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GEOMETRY ON EXOTIC HYPERBOLIC SPACES

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ABSTRACT. In this paper we briefly describe the geometry of the Cayley hyperbolic plane and we show that every uniform lattice in quaternionic space cannot be deformed in the Cayley hyperbolic 2-plane. We also describe the nongeometric bending deformation by developing the theory of the Cartan angular invariant for quaternionic hyperbolic space.

1. Introduction

By the Mostow [7] rigidity it is known that two isomorphic uniform lattices in rank one symmetric spaces of noncompact type of dimension greater than two are actually conjugate. But it does not say that a uniform lattice cannot be isomorphic to a discrete torsion free group in another symmetric space. The most well-known example is the Mickey Mouse example by Thurston [9], namely the bending of a Fuchsian group into a quasi-fuchsian group in a higher dimensional real hyperbolic space. It is also known by [10] that there is no quasi-fuchsian deformation in $H^m_{\mathbb{C}}$ of a $n(\geq 2)$-dimensional uniform lattice in $PU(n,1)$ when $n \leq m \leq 2n-1$. By [2] it is proved that a uniform lattice in the Cayley hyperbolic two plane cannot be isomorphic to the fundamental group of a compact Kähler manifold.

In this note we develop the Cartan angular invariant for quaternionic hyperbolic space to describe the non-geometric bending deformation in the Cayley hyperbolic two plane. We also show that a uniform lattice in quaternionic space cannot be deformed in the Cayley hyperbolic plane.

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2. Geometry of Cayley hyperbolic plane

2.1. Generalized Heisenberg group

General references of this section are [7],[4] and [6]. The Cayley hyperbolic plane $H^2_{\mathbb{O}}$ is one of four rank one symmetric spaces of noncompact type, namely real $\mathbb{R}$, complex $\mathbb{C}$, quaternionic $\mathbb{H}$ and the Cayley $\mathbb{O}$ hyperbolic spaces. Its full isometry group can be identified with $\mathbf{F}_4^{-20}$ and its isotropy group is $\mathbf{Spin}(9)$. In the Iwasawa decomposition, $\mathrm{Iso}(H^2_{\mathbb{O}}) = \mathbf{F}_4^{-20} = \mathbf{Spin}(9)AN$ where A is isomorphic to the real number which acts as dilation along the geodesic and N is the two step nilpotent group which we will describe in detail. The boundary of $H^2_{\mathbb{O}}$ can be identified with the one point compactification of N, $\partial H^2_{\mathbb{O}} = N \cup \infty$. In complex hyperbolic case, N is known as the Heisenberg group. N can be naturally identified with

$$\mathrm{Im}\mathbb{O} + \mathbb{O}$$

with the multiplication law

$$[t, z][s, w] = [t + s + 2\mathrm{Im} z\bar{w}, z + w].$$

See [7] (page 141). The parabolic subgroup fixing ∞ is

$$\mathbf{Spin}(7)AN.$$

Here $\mathbf{Spin}(7)$ fixes the origin of the space (using the unit ball model) and ∞. The action, called the dilation of $r \in A$ in this generalized Heisenberg coordinate, can be described as;

$$[t, z] \to [r^2 t, rz].$$

Nilpotent group N acts on itself by left multiplication fixing ∞. We record some facts about the induced action of isometries on the ideal boundary.

LEMMA 1. *If ν is a unit imaginary Cayley number, then $[\nu w][z\bar{\nu}] = \nu(wz)\nu^{-1}$. The action of an isometry on the ideal boundary of the Cayley hyperbolic plane which fixes 0 and ∞, is of the form: $[t, z] \to [\alpha(t), \beta(z)]$ and the action $[t, z] \to [l^2\nu t\nu^{-1}, l\nu z]$, where ν is a unit imaginary Cayley number, is one of those.*

Proof. Note that $\mathrm{Aut}(\mathbb{O}) = \mathbf{G}_2$ and $\mathbf{G}_2$ acts transitively on unit imaginaries. The isotropy group of i is a copy of $SU(3)$ and it acts transitively on the unit imaginaries orthogonal to i. The stabilizer of i, j which fixes

k, acts transitively on the unit imaginaries orthogonal to i, j, k and it is a copy of $SU(2)$. See [4]. So using Aut($\mathbb{O}$) we can assume that $w = (a, 0), z = (b, 0), \nu = (c, r)$ where a, b, c are quaternions and r is a real. Then a direct calculation shows:

$$\nu(wz)\nu = (cab - r^2\bar{b}\bar{a}, rcab + r\bar{b}\bar{a}\bar{c})$$
$$(\nu w)(z\nu) = (cab - r^2\bar{b}\bar{a}, rbca + r\bar{a}\bar{c}\bar{b}).$$

But $bca+\bar{a}\bar{c}\bar{b} = 2Re(bca) = 2Re(cab) = cab+\bar{b}\bar{a}\bar{c}$, so $\nu(wz)\nu = (\nu w)(z\nu)$. Since ν is a unit imaginary, $\bar{\nu} = \nu^{-1} = -\nu$. Now the claim follows from these facts.

In the Iwasawa decomposition of Iso(X) $= KAN$, the hyperbolic isometries belong to $A \times K$ where A is a maximal abelian subgroup and K is a maximal compact group. We will show that the action $[t, z] \to [\nu t \nu^{-1}, \nu z]$ is an isometry. Then since every bilipschitz map on the ideal boundary comes from an isometry of the space [8], this will finish the proof. The images of two points $[t, z], [s, w]$ under the action are $[\nu t \nu^{-1}, \nu z]$ and $[\nu s \nu^{-1}, \nu w]$. Then

$$\begin{aligned} &d([\nu t\nu^{-1}, \nu z], [\nu s\nu^{-1}, \nu w])^4 \\ &= |[\nu(t-s)\nu^{-1} - 2Im < \nu(w), \nu(z) >, \nu z - \nu w]| \\ &= |\nu(t - s - 2Im < w, z >)\nu^{-1}|^2 + |z - w|^4 \\ &= |t - s - 2Im < w, z > |^2 + |z - w|^4 \\ &= d([t, z], [s, w])^4. \end{aligned}$$

□

2.2. Totally geodesic submanifolds and their stabilizers

The general reference for this section is [3]. They worked out details for real, complex and quaternionic hyperbolic spaces. Cayley number is a pair of quaternions and define the multiplication by

$$(q_1, q_2)(p_1, p_2) = (q_1p_1 - \bar{p}_2q_2, p_2q_1 + q_2\bar{p}_1).$$

Also we define $\overline{(q_1, q_2)} = (\bar{q}_1, -q_2)$. Then it satisfies the usual properties like: $x\bar{x} = |x|^2$, $|xy| = |x||y|$, $x^{-1} = \bar{x}/|x|^2$, $\overline{xy} = \bar{y}\bar{x}$. Even though Cayley numbers are neither commutative, nor associative, by Artin's lemma a subalgebra generated by two elements is associative. Mostow ([7], p. 144 and p. 136) worked out the distance formula of two points in the unit ball model:

$$\cosh(d(x, y)) = \frac{(|1 - \langle x, y\rangle|^2 + 2R\langle x, y\rangle)^{1/2}}{(1 - \langle x, x\rangle)^{1/2}(1 - \langle y, y\rangle)^{1/2}} \tag{1}$$

where $R\langle v,w\rangle = Re(v_1\bar{v_2})(w_2\bar{w_1}) - Re(\bar{v_2}w_2)(\bar{w_1}v_1)$ for the Cayley hyperbolic case and $R\langle v,w\rangle = 0$ for other cases.

LEMMA 2. *Totally geodesic proper subspaces H passing through the origin in the unit ball model up to isometry are of the form:*

1. $H^2_{\mathbb{F}}, \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.
2. $H^k_{\mathbf{R}}, 1 \le k \le 8$.

*Furthermore, Stab(H) $= K'AN \cdot K''$ where Iso(H) $= KAN$ in its Iwasawa decomposition and $K'' \subset$ **Spin**(9) fixes H pointwise.*

Proof. Let $H^2_{\mathbb{F}} = \{(v,w) \in H^2_{\mathbf{O}} | v,w \in \mathbb{F}\}$. Then by the distance formula (1), any geodesic in $H^2_{\mathbb{F}}$ is a geodesic in $H^2_{\mathbf{O}}$. Also $H^1_{\mathbf{O}} = \{(v,0) \in H^2_{\mathbf{O}}\}$ is a totally geodesic subspace which is called $\mathbb{O}$-line by Mostow. This $\mathbb{O}$-line has constant negative sectional curvature and is isomorphic to $H^8_{\mathbf{R}}$.

For the second part of the claim, first note that since K'' fixes H pointwise it is a subset of the stabilizer of the origin, which is **Spin**(9). To prove the claim, we simply observe the action of isometries on the ideal boundary which is $N \cup \infty$ as described in the previous section. Let $\text{Iso}(H^2_{\mathbf{O}}) = \mathbf{Spin}(9)AN$ be its Iwasawa decomposition. First note that the ideal boundary of $H^2_{\mathbb{F}}$ is

$$\infty \cup \{[\text{Im}\mathbb{F}, \mathbb{F}]\}.$$

The dilation action A preserves $\{[\text{Im}\mathbb{F}, \mathbb{F}]\} \cup \infty$ so it preserves $H^2_{\mathbb{F}}$. This shows that $A \subset \text{Stab}(H)$. Also obviously the left multiplication by an element in $\{[\text{Im}\mathbb{F}, \mathbb{F}]\}$ preserves $\{[\text{Im}\mathbb{F}, \mathbb{F}]\}$. So $N \subset \text{Stab}(H)$. K' fixes the origin and leaves invariant H, which is the subgroup of **Spin**(9).

When H is the $\mathbb{O}$-line, the ideal boundary is

$$\{[\text{Im}\mathbb{O}, 0]\}.$$

By the same reasoning, the dilation and the multiplication by the elements in $\{[\text{Im}\mathbb{O}, 0]\}$ preserve the set, so $\text{Stab}(H^1_{\mathbf{O}}) = \mathbf{Spin}(8)AN \cdot K'$ where **Spin**(8) is the stabilizer of $\mathbb{O}$-line fixing the origin, K' fixes $\mathbb{O}$-line pointwise and $\text{Iso}(H^1_{\mathbf{O}}) = \text{Iso}(H^8_{\mathbf{R}}) = O(8)AN$. Since $\text{Iso}(H^8_{\mathbf{R}})$ action is transitive, the other totally geodesic subspaces of $\mathbb{O}$-line have the desired form. □

3. Proof of the theorem

It is known that a connected semisimple Lie group with trivial center has a real algebraic structure ([11]). Throughout the section, the Zariski closure means the real algebraic Zariski closure in the Lie group.

THEOREM 1. *Let Γ be a discrete torsion free subgroup of the group $\mathbf{F}_4^{-20}$. If Γ is isomorphic to a uniform lattice in $Iso(H^n_{\mathbf{H}})(n \geq 2)$, then Γ leaves invariant $H^2_{\mathbf{H}}$ and the restricted action is cocompact.*

Proof. Let π be a uniform lattice in $\mathrm{Iso}(H^n_{\mathbf{H}})$ and

$$\rho : \pi \to \Gamma \subset \mathbf{F}_4^{-20}$$

be the holonomy representation. Let $\bar{\Gamma}$ be the real Zariski closure of Γ in $\mathbf{F}_4^{-20}$. If $\bar{\Gamma} = \mathbf{F}_4^{-20}$, then by Corlette's superrigidity, ρ extends to

$$\bar{\rho} : \mathrm{Iso}(H^n_{\mathbf{H}}) \to \mathbf{F}_4^{-20}$$

with onto image. Since $\mathrm{Iso}(H^n_{\mathbf{H}})$ is a simple group, ρ is an isomorphism. This is possible only when $\mathrm{Iso}(H^n_{\mathbf{H}}) = \mathbf{F}_4^{-20}$. Then it is a contradiction.

Let Λ_Γ be the limit set of Γ and H be the smallest totally geodesic subspace whose ideal boundary contains Λ_Γ. Since Γ leaves invariant Λ_Γ, it leaves invariant H also. So we can think $\Gamma \subset \mathrm{Stab}(H) \subset \mathbf{F}_4^{-20}$. Note here that if $H = H^2_{\mathbf{O}}$, then Γ is Zariski dense in $\mathbf{F}_4^{-20}$, so we get the same contradiction as above. So we may assume that $H \neq H^2_{\mathbf{O}}$.

(Case I) $H = H^2_{\mathbf{R}}, H^2_{\mathbf{C}}, H^2_{\mathbf{H}}$.

By the results in the previous section, $\mathrm{Stab}(H) = K'AN \cdot K''$ where $K', K'' \subset \mathbf{Spin}(9), \mathrm{Iso}(H^2_{\mathbf{F}}) = KAN$ in their Iwasawa decomposition. Let Pr be the projection onto the first factor of $\mathrm{Stab}(H) = K'AN \cdot K''$. Then $Pr(\bar{\Gamma}) = K'AN$ since H is the smallest totally geodesic subspace containing Λ_Γ, i.e., $Pr(\Gamma)$ is Zariski dense in $K'AN$. Since $K'AN$ is Zariski dense in $KAN = \mathrm{Iso}(H^2_{\mathbf{F}})$, again by Corlette's theorem, $Pr \circ \rho$ extends to an isomorphism ϕ

$$\phi : \mathrm{Iso}(H^n_{\mathbf{H}}) \to \mathrm{Iso}(H^2_{\mathbf{F}})$$

since KAN is semisimple with trivial center. Then this is possible only when $H = H^2_{\mathbf{H}} = H^n_{\mathbf{H}}$. This shows that $\Gamma \subset \mathrm{Stab}(H^2_{\mathbf{H}})$.

(Case II) $H \subset H^1_{\mathbf{O}} = H^8_{\mathbf{R}}$.

Since $H \subset H^1_{\mathbf{O}}$, $\mathrm{Stab}(H) \subset \mathrm{Stab}(H^1_{\mathbf{O}}) = \mathbf{Spin}(8)AN \cdot K'$ where $\mathrm{Iso}(H^1_{\mathbf{O}}) = KAN, K' \subset \mathbf{Spin}(8)$. Since $\mathrm{Iso}(H^8_{\mathbf{R}}) = O(8)AN$, the restriction of $\mathrm{Stab}(H^1_{\mathbf{O}})$ on $H^1_{\mathbf{O}})$ is the whole isometry group of $\mathrm{Iso}(H^8_{\mathbf{R}})$.

This shows that $\mathrm{Stab}(H) = \mathrm{Iso}(H) \cdot K$ where $H \subset H^8_{\mathbb{R}}$ and K is the subgroup of **Spin**(8) which fixes H pointwise. By the same reasoning as in the case I, we get an isomorphism

$$\phi : \mathrm{Iso}(H^n_{\mathbb{H}}) \to \mathrm{Iso}(H) = \mathrm{Iso}(H^k_{\mathbb{R}})(k \leq 8)$$

which is impossible by the assumption that $n \geq 2$.

So we proved

$$\Gamma \subset \mathrm{Stab}(H^2_{\mathbb{H}}).$$

Since the map ϕ constructed in the proof of Case I is an isomorphism, the projection of Γ onto $\mathrm{Iso}(H^2_{\mathbb{H}})$ is also an isomorphism. This means that Γ acts on H by cocompact lattice by the Mostow rigidity. This proves that the restricted action of Γ on H is cocompact. □

4. Bending deformation

Even though a uniform lattice in the quaternionic space cannot be deformed in the Cayley plane, it is easy to see that a uniform lattice in the real hyperbolic space can be deformed in Cayley hyperbolic plane. In this section we want to describe the bending deformation of a uniform lattice of $H^2_{\mathbb{R}}$ in the Cayley hyperbolic plane. Since $H^2_{\mathbb{R}} \subset H^2_{\mathbb{H}} \subset H^2_{\mathbb{O}}$, we will describe the bending in $H^2_{\mathbb{H}}$. If we set $H^2_{\mathbb{R}} = \{(v, w) \in H^2_{\mathbb{O}} | v, w \in \mathbb{R}\}$, this real hyperbolic 2-plane is orthogonal to $\mathbb{O}$-line $\{(0, v) | v \in \mathbb{O}\}$. They share a real geodesic passing through the origin and $(0, 1)$ in the unit ball model. The boundary of the real 2-plane called the $\mathbb{R}$-circle is also orthogonal to the boundary of $\mathbb{O}$-line with respect to the standard spherical metric (see [7], page 147). Let $H^2_{\mathbb{H}} = \{(v, w) | v, w \in \mathbb{H}\}$. In the generalized Heisenberg model

$$\begin{aligned} \partial H^2_{\mathbb{R}} &= \{[0, \mathbb{R}]\} \\ \partial H^1_{\mathbb{O}} &= \{[\mathrm{Im}\mathbb{O}, 0]\} \\ \partial H^2_{\mathbb{H}} &= \{[\mathrm{Im}\mathbb{H}, \mathbb{H}]\}. \end{aligned}$$

There is a subgroup in $\mathrm{Iso}(H^2_{\mathbb{H}})$ which is a rotation around the $H^1_{\mathbb{H}}$, which can be described in the generalized Heisenberg coordinate as:

$$[t, z] \to [t, \mu z]$$

where $\mu \in Sp(1)$. Now we will bend $\Gamma \in \mathrm{Iso}(H^2_{\mathbb{R}})$ in $\mathrm{Iso}(H^2_{\mathbb{H}})$. Let μ_t be one-parameter group near identity in $Sp(1)$. Let $\mathbb{R}^+_t = \mu_t(\mathbb{R}^+)$ where $\mathbb{R}^+$ is the positive half real line in $\partial H^2_{\mathbb{R}} = \{[0, \mathbb{R}]\}$. Geometrically the negative half of $H^2_{\mathbf{R}}$ bounded by the geodesic γ connecting the origin and $(1,0)$ remains fixed and the positive half of $H^2_{\mathbf{R}}$ bounded by γ changes into the space bounded by $\mathbb{R}^+_t$. Note that by conjugating Γ if necessary to make some element in Γ have the invariant geodesic γ, the geodesic γ projects down to a closed geodesic in $H^2_{\mathbf{R}}/\Gamma$. Denote that element by γ also. If $\Gamma = \Gamma_1 *_{\langle\gamma\rangle} \Gamma_2$ ($\Gamma_1 *_{\langle\gamma\rangle} = \langle\Gamma_1, \alpha\rangle$, HNN extension if γ is a non-separating geodesic in the Riemann surface $H^2_{\mathbf{R}}/\Gamma$), then

$$\Gamma_t = \Gamma_1 * \mu_t \Gamma_2 {\mu_t}^{-1}(\langle\Gamma_1, \mu_t\alpha\rangle).$$

See [1]. Note that there is no quasi-conformal map inducing Γ_t, if there is, there is an isometry between them by the Pansu's rigidity [8], then Γ and Γ_t will be conjugate by this isometry. This is a difference from the real or complex hyperbolic cases.

COROLLARY 1. *Let H be the boundary of a complex hyperbolic two space. Then there is a quasi-conformal map from H into the ideal boundary of Cayley two plane, which cannot be extended quasi-conformally to the whole ideal boundary.*

Proof. By [1], if $\mu_t \in U(1)$, the bending is induced from a quasi-conformal homeomorphism between the boundary of complex hyperbolic two plane. But by Pansu's rigidity this map cannot be extended to the whole ideal boundary of the Cayley two plane. □

Note there is $(6g-6)$-parameter family of deformation coming from the Teichmüller space of the surface $H^2_{\mathbf{R}}/\Gamma$ of genus g, and some more come from μ_t. None of these are quasi-conformal each other by the Pansu's rigidity. To see that Γ_t is not a trivial deformation, i.e., not conjugate to the original Γ, we introduce the generalized Cartan angular invariant as in complex hyperbolic space.

4.1. Cartan angular invariant in quaternionic space

The general reference for this section is [5]. Let $x = (x_1, x_2, x_3) \in (H^n_{\mathbf{H}} \cup \partial H^n_{\mathbf{H}}) \times (H^n_{\mathbf{H}} \cup \partial H^n_{\mathbf{H}}) \times (H^n_{\mathbf{H}} \cup \partial H^n_{\mathbf{H}})$ be a triple of distinct three points and choose lifts $\tilde{x}_i \in H^{n,1}_{\mathbf{H}}$.

DEFINITION 1. The **Cartan angular invariant** of x, $\mathbb{A}(x)$, is the angle between $e_1 = (1,0,0,0)$ and $\langle\tilde{x_1}, \tilde{x_2}, \tilde{x_3}\rangle = \langle\tilde{x_1}, \tilde{x_2}\rangle\langle\tilde{x_2}, \tilde{x_3}\rangle\langle\tilde{x_3}, \tilde{x_1}\rangle$. Here we identify $\mathbb{H} = \mathbb{R}^4$.

If we choose another triple of lifts $\nu_i\tilde{x_i}$, then

$$\langle \nu_1\tilde{x_1}, \nu_2\tilde{x_2}, \nu_3\tilde{x_3}\rangle = \nu_1\langle \tilde{x_1}, \tilde{x_2}\rangle\langle \tilde{x_2}, \tilde{x_3}\rangle\langle \tilde{x_3}, \tilde{x_1}\rangle\bar{\nu_1}.$$

Since the action of $Sp(1)$ on $\mathbb{R}^4$ by conjugation is orthogonal and leaves invariant the real axis, the angle is independent of the choice of the lifts. Here we list several properties of the Cartan angular invariant for quaternionic hyperbolic space.

PROPOSITION 1. *The Cartan angular invariant is invariant under the permutation of* x_i.

Proof. $\langle x_2, x_1, x_3\rangle = \langle x_2, x_1\rangle\langle x_1, x_3\rangle\langle x_3, x_2\rangle = \overline{\langle x_1, x_2\rangle}\ \overline{\langle x_3, x_1\rangle}\ \overline{\langle x_2, x_3\rangle}$ $= \overline{\langle x_2, x_3\rangle\langle x_3, x_1\rangle\langle x_1, x_2\rangle} = \overline{\langle x_2, x_3, x_1\rangle}$. But the angle between e_1 and $\overline{\langle x_2, x_3, x_1\rangle}$ is the same with the one between e_1 and $\langle x_2, x_3, x_1\rangle$. But also the angle is unchanged under the conjugation by $\langle x_1, x_2\rangle$, so the angle between e_1 and $\langle x_2, x_3, x_1\rangle$ is equal to the one between e_1 and $\langle x_1, x_2\rangle\langle x_2, x_3, x_1\rangle\langle x_1, x_2\rangle^{-1} = \langle x_1, x_2, x_3\rangle$. So $\mathbb{A}(x_1, x_2, x_3) = \mathbb{A}(x_2, x_3, x_1)$ $= \mathbb{A}(x_2, x_1, x_3)$. □

PROPOSITION 2. *Let* $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ *be pairs of distinct triples such that*

$$\mathbb{A}(x) = \mathbb{A}(y).$$

Then there is an isometry f *in* $Iso(H^n_{\mathbb{H}})$ *such that* $f(x_i) = y_i$.

Proof. Let $X = \langle \tilde{x_1}, \tilde{x_2}, \tilde{x_3}\rangle, Y = \langle \tilde{y_1}, \tilde{y_2}, \tilde{y_3}\rangle$ and $|X| = |Y|$. If $\mathbb{A}(x) = \mathbb{A}(y)$, it is easy to see that there is an orthogonal matrix $M \in SO(3) \times$ identity which leaves invariant x-axis and maps X to Y. Since the conjugation action of $Sp(1)$ in $\mathbb{R}^3$ is $SO(3)$ action, there is $\mu \in Sp(1)$ such that

$$\langle \tilde{x_1}, \tilde{x_2}, \tilde{x_3}\rangle = \mu\langle \tilde{y_1}, \tilde{y_2}, \tilde{y_3}\rangle\bar{\mu}.$$

But if we can choose lifts $\tilde{x_i}, \tilde{y_i}$ so that $\langle \tilde{x_i}, \tilde{x_j}\rangle = \langle \tilde{y_i}, \tilde{y_j}\rangle$ then there is $A \in Sp(n, 1)$ such that $A(\tilde{x_i}) = \tilde{y_i}$. Then it descends to an element in $PSp(n, 1)$ sending x_i to y_i. First replacing $\tilde{y_1}$ by $\mu\tilde{y_1}$ (and still denote it by $\tilde{y_1}$), we get $\langle \tilde{x_1}, \tilde{x_2}\rangle\langle \tilde{x_2}, \tilde{x_3}\rangle\langle \tilde{x_3}, \tilde{x_1}\rangle = \langle \tilde{y_1}, \tilde{y_2}\rangle\langle \tilde{y_2}, \tilde{y_3}\rangle\langle \tilde{y_3}, \tilde{y_1}\rangle$. Replacing $\tilde{x_2}, \tilde{x_3}$ by $\mu_2\tilde{x_2}, \mu_3\tilde{x_3}$ if necessary, we can make $\langle \tilde{x_2}, \tilde{x_3}\rangle = \langle \tilde{y_2}, \tilde{y_3}\rangle, \langle \tilde{x_3}, \tilde{x_1}\rangle = \langle \tilde{y_3}, \tilde{y_1}\rangle$. Now the equation becomes

$$\langle \tilde{x_1}, \tilde{x_2}\rangle\langle \tilde{x_2}, \tilde{x_3}\rangle\langle \tilde{x_3}, \tilde{x_1}\rangle = |\mu_2|^2|\mu_3|^2\langle \tilde{y_1}, \tilde{y_2}\rangle\langle \tilde{y_2}, \tilde{y_3}\rangle\langle \tilde{y_3}, \tilde{y_1}\rangle$$

and we get $\langle \tilde{x_1}, \tilde{x_2}\rangle = r\langle \tilde{y_1}, \tilde{y_2}\rangle$ where $r = |\mu_2||\mu_3|$. Replacing $\tilde{x_1}, \tilde{x_2}, \tilde{x_3}, \tilde{y_1}$ by $r^{-1}\tilde{x_1}, r^{-1}\tilde{x_2}, r\tilde{x_3}, r^2\tilde{y_1}$ we get $\langle \tilde{x_i}, \tilde{x_j}\rangle = \langle \tilde{y_i}, \tilde{y_j}\rangle$. □

THEOREM 2. *Let $x_1, x_2, x_3 \in \partial H^n_{\mathbb{H}}$. Let σ_{12}, Σ_{12} be real and quaternionic geodesic containing x_1, x_2 and $\pi : H^n_{\mathbb{H}} \to \Sigma_{12}$ be the orthogonal projection. Then*

$$|\tan \mathbb{A}(x)| = \sinh(\rho(\pi x_3, \sigma_{12}))$$

where ρ is a distance in $H^n_{\mathbb{H}}$.

Proof. Applying an isometry to x, we can make them $x_1 = (0, -1), x_2 = (0, 1), x_3 = (z', z_n)$ and so we can lift them to

$$\tilde{x_1} = (0, -1, 1), \tilde{x_2} = (0, 1, 1), \tilde{x_3} = (z', z_n, 1).$$

In this setting $\sigma_{12} = \{(0, \mathbb{R})\}, \Sigma_{12} = \{(0, \mathbb{H})\}$. Then $\langle \tilde{x_1}, \tilde{x_2}, \tilde{x_3} \rangle = 2(\bar{z}_n - 1)(1 + z_n)$. So we get

$$|\tan \mathbb{A}(x)| = \frac{|2\mathrm{Im}(z_n)|}{1 - |z_n|^2}.$$

Note $\pi(x_3) = z_n$ and Σ_{12} is a real hyperbolic 4 space in the Poincaré ball model with curvature -1. Choose a real hyperbolic two plane containing σ_{12} and z_n. This plane is the Poincaré disk with curvature -1. Write $z_n = Re z_n + i|\mathrm{Im} z_n|$ in the Poincaré unit disk. Then it is a direct calculation to see that $\sinh(\rho(z_n, \text{real axis})) = \frac{|2\mathrm{Im}(z_n)|}{1-|z_n|^2}$. □

THEOREM 3. *Three points in the ideal boundary of a quaternionic n-space are in the boundary of real hyperbolic two plane iff $\mathbb{A}(x) = 0$.*

Proof. ($\Rightarrow$)Applying an isometry, we can make them $x_1 = (0, -1), x_2 = (0, 1), x_3 = (0, r)$ where r is real. Then $\langle \tilde{x_1}, \tilde{x_2}, \tilde{x_3} \rangle$ is real. So the angle is zero.

($\Leftarrow$) As in the proof of the previous theorem, let $x_1 = (0, -1), x_2 = (0, 1), x_3 = (z', z_n)$ and lift them to $\tilde{x_1} = (0, -1, 1), \tilde{x_2} = (0, 1, 1), \tilde{x_3} = (z', z_n, 1)$. To get $\mathbb{A}(x) = 0$, z_n must be real. Now we use the generalized projection from $N \cup \infty$ to the boundary of the unit ball model where N is a nilpotent group in the Iwasawa decomposition of the isometry group, to see that x_3 lies on the boundary of a real hyperbolic two plane. The generalized projection is (see [6])

$$[t, z] \to \left[2\frac{1 + |z|^2 + t}{|1 + |z|^2 + t|^2} z, \frac{1 + |z|^2 + t}{|1 + |z|^2 + t|^2}(1 - |z|^2 + t)\right].$$

Note that [0,0], ∞ correspond to (0,1),(0,−1) respectively. A boundary of the real hyperbolic two plane containing (0,1),(0,−1) in the unit ball model correspond to the one in $N \cup \infty$ passing through the origin and some point $[0, z]$. The reason is as follows. $\{[0, \mathbb{R}]\}$ is obviously the boundary of the real hyperbolic two plane. Now the group fixing zero and ∞ is $Sp(n-1) \times \mathbb{R} \times Sp(1)$. Then for any $[0, z], |z| = 1, z \in \mathbb{H}^{n-1}$, there is $M \in Sp(n-1)$ which maps $(1, 0, ..., 0)$ to z. So M sends $[0, 1]$ to $[0, z]$, which implies that it maps $\{[0, \mathbb{R}]\}$ to the line through the origin and $[0, z]$.

So if z_n is real, then its corresponding point in $N \cup \infty$ has $t = 0$. This shows that x_3 is on the boundary of real hyperbolic two plane containing $(0, 1), (0, -1)$. □

Now we can apply this theorem to show that the groups Γ_t is not conjugate to Γ in the previous section. For each Γ_t, we can choose three points in the limit set such that its product $\langle \tilde{x_1}, \tilde{x_2}, \tilde{x_3} \rangle$ is not real. In a similar fashion we can see that Γ_t, Γ_s are not conjugate for small t and s.

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