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THE DIMENSION OF THE CONVOLUTION OF BIPARTITE ORDERED SETS

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ABSTRACT. In this paper, for any two bipartite ordered sets P and Q, we define the convolution $P * Q$ of P and Q. For $\dim(P) = s$ and $\dim(Q) = t$, we prove that $s+t-(U+V)-2 \leq \dim(P*Q) \leq s+t-(U+V)+2$, where $U+V$ is the max-min integer of the certain realizers. In particular, we also prove that $\dim(P_{n,k}) = n+k-\lfloor \frac{n+k}{3} \rfloor$ for $2 \leq k \leq n < 2k$ and $\dim(P_{n,k}) = n$ for $n \geq 2k$, where $P_{n,k} = S_n * S_k$ is the convolution of two standard ordered sets S_n and S_k.

1. Introduction

Let X be a set. An *order* R on X is a reflexive, antisymmetric and transitive binary relation on X. Then $P = (X, R)$ is called an ordered set. In this paper, we assume that X is finite. An order R on a set is called an *extension* of another order S on the same set if $S \subseteq R$. For $a, b \in X$, we usually write $a \leq b$ for $(a, b) \in R$ and also $a < b$ when $a \leq b$ and $a \neq b$. For elements $b > a$ in an ordered set P, we write $b \succ a$ or $a \prec b$ (b *covers* a or a is *covered by* b) if $b \geq c > a$ implies $b = c$ for every element c of P. A *linear extension* of an ordered set P is a linear order $E : x_1 \prec x_2 \prec \cdots \prec x_n$ containing the order of P. Szpilarjn [3] shows that any order has a linear extension. Dushnik and Miller [2] later defined the *dimension* of an ordered set P, denoted by $\dim(P)$, to be the minimal cardinality of a family of its linear extensions whose intersection is its order itself. An incomparable pair (a, b) in an ordered set P is called a *critical pair* if $x < a$ implies $x < b$ and $x > b$ implies $x > a$ and $\mathrm{crit}(P)$ denotes the set of all critical pairs. A *bipartite* ordered set is a triple $P = (X, Y, I_P)$ where X and Y are disjoint sets and I_P is an order on $X \cup Y$

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with $\{(x,y) \in I_P \mid x \neq y\} \subseteq X \times Y$. In [5], Trotter defined *the interval dimension* of P, denoted by $\dim_I(P)$, as the least positive integer t for which there exists a family $\mathcal{R} = \{L_1, L_2, \cdots, L_t\}$ of linear extensions of P reversing all critical pairs in $\text{crit}(P) \cap (X \times Y)$. And Trotter [4] defined the ordered set P_n, for integer $n \geq 2$, as follows: $\min(P_n) = \{x_1, x_2, \cdots, x_n\}$, $\text{mid}(P_n) = \{z_{ij} \mid 1 \leq i,j \leq n\}$ and $\max(P_n) = \{b_1, b_2, \cdots, b_n\}$. For all $i,j,u,v = 1,2,\cdots,n$ $x_i < z_{uj}$ in P_n for $i \neq u$, $z_{iv} < b_j$ in P_n for $v \neq j$, and $x_i \| z_{ij} \| b_j$ in P_n. It is known [4] that $\dim(P_n) = \lceil \frac{4n}{3} \rceil$.

In this paper, we define the convolution $P * Q$ of any two bipartite ordered sets P and Q. And we prove that $\dim(P) + \dim(Q) - (U+V) - 2 \leq \dim(P * Q) \leq \dim(P) + \dim(Q) - (U+V) + 2$, where $U+V$ is the max-min integer of the certain realizers. In particular, we also prove that $\dim(P_{n,k}) = n + k - \lfloor \frac{n+k}{3} \rfloor$ for $2 \leq k \leq n < 2k$ and $\dim(P_{n,k}) = n$ for $n \geq 2k$, where $P_{n,k} = S_n * S_k$ is the convolution of two standard ordered sets S_n and S_k. Furthermore, we see that $P_n = S_n * S_n$ and $\dim(P_n) = \lceil \frac{4n}{3} \rceil = 2n - \lfloor \frac{2n}{3} \rfloor$.

2. Definitions and examples

Let G and M be the sets and let I be a binary relation between G and M. We define a *context* as a triple (G, M, I) (see Wille [7]). A relation $F \subseteq G \times M$ is called a *Ferrers relation* if

$$g_1 F m_1 \quad \text{and} \quad g_2 F m_2 \quad \text{implies} \quad g_1 F m_2 \quad \text{or} \quad g_2 F m_1$$

for all $g_1, g_2 \in G$ and $m_1, m_2 \in M$. The *Ferrers dimension* of a context (G, M, I), denoted by $\text{fdim}(G, M, I)$, is defined to be the smallest number of Ferrers relations $F_1, F_2, \cdots, F_n$ with $I = \bigcap F_i$. Observe that the complement of a Ferrers relation F is again a Ferrers relation in $G \times M - I$. Therefore, one can alternatively define $\text{fdim}(G, M, I)$ as the minimum number of Ferrers relations $F_1, F_2, \cdots, F_n$ with $F_i \subseteq G \times M - I$ such that $\bigcup F_i = G \times M - I$. Throughout this paper, for any context (G, M, I), we assume that F is a Ferrers relation in $G \times M$ is the same meaning as F is a Ferrers relation in $G \times M - I$.

Let $P = (X, \leq)$ be an ordered set and let $S \subseteq X$. Let $J(P) = \{x \in X \mid x \in \bigvee S \Rightarrow x \in S\}$ and $M(P) = \{x \in X \mid x \in \bigwedge S \Rightarrow x \in S\}$. Then it is known [7] that $(X, X, \leq)$ and $(J(P), M(P), \leq_{J(P) \times M(P)})$ are contexts and that

$$\dim(P) = \text{fdim}(X, X, \leq) = \text{fdim}(J(P), M(P), \leq_{J(P) \times M(P)}).$$

For a bipartite ordered set $P = (X, Y, I_P)$, we can easily show that

$$\dim_I(P) = \operatorname{fdim}(X, Y, I_P).$$

Trotter obtained the following result:

THEOREM 2.1 [5]. *Let $P = (X, Y, I_P)$ be a bipartite ordered set. Then*

$$\dim(P) - 1 \leq \dim_I(P) \leq \dim(P).$$

Let (G, M, I) be a context and let F be any Ferrers relation in $G \times M$. We define the two subsets $C(F)$ and $R(F)$ of G and M, respectively, as follows:

$$C(F) = \bigcup\{ a \mid (a, b) \in F\} \quad \text{and} \quad R(F) = \bigcup\{ b \mid (a, b) \in F\}.$$

In fact, we know that $C(F)$ is a set of the first coordinate of the certain longest column in F and $R(F)$ is a set of the second coordinate of the certain longest row in F.

Let $P = (X, Y, I_P)$ be a bipartite ordered set and let $\mathcal{F}$ be a family of Ferrers relations in $X \times Y$. We say that $\mathcal{F}$ is a *(optimal) realizer* of $X \times Y$ if $|\mathcal{F}| = \operatorname{fdim}(X, Y, I_P)$ and $\bigcup_{F \in \mathcal{F}} F = X \times Y - I_P$.

The following Theorem done by Bae and Lee was a corollary in [1].

THEOREM 2.2 [1]. *Let $P = (X, Y, I_P)$ be a bipartite ordered set and let $\mathcal{F}$ be a realizer of $X \times Y$ with $|\mathcal{F}| \geq 2$. If there are elements $F_i, F_j \in \mathcal{F}$ such that $R(F_i) \cap R(F_j) = \emptyset$ and $C(F_i) \cap C(F_j) = \emptyset$, then*

$$\dim(P) = \dim_I(P).$$

Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets with

$$X = \{x_1, x_2, \cdots, x_n\} \quad \text{and} \quad Y = \{y_1, y_2, \cdots, y_l\},$$
$$A = \{a_1, a_2, \cdots, a_m\} \quad \text{and} \quad B = \{b_1, b_2, \cdots, b_k\}.$$

Consider the sets as follows:

$$X = \{x_1, x_2, \cdots, x_n\},$$
$$Z = \{z_{uv} = (y_u, a_v) \mid 1 \leq u \leq l \quad \text{and} \quad 1 \leq v \leq m\},$$
$$B = \{b_1, b_2, \cdots, b_k\}.$$

Then we define a convolution $P*Q$ of P and Q induced by P and Q is an ordered set on $X \cup Z \cup B$ with the following order relations:

$$x_i < z_{uv} \text{ in } P*Q \quad \text{if } x_i < y_u \text{ in } P,$$
$$z_{uv} < b_j \text{ in } P*Q \quad \text{if } a_v < b_j \text{ in } Q.$$

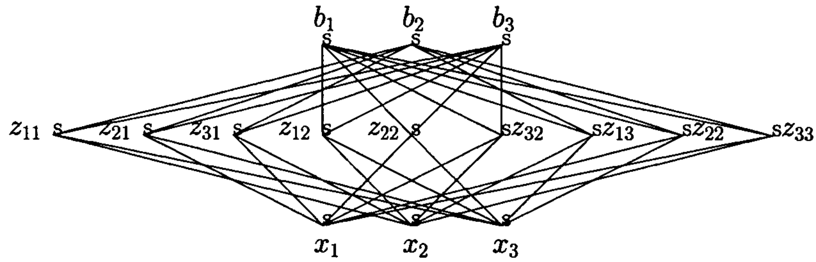

Figure 1. $S_3 * S_3$

Let $P = (X, Y, I_P)$ be a bipartite ordered set and let $\mathcal{F}$ be a realizer of $X \times Y$. A subfamily $\mathcal{F}_i$ of $\mathcal{F}$ is said to be *R-irreducible* of $\mathcal{F}$ if

(i) $\bigcap_{F\in\mathcal{F}_i} R(F) = \emptyset$,

(ii) There do not exist nonempty families $\mathcal{E}_1$ and $\mathcal{E}_2$ of Ferrers relations in $\bigcup_{F\in\mathcal{F}_i} F$ such that $\bigcap_{E\in\mathcal{E}_1} R(E) = \emptyset$ or $\bigcap_{E\in\mathcal{E}_2} R(E) = \emptyset$ with $|\mathcal{E}_1| + |\mathcal{E}_2| = |\mathcal{F}_i|$.

Similarly, a subfamily $\mathcal{G}_j$ of $\mathcal{F}$ is said to be *C-irreducible* of $\mathcal{F}$ if

(i)′ $\bigcap_{G\in\mathcal{G}_j} C(G) = \emptyset$,

(ii)′ There do not exist nonempty families $\mathcal{D}_1$ and $\mathcal{D}_2$ of Ferrers relations in $\bigcup_{G\in\mathcal{G}_j} G$ such that $\bigcap_{D\in\mathcal{D}_1} C(D) = \emptyset$ or $\bigcap_{D\in\mathcal{D}_2} C(D) = \emptyset$ with $|\mathcal{D}_1| + |\mathcal{D}_2| = |\mathcal{G}_j|$.

REMARK. Let $P = (X, Y, I_P)$ be a bipartite ordered set and let $\mathcal{F}$ be a realizer of $X \times Y$. If $\mathcal{F}_i$ and $\mathcal{G}_j$ are R-irreducible and C-irreducible, respectively, of $\mathcal{F}$, then we have

(1) $|\mathcal{F}_i| \geq 2$ and $|\mathcal{G}_j| \geq 2$,

(2) $\bigcap_{F\in\mathcal{F}_i-\{F_0\}} R(F) \neq \emptyset$ for all $F_0 \in \mathcal{F}_i$,

(3) $\bigcap_{G\in\mathcal{G}_j-\{G_0\}} C(G) \neq \emptyset$ for all $G_0 \in \mathcal{G}_j$.

Let $P = (X, Y, I_P)$ be a bipartite ordered set and let $\mathcal{F}$ be a realizer of $X \times Y$. A collection $\{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_w\}$ of subfamilies of $\mathcal{F}$ is said to be an *R-irreducible family* of $\mathcal{F}$ if

(i) each $\mathcal{F}_i (i = 1, 2, \cdots, w)$ is R-irreducible of $\mathcal{F}$,

(ii) $\bigcap\{R(F) \mid F \in \mathcal{F} - \bigcup_{i=1}^{w} \mathcal{F}_i\} \neq \emptyset$ or $\mathcal{F} = \bigcup_{i=1}^{w} \mathcal{F}_i$.

In this case, we say that $\mathcal{F}$ has an R-irreducible family. Similarly, a collection $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{w'}\}$ of subfamilies of $\mathcal{F}$ is said to be a *C-irreducible family* of $\mathcal{F}$ if

(i)′ each $\mathcal{G}_j (j = 1, 2, \cdots, w')$ is C-irreducible of $\mathcal{F}$,

(ii)′ $\bigcap\{C(F) \mid F \in \mathcal{F} - \bigcup_{j=1}^{w'} \mathcal{G}_j\} \neq \emptyset$ or $\mathcal{F} = \bigcup_{j=1}^{w'} \mathcal{G}_j$.

In this case, we say that $\mathcal{F}$ has a C-irreducible family.

$\mathcal{F}$ is said to be *max-min R-irreducible* of $X \times Y$ if

(i) There is a collection $\{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_w\}$ of subfamilies of $\mathcal{F}$ such that $\{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_w\}$ is an R-irreducible family of $\mathcal{F}$,

(ii) If $\mathcal{E}_R = \{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_u\}$ is an arbitrary R-irreducible family of any realizer $\mathcal{E}$ of $X \times Y$, then

$$u \leq w \text{ and } \sum_{i=1}^{k} |\mathcal{F}_i| \leq \min_{\substack{\mathcal{E}^* \subseteq \mathcal{E}_R \\ |\mathcal{E}^*| = k}} \left\{ \sum |\mathcal{E}'| : \mathcal{E}' \in \mathcal{E}^* \right\}$$

for all k with $1 \leq k \leq u$.

Similarly, $\mathcal{F}$ is said to be *max-min C-irreducible* of $X \times Y$ if

(i)′ There is a collection $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{w'}\}$ of subfamilies of $\mathcal{F}$ such that $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{w'}\}$ is a C-irreducible family of $\mathcal{F}$,

(ii)′ If $\mathcal{D}_C = \{\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_v\}$ is an arbitrary C-irreducible family of any realizer $\mathcal{D}$ of $X \times Y$, then

$$v \leq w' \text{ and } \sum_{j=1}^{k} |\mathcal{G}_j| \leq \min_{\substack{\mathcal{D}^* \subseteq \mathcal{D}_C \\ |\mathcal{D}^*| = k}} \left\{ \sum |\mathcal{D}'| : \mathcal{D}' \in \mathcal{D}^* \right\}$$

for all k with $1 \leq k \leq v$.

REMARK. Let $P = (X, Y, I_P)$ be a bipartite ordered set. If $\mathcal{F}$ is a max-min R-irreducible realizer of $X \times Y$, then there is a max-min R-irreducible family $\{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_w\}$ of $\mathcal{F}$ such that $|\mathcal{F}_i| \leq |\mathcal{F}_{i+1}|$ for all $i = 1, 2, \cdots, w$. Similarly, if $\mathcal{F}$ is a max-min C-irreducible realizer of $X \times Y$, then there is a max-min C-irreducible family $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{w'}\}$ of $\mathcal{F}$ such that $|\mathcal{G}_j| \leq |\mathcal{G}_{j+1}|$ for all $j = 1, 2, \cdots, w'$.

Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets. Let $\mathcal{F}$ and $\mathcal{G}$ be max-min R- and C-irreducible realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \min(P * Q), Z = \mathrm{mid}(P * Q)$ and $B = \max(P*Q)$. An integer $u+v$ is said to be an *R&C-irreducible integer* of $\mathcal{F}$ and $\mathcal{G}$ if there are max-min R-irreducible family $\{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_w\}$ of $\mathcal{F}$ and max-min C-irreducible family $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{w'}\}$ of $\mathcal{G}$ such that $v + \sum_{i=1}^{u} |\mathcal{F}_i| \leq \dim(P)$ and $u + \sum_{j=1}^{v} |\mathcal{G}_j| \leq \dim(Q)$ for some $u \in \{1, 2, \cdots, w\}$ and $v \in \{1, 2, \cdots, w'\}$. The integer $U + V$ is called the *max-min integer* of $\mathcal{F}$ and $\mathcal{G}$, which is defined by

$$U + V = \max_{\substack{0 \leq u \leq w \\ 0 \leq v \leq w'}} \{u + v \mid u + v \text{ is an } R\&C\text{-irreducible integer of } \mathcal{F} \text{ and } \mathcal{G}\}.$$

REMARK. Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets. Let $\mathcal{F}$ and $\mathcal{G}$ be max-min R- and C-irreducible realizers of $X \times Z$ and $Z \times B$, respectively, with the max-min integer $U + V$ of $\mathcal{F}$ and $\mathcal{G}$, where $X = \min(P * Q), Z = \mathrm{mid}(P * Q)$ and $B = \max(P * Q)$. Note that $|\mathcal{F}_i| \geq 2$ and $|\mathcal{G}_j| \geq 2$ for all $\mathcal{F}_i \in \mathcal{F}$ and $\mathcal{G}_j \in \mathcal{G}$. Hence we have $U + V \leq \lfloor \frac{\dim(P) + \dim(Q)}{3} \rfloor$.

EXAMPLE 1. For $2k > n \geq k \geq 2$, let $S_n = (X, Y, \leq)$ and $S_k = (A, B, \leq)$ be standard ordered sets with $X = \{x_1, x_2, \cdots, x_n\}, Y = \{y_1, y_2, \cdots, y_n\}, A = \{a_1, a_2, \cdots, a_k\}$ and $B = \{b_1, b_2, \cdots, b_k\}$. Then $\mathcal{F} = \{\{(x_i, z_{ij}) \mid 1 \leq j \leq k\} \mid 1 \leq i \leq n\}$ and $\mathcal{G} = \{\{(z_{ij}, b_j) \mid 1 \leq i \leq n\} \mid 1 \leq j \leq k\}$ are max-min R- and C-irreducible realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \min(S_n * S_k), Z = \mathrm{mid}(S_n * S_k)$ and $B = \max(S_n * S_k)$. For all $i = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$ and $j = 1, 2, \cdots, \lfloor \frac{k}{2} \rfloor$, let

$$\mathcal{F}_i = \{(x_i, z_{i1}), (x_i, z_{i2}), \cdots, (x_i, z_{ik})\} \cup \{(x_p, z_{p1}), (x_p, z_{p2}), \cdots, (x_p, z_{pk})\},$$
$$\mathcal{G}_j = \{(z_{1j}, b_j), (z_{2j}, b_j), \cdots, (z_{nj}, b_j)\} \cup \{(z_{1q}, b_q), (z_{2q}, b_q), \cdots, (z_{nq}, b_q)\}$$

where $p = \lfloor \frac{n}{2} \rfloor + i$ and $q = \lfloor \frac{k}{2} \rfloor + j$. Then we see that $\{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_{\lfloor \frac{n}{2} \rfloor}\}$ and $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{\lfloor \frac{k}{2} \rfloor}\}$ are max-min R- and C-irreducible realizers of $X \times Z$ and $Z \times B$, respectively, and that $|\mathcal{F}_i| = |\mathcal{G}_j| = 2$ for all $i = 1, 2, \cdots, \lfloor \frac{n}{2} \rfloor$ and $j = 1, 2, \cdots, \lfloor \frac{k}{2} \rfloor$. Then there exist $R\&C$ irreducible integer $u + v$ of $\mathcal{F}$ and $\mathcal{G}$ such that $1 \leq u \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq v \leq w$, where $w = \min\{n - 2u, \lfloor \frac{k-u}{2} \rfloor\}$. Since $2k > n \geq k \geq 2$, it follows from the above Remark that $U + V = \max\{u + v \mid 1 \leq u \leq \lfloor \frac{n}{2} \rfloor, 1 \leq v \leq w\} = \lfloor \frac{n+k}{3} \rfloor$.

EXAMPLE 2. Consider the relations in the below Table 1 defined as follows: for any pair $(p,q) \in (S_5 * S_3) \times (S_5 * S_3)$,

$$p \overset{i}{\sim} q \Longleftrightarrow (p,q) \in H_i,$$
$$p \overset{i,j}{\sim} q \Longleftrightarrow (p,q) \in H_i \text{ and } (p,q) \in H_j,$$
$$p \overset{O}{\sim} q \Longleftrightarrow p < q \text{ in } S_5 * S_3$$

for some Ferrers relations H_i and H_j in $(S_5 * S_3) \times (S_5 * S_3)$.

	x_1	x_2	x_3	x_4	x_5	z_{11}	z_{12}	z_{13}	z_{21}	z_{22}	z_{23}	z_{31}	z_{32}	z_{33}	z_{41}	z_{42}	z_{43}	z_{51}	z_{52}	z_{53}	b_1	b_2	b_3
x_1	O	1	1	1	1	1	1	1	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O
x_2	2	O	2	2	2	O	O	O	2	2	2	O	O	O	O	O	O	O	O	O	O	O	O
x_3	2	1	O	2	2	O	O	O	O	O	O	5	5	5	O	O	O	O	O	O	O	O	O
x_4	2	1	1	O	2	O	O	O	O	O	O	O	O	O	6	6	6	O	O	O	O	O	O
x_5	2	1	1	1	O	O	O	O	O	O	O	O	O	O	O	O	O	4	3	3	O	O	O
z_{11}	2	1	1,2	1,2	1,2	O	3	4	4	3	4	4	3	4	4	3	4	4	3	3	3	O	O
z_{12}	2	1	1,2	1,2	1,2	1,4	O	4	4	3	4	4	3	4	4	3	4	4	3	3	O	4	O
z_{13}	2	1	1,2	1,2	1,2	1	1	O	2,4	2,3	2,4	2,4	2,3	2,4	2,4	2,3	2,4	2,4	2,3	2,3	O	O	2
z_{21}	2	1	1,2	1,2	1,2	1	1,3	1	O	3	4	4	3	4	4	3	4	4	3	3	3	O	O
z_{22}	2	1	1,2	1,2	1,2	1,4	1	1,4	2,4	O	4	4	3	4	4	3	4	4	3	3	O	4	O
z_{23}	2	1	1,2	1,2	1,2	1	1	1	2	2	O	4	3	4	4	3	4	4	3	3	O	O	1
z_{31}	2	1	1,2	1,2	1,2	1	1,3	1	2	2,3	2	O	3	4	4	3	4	4	3	3	3	O	O
z_{32}	2	1	1,2	1,2	1,2	1,4	1	1,4	2,4	2	2,4	2,4	O	4	4	3	4	4	3	3	O	4	O
z_{33}	2	1	1,2	1,2	1,2	1	1	1	2	2	2	2	2	O	4	3	4	4	3	3	O	O	1
z_{41}	2	1	1,2	1,2	1,2	1	1,3	1	2	2,3	2	2	2,3	2	O	3	4	4	3	3	3	O	O
z_{42}	2	1	1,2	1,2	1,2	1,4	1	1,4	2,4	2	2,4	2,4	2	2,4	2,4	O	4	4	3	3	O	4	O
z_{43}	2	1	1,2	1,2	1,2	1	1	1	2	2	2	2	2	2	2	2	O	4	3	3	O	O	1
z_{51}	2	1	1,2	1,2	1,2	1	1,3	1	2	2,3	2	2	2,3	2	2	2,3	2	O	3	3	3	O	O
z_{52}	2	1	1,2	1,2	1,2	1,4	1	1,4	2,4	2	2,4	2,4	2	2,4	2,4	2	2,4	2,4	O	3	O	4	O
z_{53}	2	1	1,2	1,2	1,2	1	1	1	2	2	2	2	2	2	2	2	2	2	2	O	O	O	1
b_1	2	1	1,2	1,2	1,2	1	1,3	1	2	2,3	2	2	2,3	2	2	2,3	2	2	2,3	2,3	O	2	2
b_2	2	1	1,2	1,2	1,2	1	1,3	1	2	2,3	2	2	2,3	2	2	2,3	2	2	2,3	2,3	3	O	2
b_3	2	1	1,2	1,2	1,2	1	1,3	1	2	2,3	2	2	2,3	2	2	2,3	2	2	2,3	2,3	3	3	O

Table 1. Ferrers relations in $(S_5 * S_3) \times (S_5 * S_3)$

Then we see that there are distinct 6-Ferrers relations H_1, H_2, H_3, H_4, H_5 and H_6 such that $\bigcup_{i=1}^{6} H_i = (S_5 * S_3) \times (S_5 * S_3) - I$ and hence $\dim(S_5 * S_3) \leq 6$. From Example 1 we obtain $\dim(S_5 * S_3) \geq 6$. Hence we conclude that $\dim(S_5 * S_3) = 6$. Similarly, we have $\dim(S_4 * S_3) = 5$ and $\dim(S_3 * S_3) = 4$.

3. Main results

Let (G, M, I) be a context. For arbitrary subset $S \subseteq G \times M$, we denoted by

$$\operatorname{fdim}(S) = \min\{n \mid \bigcup_{i=1}^{n} F_i = S - I\},$$

where F_i is a Ferrers relation in $G \times M - I$.

LEMMA 3.1. *Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets and let $\mathcal{F}$ and $\mathcal{G}$ be realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \min(P * Q), Z = \operatorname{mid}(P * Q)$ and $B = \max(P * Q)$. If $\dim(P) = \dim_I(P)$ and $\dim(Q) = \dim_I(Q)$, then we have the following properties:*

(1) *If $\mathcal{F}_i$ is an R-irreducible of $\mathcal{F}$, then there is at most one $G_0 \in \mathcal{G}$ such that $\operatorname{fdim}(\bigcup_{F \in \mathcal{F}_i} F \cup G_0 \cup M) = |\mathcal{F}_i|$ for some $M \subset Z \times Z - I$.*

(2) *If $\mathcal{G}_j$ is a C-irreducible of $\mathcal{G}$, then there is at most one $F_0 \in \mathcal{F}$ such that $\operatorname{fdim}(\bigcup_{G \in \mathcal{G}_j} G \cup F_0 \cup M') = |\mathcal{G}_j|$ for some $M' \subset Z \times Z - I$.*

Proof. Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets. Let $\mathcal{F}$ and $\mathcal{G}$ be realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \{x_1, x_2, \cdots, x_n\}, Z = \{z_{uv} \mid 1 \leq u \leq l \text{ and } 1 \leq v \leq m\}$ and $B = \{b_1, b_2, \cdots, b_k\}$.

Consider the projection maps $q_1 : X \times Z \to X \times Y$ and $q_2 : Z \times B \to A \times B$, which are defined by

$$q_1(x_i, z_{uv}) = (x_i, y_u) \text{ and } q_2(z_{uv}, b_j) = (a_v, b_j).$$

Now, we have the following observations:

(i) For $F \in \mathcal{F}$, $q_1(F)$ is a Ferrers relation in $X \times Y$ and $\{q_1(F) \mid F \in \mathcal{F}\}$ is also a realizer of $X \times Y$,

(ii) For $G \in \mathcal{G}$, $q_2(G)$ is also a Ferrers relation in $A \times B$ and $\{q_2(G) \mid G \in \mathcal{G}\}$ is also a realizer of $A \times B$,

(iii) For $F \in \mathcal{F}$, there is $(x_i, y_u) \in X \times Y - I_P$ such that

$$\{(x_i, z_{u1}), (x_i, z_{u2}), \cdots, (x_i, z_{um})\} \subseteq F,$$

(iv) For $G \in \mathcal{G}$, there is $(a_v, b_j) \in A \times B - I_Q$ such that

$$\{(z_{1v}, b_j), (z_{2v}, b_j), \cdots, (z_{lv}, b_j)\} \subseteq G.$$

(1) Suppose that $\mathcal{F}_i$ is an R-irreducible of $\mathcal{F}$ and that there are distinct $G_1, G_2 \in \mathcal{G}$ such that $\text{fdim}(\bigcup_{F\in\mathcal{F}_i} F \cup G_1 \cup G_2 \cup M) = |\mathcal{F}_i| = w$ for some $M \subset Z \times Z - I$. Let $\mathcal{E}$ be an arbitrary realizer of $\bigcup_{F\in\mathcal{F}_i} F \cup G_1 \cup G_2 \cup M$ with $|\mathcal{E}| = |\mathcal{F}_i|$. Since $q_2(G_1)$ and $q_2(G_2)$ are distinct Ferrers relations in $A \times B - I_Q$, it follows that there are $(a_1, b_1) \in q_2(G_1)$ and $(a_2, b_2) \in q_2(G_2)$ such that $(a_1, b_2) \notin q_2(G_1 \cup G_2)$ and $(a_2, b_1) \notin q_2(G_1 \cup G_2)$. Therefore $\{E \in \mathcal{E} \mid E \cap G_1 \neq \emptyset\} \cap \{E \in \mathcal{E} \mid E \cap G_2 \neq \emptyset\} = \emptyset$. Furthermore, since $\mathcal{E}$ is a realizer of $\bigcup_{F\in\mathcal{F}_i} F \cup G_1 \cup G_2 \cup M$, it follows that $G_1 \subset \bigcup\{E \in \mathcal{E} \mid E \cap G_1 \neq \emptyset\}$ and $G_2 \subset \bigcup\{E \in \mathcal{E} \mid E \cap G_2 \neq \emptyset\}$. Hence we have $\bigcap\{R(E) \mid E \in \mathcal{E} \text{ with } E \cap G_1 \neq \emptyset\} = \emptyset$ and $\bigcap\{R(E) \mid E \in \mathcal{E} \text{ with } E \cap G_2 \neq \emptyset\} = \emptyset$, which is a contradiction as $\mathcal{F}_i$ is an R-irreducible of $\mathcal{F}$.

(2) Similarly, we can prove the result. □

THEOREM 3.2. *Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets with $\dim(P) = s$ and $\dim(Q) = t$ and let $\mathcal{F}$ and $\mathcal{G}$ be max-min R- and C-irreducible realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \min(P * Q), Z = \text{mid}(P * Q)$ and $B = \max(P * Q)$. If $\dim(P) = \dim_I(P)$ and $\dim(Q) = \dim_I(Q)$, then we have*

$$s + t - (U + V) \leq \dim(P * Q) \leq s + t - (U + V) + 2,$$

where $U + V$ is the max-min integer of $\mathcal{F}$ and $\mathcal{G}$.

Proof. Suppose that $\{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_w\}$ is a max-min R-irreducible family of a realizer $\mathcal{F}$ of $X \times Z$ and $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{w'}\}$ is a max-min C-irreducible family of a realizer $\mathcal{G}$ of $Z \times Y$, where $X = \min(P * Q), Z = \text{mid}(P * Q)$ and $B = \max(P * Q)$.

Consider the max-min integer $U + V$ of $\mathcal{F}$ and $\mathcal{G}$ with $1 \leq U \leq w$ and $1 \leq V \leq w'$. Then there are subfamilies $\{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_U\}$ and $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_V\}$ of $\{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_w\}$ and $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{w'}\}$, respectively, such that $V + \sum_{i=1}^{U} |\mathcal{F}_i| \leq \dim(P)$ and $U + \sum_{j=1}^{V} |\mathcal{G}_j| \leq \dim(Q)$.

By Lemma 3.1, for each $i = 1, 2, \cdots, U$, there is exactly one Ferrers relation $G_i \in \mathcal{G} - \bigcup_{j=1}^{V} \mathcal{G}_j$ such that $\text{fdim}(\bigcup_{F\in\mathcal{F}_i} F \cup M_i \cup G_i) = |\mathcal{F}_i|$ for some $M_i \subset Z \times Z - I$. Similarly, for each $j = 1, 2, \cdots, V$, there is exactly one Ferrers relation $F_j \in \mathcal{F} - \bigcup_{i=1}^{U} \mathcal{F}_i$ such that $\mathrm{fdim}(\bigcup_{G \in \mathcal{G}_j} G \cup M_j' \cup F_j) = |\mathcal{G}_j|$ for some $M_j' \subset Z \times Z - I$. From the definition of $U + V$, we know that $\mathcal{F} - \bigcup_{i=1}^{U} \mathcal{F}_i - \{F_i \mid 1 \le i \le V\}$ does not have an R-irreducible subfamily and $\mathcal{G} - \bigcup_{j=1}^{V} \mathcal{G}_j - \{G_j \mid 1 \le j \le U\}$ does not have a C-irreducible subfamily. Since $\dim(P) = \dim_I(P)$ and $\dim(Q) = \dim_I(Q)$, it follows that $\mathrm{fdim}((X \times Z) \times (Z \times B)) = \dim(P) + \dim(Q) - (U + V)$. But we know that $\mathrm{fdim}((X \cup Z \cup B) \times (X \cup Z \cup B) - (X \times Z) - (Z \times B)) = 2$, thus we have $(s - U) + (t - V) \le \dim(P * Q) \le (s - U) + (t - V) + 2$. □

COROLLARY 3.3. *Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets with $\dim(P) = s$ and $\dim(Q) = t$ and let $\mathcal{F}$ and $\mathcal{G}$ be max-min R- and C-irreducible realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \min(P * Q), Z = \mathrm{mid}(P * Q)$ and $B = \max(P * Q)$. Then we have the following properties:*

(1) *If $\dim(P) = \dim_I(P) + 1$ and $\dim(Q) = \dim_I(Q)$, then there is $G_0 \in \mathcal{G}$ such that $s+t-(U+V)-1 \le \dim(P*Q) \le s+t-(U+V)+1$, where $U + V$ is the max-min integer of $\mathcal{F}$ and $\mathcal{G} - \{G_0\}$.*
(2) *If $\dim(P) = \dim_I(P)$ and $\dim(Q) = \dim_I(Q) + 1$, then there is $F_0 \in \mathcal{F}$ such that $s+t-(U+V)-1 \le \dim(P*Q) \le s+t-(U+V)+1$, where $U + V$ is the max-min integer of $\mathcal{F} - \{F_0\}$ and $\mathcal{G}$.*
(3) *If $\dim(P) = \dim_I(P)+1$ and $\dim(Q) = \dim_I(Q)+1$, then there are $F_0 \in \mathcal{F}$ and $G_0 \in \mathcal{G}$ such that $s + t - (U + V) - 2 \le \dim(P * Q) \le s + t - (U + V)$, where $U + V$ is the max-min integer of $\mathcal{F} - \{F_0\}$ and $\mathcal{G} - \{G_0\}$.*

Proof. (1) Let $\dim(P) = \dim_I(P) + 1$ and $\dim(Q) = \dim_I(Q)$ and let $\mathcal{F}$ and $\mathcal{G}$ be max-min R- and C-irreducible realizers of $X \times Z$ and $Z \times B$, respectively, where $X = \min(P * Q), Z = \mathrm{mid}(P * Q)$ and $B = \max(P * Q)$. Since $\dim(P) = \dim_I(P) + 1$, it follows that there is a Ferrers relation F such that $F \cap (X \times Z) = \emptyset$. Then there is a Ferrers relation $G_0 \in \mathcal{G}$ such that $F \cup G_0$ contained a Ferrers relation in $P * Q$. Since $|\mathcal{F}| = s - 1$ and $|\mathcal{G} - \{G_0\}| = t - 1$, it follows from Theorem 3.2 that there is the max-min integer $U + V$ of $\mathcal{F}$ and $\mathcal{G} - \{G_0\}$ such that $s + t - 1 - (U + V) \le \dim(P * Q) \le s + t + 1 - (U + V)$. By the similar method in (1), we obtain results (2) and (3). □

LEMMA 3.4. *Let $P = (X, Y, I_P)$ and $Q = (A, B, I_Q)$ be bipartite ordered sets with $\dim(P) = s, \dim(Q) = t$ and $s, t \geq 2$. Then we have $\dim(P * Q) \geq s + t - 2 - \lfloor \frac{s+t-4}{3} \rfloor$.*

Proof. Let $X = \min(P * Q) = \{x_1, x_2, \cdots, x_n\}, Z = \text{mid}(P * Q) = \{z_{uv} \mid 1 \leq u \leq l \text{ and } 1 \leq v \leq m\}$ and $B = \max(P*Q) = \{b_1, b_2, \cdots, b_k\}$. By Theorem 2.1, we have two cases as follows:

$$\dim(P) = \dim_I(P) + 1 \text{ or } \dim(P) = \dim_I(P),$$
$$\dim(Q) = \dim_I(Q) + 1 \text{ or } \dim(Q) = \dim_I(Q).$$

CASE 1. $\dim(P) = \dim_I(P)$ and $\dim(Q) = \dim_I(Q)$.

Let $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ and $\mathcal{G} = \{G_1, G_2, \cdots, G_t\}$ be realizers of $(X \cup Z) \times (X \cup Z)$ and $(Z \cup B) \times (Z \cup B)$, respectively, and let $\mathcal{H} = \{H_1, H_2, \cdots, H_w\}$ be an arbitrary realizer of $P * Q$. Note that, for each $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$, there are $(x_i, y_u) \in X \times Y - I_P$ and $(a_v, b_j) \in A \times B - I_Q$ such that $\{(x_i, z_{uj}) \mid 1 \leq j \leq m\} \subseteq F_i, \{(z_{iv}, b_j) \mid 1 \leq i \leq l\} \subseteq G_j$ and $R(F_i) \cap C(G_j) \neq \emptyset$. Then, for each H of $\mathcal{H}$, there do not exist $F_{i_0} \in \mathcal{F}$ and $G_{j_0} \in \mathcal{G}$ such that $F_{i_0} \cup G_{j_0} \subseteq H$. For all $E \subseteq (X \cup Z \cup B) \times (X \cup Z \cup B) - I$, we see that

$$\text{(i) fdim}(\bigcup_{i=1}^{3r} D_i \cup E) \geq 2r \quad \text{for all } D_i \in \mathcal{F} \cup \mathcal{G} \text{ if } s + t = 3r,$$

$$\text{(ii) fdim}(\bigcup_{i=1}^{3r+1} D_i \cup E) \geq 2r + 1 \quad \text{for all } D_i \in \mathcal{F} \cup \mathcal{G} \text{ if } s + t = 3r + 1,$$

$$\text{(iii) fdim}(\bigcup_{i=1}^{3r+2} D_i \cup E) \geq 2r + 2 \quad \text{for all } D_i \in \mathcal{F} \cup \mathcal{G} \text{ if } s + t = 3r + 2.$$

Hence we have $\dim(P * Q) \geq s + t - \lfloor \frac{s+t}{3} \rfloor$.

CASE 2. $\dim(P) = \dim_I(P) + 1$ and $\dim(Q) = \dim_I(Q)$.

Let $\{F_1, F_2, \cdots, F_s\}$ and $\{G_1, G_2, \cdots, G_t\}$ be realizers of $(X \cup Z) \times (X \cup Z)$ and $(Z \cup B) \times (Z \cup B)$, respectively, and let $\mathcal{H} = \{H_1, H_2, \cdots, H_w\}$ be arbitrary realizer of $P * Q$. Since $\dim(P) = \dim_I(P) + 1$, without loss of generality, we may assume that there is at most one element F of $\{F_1, F_2, \cdots, F_s\}$, say $F = F_s$, such that $F_s \cap (X \times Z) = \emptyset$. Then there is a Ferrers relation $H \in \mathcal{H}$ such that $F_s \cup G_{j_0} \subset H$ for some $j_0 \in \{1, 2, \cdots, t\}$.

Let $\mathcal{F} = \{F_1, F_2, \cdots, F_{s-1}\}$ and let $\mathcal{G} = \{G_1, G_2, \cdots, G_{t-1}\}$ with $G_t = G_{j_0}$. Note that for each $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$, there are $(x_i, y_u) \in X \times Y - I_P$ and $(a_v, b_j) \in A \times B - I_Q$ such that $\{(x_i, z_{uj}) \mid 1 \le j \le m\} \subseteq F_i$, $\{(z_{iv}, b_j) \mid 1 \le i \le l\} \subseteq G_j$ and $R(F_i) \cap C(G_j) \ne \emptyset$. Then as in case 1, for each element H of $\mathcal{H}$, there do not exist $F_{i_0} \in \mathcal{F}$ and $G_{j_0} \in \mathcal{G}$ such that $F_{i_0} \cup G_{j_0} \subseteq H$. By the same method in Case 1, for all $E \subseteq (X \cup Z \cup B) \times (X \cup Z \cup B) - I$, $\mathrm{fdim}(\bigcup_{D \in \mathcal{F} \cup \mathcal{G}} D \cup E) \ge (s-1)+(t-1)-\lfloor\frac{(s-1)+(t-1)}{3}\rfloor$. Hence we have $\dim(P*Q) \ge (s-1)+(t-1)-\lfloor\frac{(s-1)+(t-1)}{3}\rfloor+1 = s+t-1-\lfloor\frac{s+t-2}{3}\rfloor$.

CASE 3. $\dim(P) = \dim_I(P)$ and $\dim(Q) = \dim_I(Q) + 1$.

By symmetry to the Case 2, we get $\dim(P * Q) \ge s+t-1-\lfloor\frac{s+t-2}{3}\rfloor$.

CASE 4. $\dim(P) = \dim_I(P) + 1$ and $\dim(Q) = \dim_I(Q) + 1$.

Let $\{F_1, F_2, \cdots, F_s\}$ and $\{G_1, G_2, \cdots, G_t\}$ be realizers of $(X \cup Z) \times (X \cup Z)$ and $(Z \cup B) \times (Z \cup B)$, respectively, and let $\mathcal{H} = \{H_1, H_2, \cdots, H_w\}$ be arbitrary realizer of $P * Q$. Since $\dim(P) = \dim_I(P) + 1$, without loss of generality, we may assume that there is at most one element F of $\{F_1, F_2, \cdots, F_s\}$, say $F = F_s$, such that $F_s \cap (X \times Z) = \emptyset$. Similarly, since $\dim(Q) = \dim_I(Q) + 1$, without loss of generality, we may assume that there is at most one element G of $\{G_1, G_2, \cdots, G_t\}$, say $G = G_t$, such that $G_t \cap (Z \times B) = \emptyset$. Then there are Ferrers relations $H, H' \in \mathcal{H}$ such that $F_s \cup G_{j_0} \subset H$ and $F_{i_0} \cup G_t \subset H'$ for some i_0 and j_0 with $1 \le i_0 \le s-1$ and $1 \le j_0 \le t-1$. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_{s-2}\}$ and let $\mathcal{G} = \{G_1, G_2, \cdots, G_{t-2}\}$ with $F_{i_0} = F_{s-1}, G_{j_0} = G_{t-1}$. Note that for each $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$, there are $(x_i, y_u) \in X \times Y - I_P$ and $(a_v, b_j) \in A \times B - I_Q$ such that $\{(x_i, z_{uj}) \mid 1 \le j \le m\} \subseteq F_i$, $\{(z_{iv}, b_j) \mid 1 \le i \le l\} \subseteq G_j$ and $R(F_i) \cap C(G_j) \ne \emptyset$. Then as in case 1, for each element H of $\mathcal{H}$, there do not exist $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$ such that $F_i \cup G_j \subseteq H$. By the same method in Case 1, for all $E \subseteq (X \cup Z \cup B) \times (X \cup Z \cup B) - I$, $\mathrm{fdim}(\bigcup_{D \in \mathcal{F} \cup \mathcal{G}} D \cup E) \ge (s-2)+(t-2)-\lfloor\frac{s+t-4}{3}\rfloor$. Thus we have $\dim(P*Q) \ge (s-2)+(t-2)-\lfloor\frac{(s-2)+(t-2)}{3}\rfloor+2 = s+t-2-\lfloor\frac{(s-2)+(t-2)}{3}\rfloor$.

By Cases 1, 2, 3 and 4, we get $\dim(P * Q) \ge s+t-2-\lfloor\frac{s+t-4}{3}\rfloor$. □

Consider the two n- and k-dimensional standard ordered sets S_n and S_k for $n, k \ge 3$. Let $\min(S_n * S_k) = \{x_1, x_2, \cdots, x_n\}$, $\max(S_n * S_k) = \{b_1, b_2, \cdots, b_k\}$ and $\mathrm{mid}(S_n * S_k) = \{z_{ij} \mid 1 \le i \le n \text{ and } 1 \le j \le k\}$. Then we have the following properties:

(i) $\min(S_n * S_k) < \max(S_n * S_k)$.

(ii) $x_i < z_{uv}$ and $z_{uv} < b_j$ in $S_n * S_k$ for all $1 \leq i, u \leq n$ and $1 \leq j, v \leq k$ with $i \neq u$ and $j \neq v$.
(iii) $x_i \parallel z_{ij}$ and $z_{ij} \parallel b_j$ in $S_n * S_k$ for all $1 \leq i \leq n$ and $1 \leq j \leq k$.

Now for all $n, k \geq 2$, let $P_{n,k} = S_n * S_k$(See $S_3 * S_3$ in Figure 1). We will see that $H_1 = H_{11}, H_2 = H_{12}, H_3 = H'_{11}, H_4 = H'_{12}$ and $H_5 = H_m(m = 2s + t + 2), H_6 = H_m(m = n - 1)$ in Example 2 and Case 2 of the proof of Theorem 3.5. In general, we have the following theorem:

THEOREM 3.5. *Let n and k be the positive integers with $n \geq k \geq 2$, we have*

$$\dim(P_{n,k}) = \begin{cases} n & \text{if } n \geq 2k \\ n + k - \lfloor \frac{n+k}{3} \rfloor & \text{if } n < 2k. \end{cases}$$

Proof. Let $X = \min(P_{n,k}) = \{x_1, x_2, \cdots, x_n\}, B = \max(P_{n,k}) = \{b_1, b_2, \cdots, b_k\}$ and $Z = \mathrm{mid}(P_{n,k}) = \{z_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq k\}$. Then $X < B$ and $x_i \parallel z_{ij}$ and $z_{ij} \parallel b_j$ in $P_{n,k}$ for $1 \leq i \leq n$ and $1 \leq j \leq k$. Consider the Ferrers relations $F_i = \{(x_i, z) \mid z \in Z \text{ with } x_i \not\leq z\}$ and $G_j = \{(z, b_j) \mid z \in Z \text{ with } z \not\leq b_j\}$ for all $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, k$. Then $\mathcal{F} = \{F_1, F_2, \cdots, F_n\}$ and $\mathcal{G} = \{G_1, G_2, \cdots, G_k\}$ are realizers of $X \times Z$ and $Z \times B$, respectively. Then, for all $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, k$, we have

$$C(F_i) = \{x_i\} \quad \text{and} \quad R(F_i) = \{z_{i1}, z_{i2}, \cdots, z_{ik}\}$$
$$C(G_j) = \{z_{1j}, z_{2j}, \cdots, z_{nj}\} \quad \text{and} \quad R(G_j) = \{b_j\}.$$

CASE 1. $n \geq 2k$.

Let $\mathcal{F} = \{F_1, F_n, F_i, F_{k+i-1}, F_{2k}, \cdots, F_{n-1} \mid 2 \leq i \leq k\}$. We will construct distinct n-Ferrers relations in $P_{n,k} \times P_{n,k} - I$ as follows:

(i) $H_1 = F_1 \cup [G_1 - \{(z_{11}, b_1)\}] \cup \{(b_l, b_1) \mid 2 \leq l \leq k\} \cup$
$[(Z \cup B) \times R(F_1)] \cup [(X \cup Z \cup B) \times (X - \{x_1\})] -$
$\{(z_{1u}, z_{1v}) \mid 1 \leq u \leq v \leq k\} - \{(x_i, x_j) \mid 1 \leq j \leq i \leq n\}$,

$H_n = F_n \cup \{(z_{11}, b_1)\} \cup [(X \cup Z \cup B) \times (X - \{x_n\})] \cup$
$\{(z_{ij}, z_{uv}) \mid i < u \text{ or } i = u, 1 \leq j < v \leq k \text{ with } 1 \leq i, u \leq n\} -$
$\{(x_i, x_j) \mid 1 \leq i \leq j \leq n\}$.

(ii) For each i with $2 \leq i \leq k$, we construct the two Ferrers relations H_i and H_{k+i-1} in $P_{n,k} \times P_{n,k} - I$. Then there are $F_i, F_{k+i-1} \in \mathcal{F}$ and $G_i \in \mathcal{G}$ such that

$$\begin{aligned} H_i = & F_i \cup [G_i - \{(z_{ii}, b_i)\}] \cup \{(b_j, b_i) \mid 1 \leq j \leq k \text{ with } i \neq j\} \cup \\ & [(Z \cup B) \times R(F_i)] \cup \{(z_{ni}, z_{nv}) \mid 1 \leq v \leq i-1\} - \\ & \{(z_{ij}, z_{iv}) \mid 1 \leq j \leq v \leq k\}, \\ H_{k+i-1} = & F_{k+i-1} \cup \{(z_{ii}, b_i)\} \cup [(Z \cup B) \times R(F_{k+i-1})] - \\ & \{(z_{k+i-1,j}, z_{k+i-1,v}) \mid 1 \leq j \leq v \leq k\}. \end{aligned}$$

(iii) For all m with $2k \leq m \leq n-1$, we construct the Ferrers relation H_m in $P_{n,k} \times P_{n,k} - I$. Then there are $F_m \in \mathcal{F}$ such that

$$\begin{aligned} H_m = & F_m \cup [(Z \cup B) \times R(F_m)] \cup \{(z_{nm}, z_{nl}) \mid 1 \leq l \leq m-1\} - \\ & \{(z_{mj}, z_{ml}) \mid 1 \leq j \leq l \leq k\}. \end{aligned}$$

Note that $R(F_i) \cap R(F_{k+i-1}) = \emptyset$ and $R(F_i) \cap C(G_j) = \{z_{ij}\}$ for all $F_i, F_{k+i-1} \in \mathcal{F}$ and $G_j \in \mathcal{G}$. Then $H_1, H_n, H_i, H_{k+i-1} (2 \leq i \leq k)$ and $H_m (2k \leq m \leq n-1)$ are Ferrers relations in $P_{n,k} \times P_{n,k}$. Furthermore, $(H_1 \cup H_n) \cup \bigcup_{i=2}^{k} (H_i \cup H_{k+i-1}) \cup \bigcup_{m=2k}^{n-1} H_m = (X \cup Z \cup B) \times (X \cup Z \cup B) - I$ and hence $\dim(P_{n,k}) \leq 2k + (n - 2k) = n$. Since $S_n \subset P_{n,k} - B$, it follows that $\dim(P_{n,k}) \geq \dim(P_{n,k} - B) \geq n$. Hence we conclude that $\dim(P_{n,k}) = n$.

Case 2. $k \leq n < 2k$.

Let s and t be positive integers with

$$s = 2\lfloor \frac{n+k}{3} \rfloor - k \text{ and } t = k - \lfloor \frac{n+k}{3} \rfloor.$$

We will construct the distinct $n + k - \lfloor \frac{n+k}{3} \rfloor$-Ferrers relations in $P_{n,k} \times P_{n,k} - I$ as follows:

$$\begin{aligned} \text{(i) } H_{11} = & F_1 \cup [G_k - \{(z_{1k}, b_k)\}] \cup [(X \cup Z \cup B) \times (X - \{x_1\})] \cup \\ & [(Z \cup B) \times R(F_1)] - \{(x_i, x_j) \mid 2 \leq i \leq j \leq n\} - \\ & \{(z_{1u}, z_{1v}) \mid 1 \leq u \leq v \leq k\}, \\ H_{12} = & F_2 \cup \{(z_{1k}, b_k)\} \cup [(X \cup Z \cup B - \{x_1\}) \times (X - \{x_2\})] \cup B \times B \cup \\ & [(Z \cup B - R(F_1)) \times (Z - R(F_1))] \cup [\{z_{1k}\} \times (Z - R(F_1))] - \\ & \{(x_i, x_j) \mid 3 \leq j \leq i \leq n\} - \{(b_i, b_j) \mid 1 \leq j \leq i \leq k\} - \\ & \{(z_{ij}, z_{uv}) \mid i < u \text{ or } i = u, 2 \leq j \leq v \leq k \text{ with } 1 \leq i, u \leq n\}, \end{aligned}$$

$$\begin{aligned} H'_{11} = {} & G_1 \cup [F_n - \{(x_n, z_{n1})\}] \cup [(Z \cup B) \times (R(F_n) - \{(x_n, z_{n1})\})] \cup \\ & [\bigcup_{i=1}^{n-2} (R(F_1) \cup \cdots \cup R(F_i)) \times \{z_{i+1,2}\}] \cup B \times B \cup \\ & [(C(G_1) \cup B) \times C(G_2)] - \\ & \{(z_{nu}, z_{nv}) \mid 1 \leq v \leq u \leq k\} - \{(b_i, b_j) \mid 1 \leq i \leq j \leq k\}, \\ H'_{12} = {} & G_2 \cup \{(x_n, z_{n1})\} \cup [C(G_2) \times Z] \cup [(Z - R(F_n)) \times \{z_{n1}\}] \cup \\ & [(Z - R(F_n)) \times (Z - R(F_n)) - \\ & \{(z_{ij}, z_{uv}) \mid u \leq i \text{ or } i = u, 1 \leq v \leq j \leq k \text{ with } 1 \leq i, u \leq n\}] - \\ & [Z \times (C(G_2) \cup \{z_{n3}, z_{n4}, \cdots, z_{nk}\})]. \end{aligned}$$

(ii) For $i = 2, 3, \cdots, s+1$, let

$$\begin{aligned} H_{i1} = {} & F_{i+1} \cup [G_{2t+i-1} - \{(z_{i+1,2t+i-1}, b_{2t+i-1})\}] \cup \\ & [(C(G_{2t+i-1}) - \{z_{i+1,2t+i-1}\}) \times R(F_{i+1})], \\ H_{i2} = {} & F_{s+i+1} \cup \{(z_{i+1,2t+i-1}, b_{2t+i-1})\} \cup \{z_{i+1,2t+i-1}\} \times R(F_{s+i+1}). \end{aligned}$$

(iii) For $j = 2, 3, \cdots, t$, let

$$\begin{aligned} H'_{j1} = {} & G_{j+1} \cup [F_{2s+j+1} - \{(x_{2s+j+1}, z_{2s+j+1,j+1}\}] \cup \\ & [C(G_{j+1}) \times (R(F_{2s+j+1}) - \{z_{2s+j+1,j+1}\})], \\ H'_{j2} = {} & G_{t+j} \cup \{(x_{2s+j+1}, z_{2s+j+1,j+1})\} \cup [C(G_{t+j}) \times \{z_{2s+j+1,j+1}\}]. \end{aligned}$$

(iv) For all $m = 2s+t+2, 2s+t+3, \cdots, n-1$, let $H_m = F_m$.

Note that

$(*) \cdots$ $R(F_i) \cap R(F_j) = \emptyset, C(G_u) \cap C(G_v) = \emptyset$ and $R(F_i) \cap C(G_u) = \{z_{iu}\}$ for all $i, j = 1, 2, \cdots, s$ with $i \neq j$ and $u, v = 1, 2, \cdots, t$ with $u \neq v$.

Then $H_{i1}, H_{i2} (i = 1, 2, \cdots, s+1), H'_{j1}, H'_{j2} (j = 1, 2, \cdots, t)$ and $H_m (m = 2s+t+2, 2s+t+3, \cdots, n-1)$ are Ferrers relations in $P_{n,k} \times P_{n,k}$ and

$$\bigcup_{i=1}^{s+1} (H_{i1} \cup H_{i2}) \cup \bigcup_{j=1}^{t} (H'_{j1} \cup H'_{j2}) \cup \bigcup_{r=2s+t+2}^{n-1} H_r = P_{n,k} \times P_{n,k} - I.$$

Note that $2 \times (s+1) + 2 \times t + (n - 1 - 2s - t - 1) = n + k - \lfloor \frac{n+k}{3} \rfloor$. Hence we have $\dim(P_{n,k}) \leq n + k - \lfloor \frac{n+k}{3} \rfloor$.

From $(*)$, it follows by Theorem 2.2 that $\mathrm{fdim}((X \cup Z) \times (X \cup Z)) = \dim_I(X, Z, I)$ and $\mathrm{fdim}((Z \cup B) \times (Z \cup B)) = \dim_I(Z, B, I)$. By Theorem

3.2, we have $\dim(P_{n,k}) \geq n + k - \lfloor \frac{n+k}{3} \rfloor$. We conclude that

$$\dim(P_{n,k}) = n + k - \lfloor \frac{n+k}{3} \rfloor. \qquad \square$$

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