex upper half plane. Then $SL_2(\mathbb{Z})$ acts on $\mathfrak{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \tau = \frac{a\tau+b}{c\tau+d}$ for $\tau \in \mathfrak{H}$. Let $\Gamma(N)$ $(N = 1, 2, 3, \cdots)$ be the principal congruence subgroups of $SL_2(\mathbb{Z})$ of level N and let $\mathfrak{H}^*$ be the union of $\mathfrak{H}$ and $\mathbb{P}^1(\mathbb{Q})$. The modular curve $\Gamma(N)\backslash\mathfrak{H}^*$ is a projective closure of the smooth affine curve $\Gamma(N)\backslash\mathfrak{H}$, which we denote by $X(N)$ with genus g_N. We identify the function field $K(X(N))$ on the modular curve $X(N)$ with the field of modular functions of level N. By the genus formula ([16] Ch.IV §7, or [17] Proposition 1.40), $g_N = 0$ only for the five cases $1 \leq N \leq 5$. Hence the field $K(X(4))$ is a rational function field $\mathbb{C}(j_4)$, where a field generator j_4 (§2, Theorem 4) can be constructed by using the theory of half integral modular forms. For generalities of half integral forms, we refer to [10] and [18].

In §3 we shall derive a recursion formula for the Fourier coefficients of j_4. Observing that the Fourier coefficients of the normalized generator $N(j_4)$ vanish periodically, we shall prove this phenomenon in Theorem 8 rather generally.

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In §4 we shall show that the normalized generator $N(j_4)$ induces a Thompson series of type $16B$ and derives a recursion formula. In §5 we shall explicitly construct some class fields over an imaginary quadratic field from the modular function j_4 by making use of Shimura theory and standard results of complex multiplication.

Through this article we adopt the following notations:

$\mathfrak{H}^*$ the extended complex upper half plane
Γ_s the isotropy subgroup of s
$\Gamma(N) \ = \{\gamma \in SL_2(\mathbb{Z}) | \ \gamma \equiv I \mod N\}$
$\Gamma_0(N)$ the Hecke subgroup $\{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma(1) | \ c \equiv 0 \mod N\}$
$\Gamma_1(N) \ = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma(1) | \ a \equiv d \equiv 1, \ c \equiv 0 \mod N\}$
$\Gamma_0^0(N) \ = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma(1) | \ b \equiv c \equiv 0 \mod N\}$
$X(N) = \Gamma(N) \backslash \mathfrak{H}^*$
$X_0(N) = \Gamma_0(N) \backslash \mathfrak{H}^*$
$\overline{\Gamma}$ the inhomogeneous group of $\Gamma (= \Gamma / \pm I)$
$q_h = e^{2\pi i z/h}, \ z \in \mathfrak{H}$
$\zeta_N = e^{2\pi i/N}$
$\mathbb{Z}_p$ the ring of p-adic integers
$\mathbb{Q}_p$ the field of p-adic numbers
$a \sim b$ means that a is equivalent to b.
$f(z) = g(z) + O(1)$ means that $f(z) - g(z)$ is bounded as z goes to $i\infty$.
$z \longrightarrow i\infty$ denotes that z goes to $i\infty$.
f is on Γ means that f is a modular function with respect to a group Γ.

2. Hauptfunktionen of level 4 as a quotient of Jacobi theta functions

For $\mu, \nu \in \mathbb{R}$ and $z \in \mathfrak{H}$, put

$$\Theta_{\mu,\nu}(z) = \sum_{n \in \mathbb{Z}} \exp \left\{ \pi i \left(n + \frac{1}{2}\mu \right)^2 z + \pi i n \nu \right\}.$$

This series uniformly converges for $\mathrm{Im}(z) \geq \eta > 0$, and hence defines a holomorphic function on $\mathfrak{H}$.

THEOREM 1. *If* $z \in \mathfrak{H}$, *then* $\Theta_{\mu,\nu}(z) = \frac{e^{-\frac{1}{2}\pi i \mu\nu}}{(-iz)^{\frac{1}{2}}} \Theta_{\nu,-\mu}(-1/z)$.

Proof. Theorem 7:1.1 [15]. □

We recall the Jacobi theta functions $\theta_2, \theta_3, \theta_4$ defined by

$$\theta_2(z) := \Theta_{1,0}(z) = \sum_{n\in\mathbb{Z}} q_2^{\left(n+\frac{1}{2}\right)^2}$$
$$\theta_3(z) := \Theta_{0,0}(z) = \sum_{n\in\mathbb{Z}} q_2^{n^2}$$
$$\theta_4(z) := \Theta_{0,1}(z) = \sum_{n\in\mathbb{Z}} (-1)^n q_2^{n^2}$$

Then we have the following transformation formulas.

THEOREM 2. *For all* $z \in \mathfrak{H}$,

$$(i) \quad \theta_2(z+1) = e^{\frac{1}{4}\pi i}\theta_2(z) \qquad (ii) \quad \theta_2(-1/z) = (-iz)^{\frac{1}{2}}\theta_4(z)$$
$$\theta_3(z+1) = \theta_4(z) \qquad \theta_3(-1/z) = (-iz)^{\frac{1}{2}}\theta_3(z)$$
$$\theta_4(z+1) = \theta_3(z) \qquad \theta_4(-1/z) = (-iz)^{\frac{1}{2}}\theta_2(z).$$

Proof. Theorem 7.1.2 [15]. □

Let N be a multiple of 4. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we define an automorphy factor $j(\gamma, z)$ as follows:

$$j(\gamma, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz+d}$$

where $\varepsilon_d = 1$ if $d \equiv 1 \mod 4$ and i otherwise. Let Γ be a congruence subgroup of $\Gamma_0(N)$ and f be a holomorphic function on $\mathfrak{H}$ such that

$$f|_{[\widetilde{\gamma}]_{\frac{k}{2}}} \stackrel{\text{def}}{=} f(z)$$

for all $\gamma \in \Gamma$. Such a function is called a *modular form of half-integral weight* $k/2$ *for* Γ when it satisfies some bounded condition at the cusps, as described in [10], p. 182 or [18], p. 444. We denote by $M_{\frac{k}{2}}(\widetilde{\Gamma})$ the vector space consisting of all such f.

THEOREM 3. $\theta_3(\frac{z}{2})$ *and* $\theta_4(\frac{z}{2})$ *belong to* $M_{\frac{1}{2}}(\widetilde{\Gamma}(4))$.

Proof. [9], Theorem 6. □

Put

$$j_4(z) = \theta_3(\frac{z}{2})/\theta_4(\frac{z}{2}) = 1 + 4q_4 + 8{q_4}^2 + 16{q_4}^3 + 32{q_4}^4 + 56{q_4}^5 + 96{q_4}^6 + 160{q_4}^7 + \cdots .$$

THEOREM 4. *$K(X(4)) = \mathbb{C}(j_4)$ and j_4 has the following value at each cusp: $j_4(\infty) = 1$, $j_4(0) = \infty$ (a simple pole), $j_4(1) = i$, $j_4(-1) = -i$, $j_4(-2) = 0$ (a simple zero), $j_4(\frac{1}{2}) = -1$.*

Proof. [9], Theorem 7. □

3. Some remarks on Fourier coefficients of j_4 and $N(j_4)$

As before we let

$$j_4(z) = \frac{\theta_3(\frac{z}{2})}{\theta_4(\frac{z}{2})} = \frac{\sum_{n\in\mathbb{Z}} q_4^{n^2}}{\sum_{n\in\mathbb{Z}}(-1)^n q_4^{n^2}}$$
$$= 1 + 4q_4 + 8{q_4}^2 + 16{q_4}^3 + 32{q_4}^4 + 56{q_4}^5 + 96{q_4}^6 + 160{q_4}^7 + 256q_4^8 + 404{q_4}^9 + 624{q_4}^{10} + 944{q_4}^{11} + 1408{q_4}^{12} + 2072{q_4}^{13} + 3008{q_4}^{14} + 4320q_4^{15} + \cdots .$$

We will derive a recursion formula for the Fourier coefficients of the modular function j_4. First we need two lemmas.

LEMMA 5. $\pm\left(\begin{smallmatrix} 4 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \Gamma(4) \left(\begin{smallmatrix} 4 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \Gamma_0(16)$

Proof. Straightforward. □

LEMMA 6. *For N even, if f is on $\Gamma_0(N)$, then so is $\frac{1}{2}\left(f|_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)} + f|_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)}\right)$. Here the meaning of $f|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}$ is just $f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot z\right)$.*

Proof. [1], Lemma 6. □

PROPOSITION 7. *Let $j_4(z) = \sum_{m\geq 0} b_m q_4^m$. Then for $k \geq 1$,*

$$b_{4k-1} = \frac{1}{b_1}\left(2\sum_{0\leq j<k} b_j b_{2k-j} + {b_k}^2 + \sum_{2\leq j\leq 2k-1}(-1)^j b_j b_{4k-j} + {b_{2k}}^2/2\right),$$

$$b_{4k} = 2\sum_{0\leq j<k} b_j b_{2k-j} + {b_k}^2,$$

$$b_{4k+1} = \frac{1}{b_1}\left(2\sum_{0\le j\le k} b_j b_{2k-j+1} + \sum_{2\le j\le 2k}(-1)^j b_j b_{4k-j+2} - {b_{2k+1}}^2/2\right),$$

$$b_{4k+2} = 2\sum_{0\le j\le k} b_j b_{2k-j+1}.$$

With the initial values $b_0 = 1$, $b_1 = 4$ *and* $b_2 = 8$, *we are able to determine all* b_m.

Proof. First we consider the identity

$$j_4|_{\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)} = \frac{\theta_3(\frac{z+2}{2})}{\theta_4(\frac{z+2}{2})} = \frac{\theta_4(\frac{z}{2})}{\theta_3(\frac{z}{2})} = 1/j_4 \qquad \text{by Theorem 2-(i).}$$

Then $j_4 \times j_4|_{\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)} = 1$. This implies that $\sum_{m\ge 0} b_m q_4^m \times \sum_{m\ge 0}(-1)^m b_m q_4^m = 1$. For $k \ge 1$, comparing the coefficients of the terms q_4^{4k} and q_4^{4k+2} on both sides, we get

$$(3.1)\qquad b_{4k} - b_1 b_{4k-1} + \sum_{2\le j\le 2k-1}(-1)^j b_j b_{4k-j} + {b_{2k}}^2/2 = 0$$

and

$$(3.2)\qquad b_{4k+2} - b_1 b_{4k+1} + \sum_{2\le j\le 2k}(-1)^j b_j b_{4k-j+2} - {b_{2k+1}}^2/2 = 0.$$

Now we define

$$j_4|_{U_2} \stackrel{\text{def}}{=} \frac{1}{2}\left(j_4|_{\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)} + j_4|_{\left(\begin{smallmatrix}1&4\\0&2\end{smallmatrix}\right)}\right).$$

Then $j_4|_{U_2} = \frac{1}{2}(j_4|_{\left(\begin{smallmatrix}4&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)\left(\begin{smallmatrix}4^{-1}&0\\0&1\end{smallmatrix}\right)} + j_4|_{\left(\begin{smallmatrix}4&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)\left(\begin{smallmatrix}4^{-1}&0\\0&1\end{smallmatrix}\right)})$. It follows from Lemma 5 and 6 that $j_4|_{U_2}$ is again on $\Gamma(4)$. And its Fourier expansion is $\sum_{m\ge 0} b_{2m} q_4^m$. Here we shall examine the poles of $j_4|_{U_2}$. Since j_4 has poles only at the cusps equivalent to 0, $j_4|_{U_2}$ can have poles only at $\left(\begin{smallmatrix}1&4i\\0&2\end{smallmatrix}\right)^{-1}\Gamma(4)\cdot 0$ for $i = 0, 1$. Moreover, we have

$$\begin{aligned}\left(\begin{smallmatrix}1&4i\\0&2\end{smallmatrix}\right)^{-1}\Gamma(4)\cdot 0 &= 2^{-1}\left(\begin{smallmatrix}2&-4i\\0&1\end{smallmatrix}\right)\Gamma(4)\cdot 0\\ &\sim \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&-2i\\0&1\end{smallmatrix}\right)\Gamma(4)\cdot 0\\ &\sim \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)\Gamma(4)\left(\begin{smallmatrix}1&-2i\\0&1\end{smallmatrix}\right)\cdot 0 = \left(\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right)\Gamma(4)\cdot(-2i).\end{aligned}$$

Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element in $\Gamma(4)$. Then $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot(-2i) = \frac{-4ai+2b}{-2ci+d}$ in lowest terms. But $\left(\begin{smallmatrix} -4ai+2b \\ -2ci+d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \mod 4$. Hence $j_4|_{U_2}$ can have poles only at the cusps equivalent to 0. We note from Theorem 2-(ii) that

$$(3.3) \qquad j_4|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)} = \frac{\theta_3(\frac{z}{2})}{\theta_4(\frac{z}{2})}\Bigg|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)} = \frac{\theta_3(2z)}{\theta_2(2z)} \in \frac{1}{2q_4} + q_4\mathbb{C}[[q_4]].$$

Then

$$\begin{aligned}(j_4|_{U_2})|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)} &= \frac{1}{2}\left(j_4|_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)} + j_4|_{\left(\begin{smallmatrix} 1 & 4 \\ 0 & 2 \end{smallmatrix}\right)}\right)\Bigg|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)} \\ &= \frac{1}{2}\left(j_4|_{\left(\begin{smallmatrix} 0 & -1 \\ 2 & 0 \end{smallmatrix}\right)} + j_4|_{\left(\begin{smallmatrix} 4 & -1 \\ 2 & 0 \end{smallmatrix}\right)}\right) \\ &= \frac{1}{2}\left(j_4|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\right) + O(1) \\ &= \frac{1}{4q_2} + O(1) \quad \text{by (3.3).}\end{aligned}$$

On the other hand, ${j_4}^2$ has poles only at 0 and

$${j_4}^2|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)} = \left(j_4|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}\right)^2 = \frac{1}{4q_2} + O(1).$$

Hence we get the following identity:

$$j_4|_{U_2} = {j_4}^2.$$

After replacing $j_4|_{U_2}$ (resp. j_4) by $\sum_{m\geq 0} b_{2m}q_4^m$ (resp. $\sum_{m\geq 0} b_m q_4^m$), if we compare the coefficients of the terms q_4^{2k} and q_4^{2k+1} on both sides for $k \geq 1$, we obtain

$$(3.4) \qquad b_{4k} = 2\sum_{0\leq j<k} b_j b_{2k-j} + {b_k}^2$$

and

$$(3.5) \qquad b_{4k+2} = 2\sum_{0\leq j\leq k} b_j b_{2k-j+1}.$$

By equating (3.1) and (3.4) (resp. (3.2) and (3.5)) we come up with b_{4k-1} (resp. b_{4k+1}) as desired. □

Let Γ be a Fuchsian group of the first kind with $\pm\Gamma_\infty = \{\pm\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)^n | n \in \mathbb{Z}\}$. For $f \in K(X(\Gamma))$, we call f "normalized" if its q series begins $\frac{1}{q_h} + 0 + a_1 q_h + a_2 q_h^2 + \cdots$. When $\Gamma = \Gamma(4)$, we will construct the normalized generator from the modular function j_4 described in Theorem 4 as follows.

$$\begin{aligned}\frac{4}{j_4-1} &= \frac{4\ \theta_4(\frac{z}{2})}{\theta_3(\frac{z}{2})-\theta_4(\frac{z}{2})}\\ &= \frac{1-2q_4+2q_4^4-2q_4^9+2q_4^{16}+\cdots}{q_4+q_4^9+q_4^{25}+\cdots}\\ &= \frac{1}{q_4}-2+2q_4^3-q_4^7-2q_4^{11}+3q_4^{15}+2q_4^{19}\\ &\quad -4q_4^{23}-4q_4^{27}+5q_4^{31}+8q_4^{35}-8q_4^{39}+\cdots,\end{aligned}$$

which is in $q_4{}^{-1}\mathbb{Z}[[q_4]]$ because $q_4+q_4^9+\cdots+q_4^{(2n-1)^2}+\cdots\in q_4\mathbb{Z}[[q_4]]^\times$. Let $N(j_4)=\frac{4}{j_4-1}+2$. Then $N(j_4)$ is normalized and unique ([Lemma 10]).

Write $N(j_4)=q_4^{-1}+\sum_{m\geq 1}H_m q_4^m$. We then observe from the series expansion that $H_m=0$ unless $m\equiv 3 \mod 4$. We will explain this cycle of nonvanishing in the following. Let t be a normalized modular function. Then for each $n\geq 1$ there exists a unique polynomial $X_n(t)$ in t such that $X_n(t)\equiv\frac{1}{n}q_h^{-n} \mod q_h\mathbb{C}[[q_h]]$. In particular, $X_1(t)=t$.

THEOREM 8. *Let t be the normalized generator of $K(X(N))$ for $2\leq N\leq 5$. If we write $X_n(t)=\frac{1}{n}q_N^{-n}+\sum_{m\geq 1}H_{m,n}q_N^m$, then $H_{m,n}=0$ unless $m\equiv -n \mod N$.*

Proof. Since $\Gamma(N)$ is a normal subgroup of $SL_2(\mathbb{Z})$, it follows that $X_n(t)|_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}$ is again on $\Gamma(N)$. We investigate the poles of $X_n(t)|_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}$. Since $X_n(t)$ has poles only at $\Gamma(N)\infty$, $X_n(t)|_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}$ has poles only at $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)^{-1}\Gamma(N)\infty$. But

$$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)^{-1}\Gamma(N)\infty=\Gamma(N)\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)^{-1}\infty=\Gamma(N)\infty.$$

At a neighborhood of ∞, $X_n(t)|_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}$ has the following expansion:

$$X_n(t)|_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}=\frac{1}{n}\zeta_N^{-n}q_N^{-n}+\sum_{m\geq 1}H_{m,n}(\zeta_N)^m q_N^m.$$

Moreover both $X_n(t)|_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}$ and $\zeta_N^{-n}X_n(t)$ have poles only at $\Gamma(N)\infty$ and the same residues at ∞. Hence $X_n(t)|_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}-\zeta_N^{-n}X_n(t)$ has no poles in $\mathfrak{H}^*$ so that $X_n(t)|_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}=\zeta_N^{-n}X_n(t)$. Considering their q_N-expansions we get

$$\frac{1}{n}\zeta_N^{-n}q_N^{-n} + \sum_{m\geq 1} H_{m,n}\zeta_N^m q_N^m = \frac{1}{n}\zeta_N^{-n}q_N^{-n} + \sum_{m\geq 1} \zeta_N^{-n} H_{m,n} q_N^m.$$

This implies that $(\zeta_N^m - \zeta_N^{-n}) \times H_{m,n} = 0$, from which the assertion follows. □

REMARK 9. We note that

$$\left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \Gamma_0^0(N) \left(\begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix}\right) = \Gamma_0(N^2). \tag{3.6}$$

From the index formulas (p.76, 79 in [16]) we can check that $\overline{\Gamma}(N) = \overline{\Gamma}_0^0(N)$ for $N = 2, 3, 4$ and $\overline{\Gamma}(5)$ is a subgroup of index 2 in $\overline{\Gamma}_0^0(5)$. Let t be the normalized generator of $K(X(N))$ as in Theorem 8. When $N = 2, 3, 4$, it follows from (3.6) that $t(Nz)$ is the normalized generator of $\Gamma_0(N^2)$. By [2] and [4] it corresponds to the Thompson series of type 4C (resp. 9B, 16B) if $N = 2$ (resp. 3, 4). Hence for $N = 2, 3, 4$, the Fourier coefficients of $t(Nz)$ has the same cycle of nonvanishing as stated in Theorem 8, that is, if $t(Nz) = \frac{1}{q} + \sum_{m\geq 1} H_m q^m$, then $H_m = H_{m,1} = 0$ unless $m \equiv -1 \mod N$ (see: Table 4 in [4]).

4. Application to Thompson series

In this section we shall relate the Fourier coefficients of $N(j_4)(4z)$ to representations of the monster group, and derive a recursion formula for the Fourier coefficients.

LEMMA 10. *The normalized generator of a genus zero function field is unique.*

Proof. Let Γ be a Fuchsian group such that the genus of the curve $\Gamma\backslash\mathfrak{H}^*$ is zero. Assume that $K(X(\Gamma)) = \mathbb{C}(J_1) = \mathbb{C}(J_2)$ where J_1 and J_2 are normalized. We can then write their Fourier expansions as $J_1 = \dfrac{1}{q} + 0 + a_1 q + a_2 q^2 + \cdots$ and $J_2 = \dfrac{1}{q} + 0 + b_1 q + b_2 q^2 + \cdots$. Observe that $1 = [K(X(\Gamma)) : \mathbb{C}(J_i)] = \nu_0(J_i) = \nu_\infty(J_i)$ for $i = 1, 2$. Hence, J_1 and J_2 have only one zero and one pole whose orders are simple. We see that the only poles of J_i occur at ∞. Then, $J_1 - J_2$ has no poles because the two series start with $\dfrac{1}{q}$. So, it should be a constant. Since $J_1 - J_2 = (a_1 - b_1)q + \cdots$, this constant must be zero. This proves the lemma. □

Let $\mathfrak{F}$ be the set of functions $f(z)$ satisfying the following conditions:

(i) $f(z) \in K(X(\Gamma))$ for some discrete subgroup Γ of $SL_2(\mathbb{R})$ that contains $\Gamma_1(N)$ for some N.
(ii) The genus of the curve $X(\Gamma)$ is 0 and its function field $K(X(\Gamma))$ is equal to $\mathbb{C}(f)$.
(iii) In a neighborhood of ∞, $f(z)$ is expressed in the form

$$f(z) = \frac{1}{q} + \sum_{n=0}^{\infty} a_n q^n, \quad a_n \in \mathbb{C}.$$

We say that a pair (G, ϕ) is a "moonshine" for a finite group G if ϕ is a function from G to $\mathfrak{F}$ defined by $\phi_\sigma(z) = \frac{1}{q} + a_0(\sigma) + \sum_{n=1}^{\infty} a_n(\sigma) q^n$ for $\sigma \in G$ and the mapping $\sigma \rightarrow a_n(\sigma)$ from G to $\mathbb{C}$ for each n is a generalized character of G. In particular, ϕ_σ is a class function of G. Finding or constructing a moonshine (G, ϕ) for a given group G, however, involves some nontrivial work. It is because that for each element σ of G, we have to find a natural number N and a Fuchsian group Γ containing $\Gamma_1(N)$ in such a way that its function field $K(X(\Gamma))$ is equal to $\mathbb{C}(\phi_\sigma)$ and the coefficients $a_n(\sigma)$ of the expansion of $\phi_\sigma(z)$ at ∞ induce generalized characters for all $n \geq 1$.

However, the following theorem conjectured by Thompson and proved by Borcherds shows that there exists a "moonshine" for the monster group M whose order is approximately 8×10^{53}. Let j be the modular invariant of $\Gamma(1)$ whose q-series is

$$j = q^{-1} + 744 + 196884\, q + \cdots = \sum_r c_r\, q^r \tag{4.7}$$

Then $j - 744$ is the normalized generator of $\Gamma(1)$. Thompson proposed that the coefficients in the q-series for $j - 744$ be replaced by the representations of M so that we obtain a formal series

$$H_{-1}\, q^{-1} + 0 + H_1\, q + H_2\, q^2 + \cdots$$

in which the H_r are certain representations of M called *head representations.* H_r has degree c_r as in (4.7), for example, H_{-1} is the trivial representation (degree 1), while H_1 is the sum of this and the degree 196883 representation and H_2 is the sum of former two and the degree 21296876 representation ([20]).

THEOREM 11. *The series*

$$T_m = \frac{1}{q} + 0 + H_1(m)q + H_2(m)q^2 + \cdots$$

is the normalized generator of a genus zero function field arising from a group between $\Gamma_0(N)$ *and its normalizer in* $PSL_2(\mathbb{R})$, *where* m *is an element of* M *and* $H_r(m)$ *is the character value of head representation* H_r *at* m ([1], [2]). *We call* T_m *the Thompson series of type* m.

By Lemma 5 the map which sends f to $f(4z)$ $(= f|_{\left(\begin{smallmatrix}4&0\\0&1\end{smallmatrix}\right)})$ defines an isomorphism between the fields $K(X(4))$ and $K(X_0(16))$. Note that the image of a generator under an isomorphism is again a generator. Hence $N(j_4)(4z)$ generates the field $K(X_0(16))$ over $\mathbb{C}$ and is still normalized. Now by Lemma 10, Table 3 and 4 in [4] we have

THEOREM 12. $N(j_4)(4z)$ *is the normalized generator of* $K(X_0(16))$ *which corresponds to the Thompson series of type* $16B$.

REMARK 13. Let m be the conjugacy class of M of order 16 and type B in Atlas notation ([3]). Since $N(j_4)(4z)$ is the Thompson series of type m by Theorem 12, we can write it as $\frac{1}{q}+\sum_{r\geq 1} H_r(m)q^r$ with $H_r(m)$ the character value of head representation H_r. Let χ_r $(r = 1, 2, \cdots, 194)$ be the irreducible characters of the monster group M. Since we know the Fourier coefficients of $N(j_4)(4z)$ and the character values $\chi_r(m)$ ([3], p.221) together, we can check the following relations from the decomposition of head character into irreducible characters ([4] Table 1a, [20]):

$H_{-1}(m) = \chi_1(m) = 1$

$H_1(m) = \chi_1(m) + \chi_2(m)$

$H_2(m) = \chi_1(m) + \chi_2(m) + \chi_3(m)$

$H_3(m) = 2\chi_1(m) + 2\chi_2(m) + \chi_3(m) + \chi_4(m)$

$H_4(m) = 2\chi_1(m) + 3\chi_2(m) + 2\chi_3(m) + \chi_4(m) + 0 \cdot \chi_5(m) + \chi_6(m)$

$H_5(m) = 4\chi_1(m) + 5\chi_2(m) + 3\chi_3(m) + 2\chi_4(m) + \chi_5(m) + \chi_6(m) + \chi_7(m)$, etc.

Let N be a positive integer and S be a set of Hall divisors of N. By $N + S$ we mean the subgroup of $PSL_2(\mathbb{R})$ generated by $\Gamma_0(N)$ and all the Atkin-Lehner involutions $W_{Q,N}$ for $Q \in S$. We assume that the genus of the curve $X(N + S)$ is zero. Let $t = q^{-1} + \sum_{m\geq 1} H_m q^m$ be the normalized generator of the function field of $X(N + S)$ as a completely replicable function. Let $t^{(2)}$ be the normalized generator of the function field of $X(N^{(2)} + S^{(2)})$ where $N^{(2)} = N/(2, N)$ and $S^{(2)}$ is the set of all Q in S which divide $N^{(2)}$. Define $t^{(2^l)}$ to be $(t^{(2^{l-1})})^{(2)}$. Write $t^{(s)} = q^{-1} + \sum_{m\geq 1} H_m^{(s)} q^m$. Using Norton's idea ([14], also see [2], [4] and [11]), we can derive a recursion formula in terms of the coefficients of t and $t^{(2)}$, which is shown in [7] step by step. For the sake of convenience of the reader, we will state the formula in the following:

(4.8)

$$H_{4k} = H_{2k+1} + \frac{H_k^2 - H_k^{(2)}}{2} + \sum_{1\leq j<k} H_j H_{2k-j}$$

$$H_{4k+1} = H_{2k+3} - H_2 H_{2k} + \frac{H_{2k}^2 + H_{2k}^{(2)}}{2} + \frac{H_{k+1}^2 - H_{k+1}^{(2)}}{2} + \sum_{1\leq j\leq k} H_j H_{2k-j+2} + \sum_{1\leq j<k} H_j^{(2)} H_{4k-4j} + \sum_{1\leq j<2k} (-1)^j H_j H_{4k-j}$$

$$H_{4k+2} = H_{2k+2} + \sum_{1\leq j\leq k} H_j H_{2k-j+1}$$

$$H_{4k+3} = H_{2k+4} - H_2 H_{2k+1} - \frac{H_{2k+1}^2 - H_{2k+1}^{(2)}}{2} + \sum_{1\leq j\leq k+1} H_j H_{2k-j+3} + \sum_{1\leq j\leq k} H_j^{(2)} H_{4k-4j+2} + \sum_{1\leq j\leq 2k} (-1)^j H_j H_{4k-j+2}.$$

From the above formulas, we see that if $m = 4$ or $m > 5$ then H_m can be determined by the coefficients H_i and $H_i^{(2)}$ for $1 \leq i < m$, and so if we know all $H_m^{(s)}$ for $m = 1, 2, 3$, and 5 together with $s = 2^l$ then we can work out all the coefficients H_m. Now we take $N = 16$ and $S = \{1\}$. Then $t = N(j_4)(4z)$, and $t^{(2^l)}$ is the normalized generator of the function field of $X_0(16/2^l)$ for $1 \leq l \leq 3$. And for $l \geq 4$, $t^{(2^l)}$ is the normalized generator of the function field of $X_0(1)$. We summarize the above as follows.

THEOREM 14. *If we know the 20 coefficients $\{H_i^{(2^l)} \mid i = 1, 2, 3$ and 5, $0 \leq l \leq 4\}$, then all the coefficients H_m of the modular function $N(j_4)(4z)$ can be determined.*

Observe that we actually know all the coefficients mentioned above, which would be as follows:

$H_1 = 0$, $H_2 = 0$, $H_3 = 2$, $H_5 = 0$ by the definition of $N(j_4)(4z)$,
$H_1^{(2)} = 4$, $H_2^{(2)} = 0$, $H_3^{(2)} = 2$, $H_5^{(2)} = -8$ by Table 3 and 4 in [4],

$H_1^{(4)} = 20$, $H_2^{(4)} = 0$, $H_3^{(4)} = -62$ $H_5^{(4)} = 216$ by [7],
$H_1^{(8)} = 276$, $H_2^{(8)} = -2048$, $H_3^{(8)} = 11202$ $H_5^{(8)} = 184024$ by [8],
$H_1^{(16)} = 196884$, $H_2^{(16)} = 21493760$, $H_3^{(16)} = 864299970$ $H_5^{(16)} = 333202640600$ by [4]. Here, the modular functions $j_{1,2}$ and $j_{1,4}$ are given by $j_{1,2}(z) = \theta_2(z)^8/\theta_4(2z)^8$ and $j_{1,4}(z) = \theta_2(2z)^4/\theta_3(2z)^4$, respectively for $z \in \mathfrak{H}$.

5. Application to Class Fields

Let Γ be a Fuchsian group of the first kind. Then $\Gamma\backslash\mathfrak{H}^*$ $(= X(\Gamma))$ is a compact Riemann surface. Hence, there exists a projective nonsingular algebraic curve V, defined over $\mathbb{C}$, that is biregularly isomorphic to $\Gamma\backslash\mathfrak{H}^*$. We specify a Γ-invariant holomorphic map φ of $\mathfrak{H}^*$ to V which gives a biregular isomorphism of $\Gamma\backslash\mathfrak{H}^*$ to V. In that situation, we call (V, φ) a *model* of $\Gamma\backslash\mathfrak{H}^*$. Now we assume that the genus of $\Gamma\backslash\mathfrak{H}^*$ is zero. Then its function field $K(X(\Gamma))$ is equal to $\mathbb{C}(J')$ for some J' in $K(X(\Gamma))$.

LEMMA 15. *$(\mathbb{P}^1(\mathbb{C}), J')$ is a model of $\Gamma\backslash\mathfrak{H}^*$.*

Proof. [7], Lemma 14. □

Let $G_{\mathbb{A}}$ be the adelization of an algebraic group $G = GL_2$ defined over $\mathbb{Q}$. Put

$$\begin{aligned} G_p &= GL_2(\mathbb{Q}_p) \ (p : \text{rational prime}),\\ G_\infty &= GL_2(\mathbb{R}),\\ G_{\infty+} &= \{x \in G_\infty \mid \det(x) > 0\},\\ G_{\mathbb{Q}_+} &= \{x \in GL_2(\mathbb{Q}) \mid \det(x) > 0\}. \end{aligned}$$

We define the topology of $G_{\mathbb{A}}$ by taking $U = \prod_p GL_2(\mathbb{Z}_p) \times G_{\infty+}$ to be an open subgroup of $G_{\mathbb{A}}$. Let K be an imaginary quadratic field and ξ be an embedding of K into $M_2(\mathbb{Q})$. We call ξ *normalized* if it is defined by $a\binom{z}{1} = \xi(a)\binom{z}{1}$ for $a \in K$ where z is the fixed point of $\xi(K^\times)$ $(\subset G_{\mathbb{Q}_+})$ in $\mathfrak{H}$. The embedding ξ defines a continuous homomorphism of $K_{\mathbb{A}}^\times$ into $G_{\mathbb{A}+}$, which we denote again by ξ. Here $G_{\mathbb{A}+}$ is the group $G_0G_{\infty+}$ with G_0 the non-archimedean part of $G_{\mathbb{A}}$ and $K_{\mathbb{A}}{}^\times$ the idele group of K. Let $\mathcal{Z}$ be the set of open subgroups S of $G_{\mathbb{A}+}$ containing $\mathbb{Q}^\times G_{\infty+}$ such that $S/\mathbb{Q}^\times G_{\infty+}$ is compact. For $S \in \mathcal{Z}$, we see that $\det(S)$ is open in $\mathbb{Q}_{\mathbb{A}}^\times$. Therefore the subgroup $\mathbb{Q}^\times \cdot \det(S)$ of $\mathbb{Q}_{\mathbb{A}}^\times$ corresponds to a finite abelian extension of $\mathbb{Q}$, which we write k_S. Put $\Gamma_S = S \cap G_{\mathbb{Q}_+}$

for $S \in \mathcal{Z}$. Then it is well known ([17], Proposition 6.27) that $\Gamma_S/\mathbb{Q}^\times$ is a Fuchsian group of the first kind commensurable with $\Gamma(1)/\{\pm 1\}$. Let $U_N = \{x = (x_p) \in U \mid x_p \equiv I \mod N \cdot M_2(\mathbb{Z}_p)\}$. We then have

LEMMA 16. (i) $\mathbb{Q}^\times U_N \in \mathcal{Z}$.
(ii) $k_S = \mathbb{Q}(\zeta_N)$, *if* $S = \mathbb{Q}^\times U_N$.
(iii) $\Gamma_S = \mathbb{Q}^\times \Gamma(N)$ *if* $S = \mathbb{Q}^\times U_N$.

Proof. First, we observe that $\mathbb{Q}^\times U_N$ is an open subgroup of $\mathbb{Q}^\times U$. Hence, for (i) it is enough to show that $\mathbb{Q}^\times U/\mathbb{Q}^\times G_{\infty_+}$ is compact. But, we know that $\mathbb{Q}^\times U/\mathbb{Q}^\times G_{\infty_+} = \prod GL_2(\mathbb{Z}_p)$ is compact. Let $V_{Np_\infty} = \{\alpha = (\alpha_p) \in \mathbb{Q}_{\mathbb{A}}^\times \mid \alpha \equiv 1 \mod^* Np_\infty, \alpha_p \in \mathbb{Z}_p^\times \text{ for } p \nmid N\}$ where p_∞ denotes the infinite $\mathbb{Q}$-prime. Here $\alpha \equiv 1 \mod^* Np_\infty$ means that each α_{p_i} is congruent to $1 \mod p_i^{n^i}\mathbb{Z}_{p_i}$ if $N = p_1^{n_1} \cdots p_r^{n^r}$ and $\alpha_{p_\infty} > 0$. As is well known ([6], Theorem 13-1-4), $\mathbb{Q}(\zeta_N)$ is the class field corresponding to $\mathbb{Q}^\times V_{Np_\infty}$. Now as for (ii), it suffices to show that $\det(U_N) = V_{Np_\infty}$. For $(x_p) \in U_N$, $\det x_p \equiv 1 \mod N\mathbb{Z}_p \equiv 1 \mod p^n\mathbb{Z}_p$ when $p^n \| N$. Hence, $\det U_N \subset V_{Np_\infty}$. Conversely, for $(\alpha_p) \in V_{Np_\infty}$, take $x_p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha_p \end{smallmatrix}\right)$. Since $N\mathbb{Z}_p = p^n\mathbb{Z}_p$ and $\alpha_p \equiv 1 \mod p^n\mathbb{Z}_p$ for $p^n \| N$, it is clear that $(x_p) \in U_N$ and $\det x_p = \alpha_p$. Finally, if $S = \mathbb{Q}^\times U_N$, we have $\Gamma_S = \mathbb{Q}^\times U_N \cap G_{\mathbb{Q}_+} = \mathbb{Q}^\times (U_N \cap G_{\mathbb{Q}_+}) = \mathbb{Q}^\times \Gamma(N)$. □

REMARK 17. For $z \in K \cap \mathfrak{H}$, we consider a normalized embedding $\xi_z : K \longrightarrow M_2(\mathbb{Q})$ defined by $a\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = \xi_z(a)\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)$ for $a \in K$. Then z is the fixed point of $\xi_z(K^\times)$ in $\mathfrak{H}$. Let (V_S, φ_S) be a model of $\Gamma_S \backslash \mathfrak{H}^*$. By Lemma 16-(iii), $\Gamma_S = \mathbb{Q}^\times \Gamma(4)$ when $S = \mathbb{Q}^\times U_N$ with $N = 4$. By Theorem 4 and Lemma 15, we can take $\varphi_S = j_4$ and $V_S = \mathbb{P}^1$. It follows from the fact ([17], Proposition 6.31-(ii)) that $j_4(z)$ belongs to $\mathbb{P}^1(K^{ab})$ where K^{ab} is the maximal abelian extension of K. Furthermore, it is true that $\theta_i(z)$ has no zeros in $\mathfrak{H}$ for i=2, 3, 4. Hence, we conclude that $j_4(z)$ in fact sits in K^{ab} for $z \in K \cap \mathfrak{H}$.

THEOREM 18. *Let K be an imaginary quadratic field and let ξ_z be the normalized embedding for $z \in K \cap \mathfrak{H}$. Then $j_4(z) \in K^{ab}$ and $K(i, j_4(z))$ if $i\ (= \sqrt{-1}) \notin K$ (or $K(j_4(z))$ if $i \in K$) is a class field of K corresponding to the subgroup $K^\times \cdot \xi_z^{-1}(\mathbb{Q}^\times U_4)$ of $K_{\mathbb{A}}^\times$.*

Proof. It follows from Lemma 16-(ii) and (iii) that $k_S = \mathbb{Q}(\zeta_4) = \mathbb{Q}(i)$ and $\Gamma_S = \mathbb{Q}^\times \Gamma(4)$ when $S = \mathbb{Q}^\times U_N$ with $N = 4$. Since j_4 gives a model of the curve $X(4)$, we can take $\varphi_S = j_4$. Then the assertion follows from [17], Proposition 6.33 and Remark 17. □

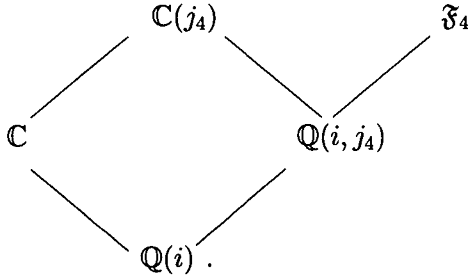
In view of standard results of complex multiplication, it is interesting to investigate whether the value $j_4(\alpha)$ is a generator for a certain full ray class field if α is the quotient of a basis of an ideal belonging to the maximal order in an imaginary quadratic field. We first need a result of complex multiplication.

THEOREM 19. *Let $\mathfrak{F}_N$ be the field of modular functions of level N rational over $\mathbb{Q}(e^{2\pi i/N})$, and let K be an imaginary quadratic field. Let $\mathfrak{O}_K$ be the maximal order of K and $\mathfrak{A}$ be an $\mathfrak{O}_K$-ideal such that $\mathfrak{A} = [z_1, z_2]$ and $z = z_1/z_2 \in \mathfrak{H}$. Then the field $K\mathfrak{F}_N(z)$ generated over K by all values $f(z)$ with $f \in \mathfrak{F}_N$ and f defined at z, is the ray class field over K with conductor N.*

Proof. [12], Ch. 10 Corollary of Theorem 2. □

LEMMA 20. $\mathfrak{F}_4 = \mathbb{Q}(i, j_4)$.

Proof. First, note that $\mathfrak{F}_4$ and $\mathbb{C}$ are linearly disjoint over $\mathbb{Q}(i)$. Indeed, let $\mu_1, \cdots, \mu_m$ be the elements of $\mathbb{C}$ which are linearly independent over $\mathbb{Q}(i)$. Assume that $\sum_i \mu_i g_i = 0$ with g_i in $\mathfrak{F}_4$. Let $g_i = \sum_n c_{in} {q_4}^n$ with $c_{in} \in \mathbb{Q}(i)$. Then $\sum_i \mu_i c_{in} = 0$ for every n, which implies $c_{in} = 0$ for all i and n. Hence $g_1 = \cdots = g_m = 0$. We then have the field tower



From the tower ([13], p. 361) we see that $\mathfrak{F}_4$ and $\mathbb{C}(j_4)$ are linearly disjoint over $\mathbb{Q}(i, j_4)$. Hence, again by Theorem 4

$$1 \leq [\mathfrak{F}_4 : \mathbb{Q}(i, j_4)] \leq [\mathbb{C}\mathfrak{F}_4 : \mathbb{C}(j_4)] \leq [K(X(4)) : K(X(4))] = 1$$

which yields that $\mathfrak{F}_4 = \mathbb{Q}(i, j_4)$. □

THEOREM 21. *Let K and z be as in* Theorem 19. *Then the field $K(i, j_4(z))$ (or $K(j_4(z))$) described in* Theorem 18 *is the ray class field over K with conductor* 4.

Proof. Immediate from Theorem 19 and Lemma 20. □

As its examples, we deal with the two cases when $K = \mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$. To this end we need a lemma.

LEMMA 22. (i) *For a positive real x, $j_4(xi) > 0$.*
(ii) *For $z \in \mathfrak{H}$, $j_4(2z)^2 = \frac{1}{2}(j_4(z) + j_4(z)^{-1})$.*
(iii) $j_4(\frac{i}{2^n}) = \dfrac{j_4(2^n i) + 1}{j_4(2^n i) - 1}$ *for $n \in \mathbb{N} \cup \{0\}$.*
(iv) $j_4(2z)^4 = \frac{1}{1-\lambda(z)}$ *where* $\lambda(z) = \frac{\theta_2{}^4(z)}{\theta_3{}^4(z)}$.

Proof. First, we observe that from the formula (23), p. 104 in [5]

$$\theta_2(2z) = \frac{1}{2}\left(\theta_3\left(\frac{z}{2}\right) - \theta_4\left(\frac{z}{2}\right)\right), \tag{5.9}$$

$$\theta_3(2z) = \frac{1}{2}\left(\theta_3\left(\frac{z}{2}\right) + \theta_4\left(\frac{z}{2}\right)\right). \tag{5.10}$$

It follows from the definition that $\theta_3(\frac{xi}{2}) = \sum_{n\in\mathbb{Z}} e^{\pi i(\frac{xi}{2})n^2} = \sum_{n\in\mathbb{Z}} e^{\frac{-\pi x n^2}{2}} > 0$. And by Theorem 2-(ii) and (5.9), $\theta_4\left(\frac{xi}{2}\right) = \theta_4\left(-\frac{x}{2i}\right) = \left(-i\frac{2i}{x}\right)^{\frac{1}{2}}\theta_2\left(\frac{2i}{x}\right)$ $= \sqrt{\frac{2}{x}}\,\frac{1}{2}\left(\theta_3\left(\frac{i}{2x}\right) - \theta_4\left(\frac{i}{2x}\right)\right) > 0$. This implies (i). For the second, we readily get that

$$\begin{aligned} j_4(2z)^2 = \frac{\theta_3(z)^2}{\theta_4(z)^2} &= \frac{\theta_3(\frac{z}{2})^2 + \theta_4(\frac{z}{2})^2}{2\,\theta_3(\frac{z}{2})\,\theta_4(\frac{z}{2})} \quad \text{by [15], Theorem 7.1.8} \\ &= \frac{1}{2}\,(j_4(z) + j_4(z)^{-1}). \end{aligned}$$

Thirdly, for $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} j_4\left(\frac{i}{2^n}\right) = \frac{\theta_3(\frac{i}{2^{n+1}})}{\theta_4(\frac{i}{2^{n+1}})} &= \frac{\theta_3(2^{n+1}i)}{\theta_2(2^{n+1}i)} \quad \text{by Theorem 2-(ii)} \\ &= \frac{\theta_3(2^{n-1}i) + \theta_4(2^{n-1}i)}{\theta_3(2^{n-1}i) - \theta_4(2^{n-1}i)} \quad \text{by (5.9) and (5.10)} \\ &= \frac{j_4(2^n i) + 1}{j_4(2^n i) - 1}. \end{aligned}$$

Finally, $j_4(2z)^4 = \frac{\theta_3(z)^4}{\theta_4(z)^4} = \frac{\theta_3(z)^4}{\theta_3(z)^4 - \theta_2(z)^4} = \frac{1}{1-\lambda(z)}$. This completes the lemma. □

PROPOSITION 23. *Let* $K_{(4)}$ *denote the ray class field over* K *with conductor 4.*

(i) *If* $K = \mathbb{Q}(i)$, *then* $K_{(4)} = K(\sqrt{2})$.

(ii) *If* $K = \mathbb{Q}(\sqrt{-3})$, *then* $K_{(4)} = K(\sqrt{3})$.

One can compare these with Exercises 2.13 (a) *and* 2.14 (a) *in* [19].

Proof. (i) If $K = \mathbb{Q}(i)$, $\mathfrak{O}_K = \mathbb{Z}i + \mathbb{Z}$. Hence by Theorem 21, $K_{(4)} = K(j_4(i))$. In Lemma 22-(iii), let us take $n = 0$. Then we come up with $j_4(i) = 1 \pm \sqrt{2}$. By Lemma 22-(i), $j_4(i) > 0$ and so $j_4(i) = 1 + \sqrt{2}$. Hence $K_{(4)} = K(\sqrt{2})$.

(ii) If $K = \mathbb{Q}(\sqrt{-3})$, $\mathfrak{O}_K = \mathbb{Z}\rho + \mathbb{Z}$ where $\rho = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then again by Theorem 21,

$$K_{(4)} = K(i, j_4(\rho)). \tag{5.11}$$

It is well-known ([15], p. 228) that $\lambda(\rho) = -\rho = {\zeta_6}^{-1}$. Using Lemma 22-(iv), we have $j_4(2\rho) = \pm\zeta_{24}^{-1}$ or $\pm i\zeta_{24}^{-1}$. On the other hand,

$$j_4(2\rho) = \frac{\theta_3(\rho)}{\theta_4(\rho)} = \frac{1 + 2\sum_{n\geq 1} e^{\pi i(2n)^2\cdot(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} + 2\sum_{n\geq 1} e^{\pi i(2n+1)^2\cdot(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)}}{1 + 2\sum_{n\geq 1} e^{\pi i(2n)^2\cdot(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} - 2\sum_{n\geq 1} e^{\pi i(2n+1)^2\cdot(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)}}.$$

Here we observe that $\sum_{n\geq 1} e^{\pi i(2n)^2\cdot(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} = \sum_{n\geq 1} e^{-\frac{\sqrt{3}}{2}\pi(2n)^2}$ and $\sum_{n\geq 1} e^{\pi i(2n+1)^2\cdot(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} = -i\sum_{n\geq 1} e^{-\frac{\sqrt{3}}{2}\pi(2n+1)^2}$. Let

$$p = 1 + 2\sum_{n\geq 1} e^{-\frac{\sqrt{3}}{2}\pi(2n)^2} \quad \text{and} \quad q = 2\sum_{n\geq 1} e^{-\frac{\sqrt{3}}{2}\pi(2n+1)^2}.$$

Then $j_4(2\rho) = \frac{p-iq}{p+iq}$. We note that $p, q > 0$ and $p - q = \theta_4(\frac{\sqrt{3}}{2}i) > 0$ which is shown in the proof of Lemma 22-(i). Hence $j_4(2\rho)$ lies in the 4-th quadrant of complex plane, from which we conclude that $j_4(2\rho) = \zeta_{24}^{-1}$. Now taking $z = \rho$ in Lemma 22-(ii) and substituting $j_4(2\rho)$ with ζ_{24}^{-1}, we derive

$$j_4(\rho) = \zeta_{12}^{-1} \pm \zeta_6^{-1} = \frac{\sqrt{3}}{2} - \frac{i}{2} \pm \left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right).$$

In any cases $j_4(\rho) \in \mathbb{Q}(\sqrt{3}, i)$. Then it is easy to see that $\mathbb{Q}(\sqrt{3}, i) = K(i) = K(\sqrt{3})$. Therefore by (5.11) we have $K_{(4)} = K(\sqrt{3})$, as desired. $\square$

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