J. Korean Math. Soc. **36** (1999), No. 3, pp. 593–607

REPRESENTATIONS FOR LIE SUPERALGEBRA $spo(2m, 1)$

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ABSTRACT. Let $\mathcal{G}$ denote the orthosymplectic Lie superalgebra spo $(2m, 1)$. For each irreducible $\mathcal{G}$-module, we describe its character in terms of tableaux. Using this result, we decompose $\otimes^k V$, the k-fold tensor product of the natural representation V of $\mathcal{G}$, into its irreducible $\mathcal{G}$-submodules, and prove that the Brauer algebra $B_k(1-2m)$ is isomorphic to the centralizer algebra of $spo(2m, 1)$ on $\otimes^k V$ for $m \geq k$.

1. Introduction

One of the fundamental problems in representation theory is to decompose $\otimes^k U$, the k-fold tensor product of an irreducible module U, into its irreducible submodules. For the classical groups $GL(n, \mathbf{C})$, $Sp(n, \mathbf{C})$, and $O(n, \mathbf{C})$ (or the corresponding Lie algebras), the decomposition of $T = \otimes^k V$ where V is the natural representation is obtained by the work of Schur, Weyl, and Brauer. Their work also has shown that the centralizer algebra, $\mathcal{C} = End_G(T) = \{f \in End(T) \mid f(x \cdot t) = x \cdot f(t)$ for all $x \in G$, $t \in T\}$, is intimately related with the decomposition.

Berele and Regev [2] extended Schur's results for the general linear Lie superalgebra $gl(m, n)$. In particular, [2] introduced a graded-action of the symmetric group S_k on $\otimes^k V$ where V is the natural representation of $gl(m, n)$ which commutes with the graded-derivation action of $gl(m, n)$, proved that these graded-actions provide the centralizer algebras to each other, and obtained an explicit decomposition of $\otimes^k V$ into its irreducibles.

Received December 28, 1998. Revised March 5, 1999.
1991 Mathematics Subject Classification: 17B10, 05E10.
Key words and phrases: orthosymplectic Lie superalgebra, tensor product representation, centralizer algebra.

For the orthosymplectic Lie superalgebra $spo(2m,N)$, [1] introduced a graded-action of the Brauer algebra $B_k(N-2m)$ on $\otimes^k V$ where V is the natural representation of $spo(2m,N)$ which commutes with the action of $spo(2m,N)$. Using these commuting actions, [1] decomposed $\otimes^k V$ into certain submodules and described their characters. However, in general, these submodules and their characters are not irreducible, and the centralizer algebras are not determined.

In this paper, we consider $spo(2m,1)$ and its tensor representation $\otimes^k V$ where V is the natural representation of $spo(2m,1)$. We prove that the decomposition in [1] is indeed a decomposition into irreducibles, the characters are equivalent to the characters given by Kac [4] (hence they are irreducible), and determine the centralizer algebra as the isomorphic image of the graded-action of the Brauer algebra $B_k(1-2m)$ for $m \geq k$. These results provide a super-analogue of the Brauer's results for $spo(2m,1)$.

Rittenberg and Scheunert [7] showed that the non-spinor irreducible representations of the orthogonal Lie algebra $o(2m+1)$ and the irreducible representations of $spo(2m,1)$ of the same highest weights have the equivalent characters by computing the weight multiplicities. Thus the representations of $spo(2m,1)$ can be studied from those of $o(2m+1)$ in terms of their characters.

In this paper we adopt a more constructive approach. This provides more concrete information about the representations such as an explicit construction of each finite dimensional irreducible representation as a submodule of a tensor representation, an explicit decomposition of the tensor representation into its irreducibles, and an explicit description of the centralizer algebra in terms of the graded-action of the Brauer algebra $B_k(1-2m)$. This approach is a specialization of general techniques developed for the representations of $spo(2m,N)$ rather than a translation of the results known for the representations of $o(2m+1)$ in terms of the characters. This approach allows us to understand the representations of $spo(2m,1)$ in a general context of $spo(2m,N)$ and shows how some of the results and proofs presented here can be generalized for the representations of $spo(2m,N)$.

The rest of the paper is organized as follows: In section 2, we briefly discuss basic notions and notations for $\mathcal{G} = spo(2m,1)$. In section 3, we describe the actions of $\mathcal{G}$ and $B_k(1-2m)$ on $\otimes^k V$, and prove that the action of $B_k(1-2m)$ is faithful for $m \geq k$. In section 4, we construct irreducible submodules of $\otimes^k V$, and describe their characters in terms of tableaux. In section 5, we obtain the decomposition of $\otimes^k V$ into irreducible $\mathcal{G}$-modules and prove that the centralizer algebra is isomorphic to $B_k(1-2m)$ for $m \geq k$.

2. Preliminaries

The *general linear Lie superalgebra* $gl(r,s) = gl(r,s)_0 \oplus gl(r,s)_1$ over $\mathbf{C}$ consists of all $(r+s)\times(r+s)$ complex matrices under the commutator product $[x,y] = xy - (-1)^{ab}yx$ for $x \in gl(r,s)_a$, $y \in gl(r,s)_b$, and $a,b \in \mathbf{Z}_2$, where

$$gl(r,s)_0 = \left\{ \begin{pmatrix} Y_1 & 0 \\ 0 & Y_4 \end{pmatrix} \mid Y_1 \in M_{r,r}(\mathbf{C}), Y_4 \in M_{s,s}(\mathbf{C}) \right\},$$

$$gl(r,s)_1 = \left\{ \begin{pmatrix} 0 & Y_2 \\ Y_3 & 0 \end{pmatrix} \mid Y_2 \in M_{r,s}(\mathbf{C}), Y_3 \in M_{s,r}(\mathbf{C}) \right\},$$

and $M_{k,\ell}(\mathbf{C})$ denotes the $k \times \ell$ complex matrices.

Let $V = \mathbf{C}^{2m+1}$ and $\{v_i \mid i = 1, \ldots, 2m+1\}$ be the standard basis for V. Let $V = V_0 \oplus V_1$ where $V_0 = \text{span}\{v_1, \ldots, v_{2m}\}$ and $V_1 = \text{span}\{v_{2m+1}\}$. Let $b(\ ,\ )$ denote the bilinear form on V defined by $b(v,w) = v^t J w$ for $v, w \in V$ where "t" is the usual transpose, and

$$J = \begin{pmatrix} J' & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad J' = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

where I_m is the $m \times m$ identity matrix. The *orthosymplectic Lie superalgebra* $spo(2m,1)$ is defined to be $spo(2m,1) = spo(2m,1)_0 \oplus spo(2m,1)_1$ where

$$\begin{aligned} &spo(2m,1)_a \\ &= \{x \in gl(2m,1)_a \mid b(xv,w) + (-1)^{ab} b(v,xw) = 0 \text{ for } v \in V_b, w \in V\}. \end{aligned}$$

We note that $spo(2m,1)_0 \cong sp(2m) \oplus o(1) \cong sp(2m)$.

For $i = 1, \ldots, m$, we define v_i^* to be the vector such that $b(v_i, v_j^*) = \delta_{i,j}$. Then $v_i^* = v_{m+i}$, $v_{m+i}^* = -v_i$ for $i = 1, \ldots, m$ and $v_{2m+1}^* = v_{2m+1}$. Thus $B = B_0 \cup B_1$ where $B_0 = \{v_1, \ldots, v_m, v_1^*, \ldots, v_m^*\}$ and $B_1 = \{v_{2m+1}\}$. Set $\deg(b) = 0$ if $b \in B_0$ and $\deg(b) = 1$ if $b \in B_1$.

Let $\mathcal{H}$ be the Cartan subalgebra of $\mathcal{G} = spo(2m,1)$ consisting of diagonal matrices. Relative to the adjoint action of $\mathcal{H}$, $\mathcal{G}$ admits a root space decomposition $\mathcal{G} = \mathcal{H} \oplus \sum_{\alpha\in\mathcal{H}^*} \mathcal{G}_\alpha$. A root α is *even* if $\mathcal{G}_\alpha \cap \mathcal{G}_0 \neq 0$, and is *odd* if $\mathcal{G}_\alpha \cap \mathcal{G}_1 \neq 0$. We let Δ, Δ_0, and Δ_1 denote the set of roots, even roots, and odd roots, respectively. For $i = 1, \dots, 2m+1$, let ϵ_i denote the element in $\mathcal{H}^*$ which takes a matrix in $\mathcal{H}$ to its (i,i)-entry. Then the roots of $\mathcal{G}$ are:

$$\begin{aligned}\Delta_0 &= \{\pm(\epsilon_i \pm \epsilon_j), \pm 2\epsilon_i \mid 1 \le i,j \le m,\ i<j\},\\ \Delta_1 &= \{\pm\epsilon_i \mid 1 \le i \le m\}.\end{aligned}$$

The simple roots can be chosen as:

$$(1)\qquad \alpha_i = \epsilon_i - \epsilon_{i+1} \quad \text{for} \quad i = 1, \dots, m-1, \qquad \alpha_m = \epsilon_m.$$

All simple roots except α_m is even, and α_m is odd. For each i, we choose a root vector $e_i \in \mathcal{G}_{\alpha_i}$ as:

$$e_i = E_{i,i+1} - E_{m+i+1,m+i} \quad \text{for} \quad i = 1, \dots, m-1,$$

$$(2)\qquad e_m = \sqrt{2}(E_{m,2m+1} + E_{2m+1,2m})$$

A $\mathcal{G}$-module M is said to have a *weight space decomposition* if $M = \oplus \sum_{\mu\in\mathcal{H}^*} M_\mu$ where $M_\mu = \{v \in M \mid hv = \mu(h)v \text{ for all } h \in \mathcal{H}\}$. A vector v in M is a *maximal vector* if $\mathcal{G}_+ v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathcal{H}$. If M is irreducible, then M contains a unique (up to scalar) maximal vector v which generates M, and the weight of v is called the highest weight of M. An irreducible $\mathcal{G}$-module is uniquely determined (up to isomorphism) by its highest weight. Let $V(\lambda)$ denote the irreducible $\mathcal{G}$-module with the highest weight λ.

Let $\{\omega_i \mid i = 1, \dots, m\}$ denote the fundamental weights. By [4], $a_1\omega_1 + a_2\omega_2 + \cdots + a_m\omega_m$ is a highest weight for a finite dimensional irreducible $\mathcal{G}$-module if and only if $a_1, \dots, a_{m-1}$, and $(1/2)a_m$ are nonnegative integers. Equivalently, $\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_m\epsilon_m$ is a highest weight of a finite dimensional irreducible $\mathcal{G}$-module if and only if $\{\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0\}$ and λ_i are nonnegative integers for all i (see [6] Theorem 2.10).

3. Actions on $T = \otimes^k V$

Let $V = \mathbf{C}^{2m+1}$ and $T = \otimes^k V$, the k-fold tensor product of V. In this section, we describe commuting actions of $\mathcal{G}$ and the Brauer algebra

$B_k(1-2m)$ on T, and prove that the action of $B_k(1-2m)$ is faithful for $m \geq k$.

Let $\mathcal{G}$ act on V as the usual matrix multiplication. Then $V = V(\{1\})$, the irreducible $\mathcal{G}$-module with the highest weight ϵ_1. This V is the *natural representation* of $\mathcal{G}$. The $\mathcal{G}$-action on V can be extended to $T = \otimes^k V$ by the graded-derivation:

$$xw = \sum_{j=1}^{k} (-1)^{a \ \deg_j(w)} w_1 \otimes \cdots \otimes w_{j-1} \otimes xw_j \otimes w_{j+1} \otimes \cdots \otimes w_k,$$

where $x \in \mathcal{G}_a$, $w = w_1 \otimes \cdots \otimes w_k$ is a simple tensor in T, $\deg_1(w) = 0$, and $\deg_j(w) = \deg(w_1) + \cdots + \deg(w_{j-1})$ for $j > 1$. This makes T into a $\mathcal{G}$-module.

We now describe the Brauer algebra $B_k(1-2m)$ and its action on T. A *k-diagram* is a graph with two rows of k vertices each, one above the other, and k edges such that each vertex is incident to precisely one edge. Assume $\eta \in \mathbf{C}$. We define the product of two k-diagrams d_1 and d_2 to be the k-diagram obtained by placing d_1 above d_2 and identifying the vertices in the bottom row of d_1 with the corresponding vertices in the top row of d_2. The resulting graph contains k paths and some number of cycles. If d is the k-diagram with the edges that are paths in this graph but with the cycles removed, then the product d_1d_2 is given by $d_1d_2 = \eta^c d$ where c is the number of closed loops. The *Brauer algebra* $B_k(\eta)$ is the $\mathbf{C}$-span of the k-diagrams. The $\mathbf{C}$-linear extension of the diagram multiplication makes $B_k(\eta)$ into an associative algebra.

Let d be a k-diagram. Recall $B = B_0 \cup B_1$ where $B_0 = \{v_1, \ldots, v_m, v_1^*, \ldots, v_m^*\}$ and $B_1 = \{v_{2m+1}\}$. Label the top vertices (left to right) with a sequence $\underline{a} = (a_1, a_2, \ldots, a_k)$ of basis elements $a_i \in B$, and the bottom vertices (left to right) with a sequence $\underline{c} = (c_1, c_2, \ldots, c_k)$ of basis element $c_i \in B$. Assign a weight to each edge and each crossing of this labeled k-diagram as follows:

(1) For a horizontal edge connecting a and a' with a to the left of a', assign the weight $b(a,a')$ to it.

(2) For a horizontal edge connecting c and c' with c to the left of c', assign the weight $-b(c,c')$.

(3) For each vertical edge connecting a to c, assign the weight $\delta_{a,c}$ (Kronecker-delta),

(4) For each crossing, assign the weight $-(-1)^{deg(\ell_1)deg(\ell_2)}$ where ℓ_1 is the label of vertex adjacent to the first edge, and ℓ_2 is a vertex adjacent to the second edge in the crossing. Of the four vertices adjacent to the two edges that crosss, ℓ_1 and ℓ_2 are chosen to be the last two vertices (in order) when counting off the vertices in a counterclockwise fashion beginning from the bottom left corner of the diagram.

The *weight* of the k-diagram d labeled with $\underline{a}$ and $\underline{c}$, denoted by $d(\underline{a}, \underline{c})$, is the product of the weights over all edges and crossings.

Let d be a k-diagram and define ϕ_d to be the endomorphism in $End(\otimes^k V)$ such that

$$\phi_d(a_1 \otimes \cdots \otimes a_k) = \sum_{c_1,\dots,c_k \in B} d(\underline{a}, \underline{c}) c_1 \otimes \cdots \otimes c_k$$

where $d(\underline{a}, \underline{c})$ is the weight of the k-diagram d with top vertices labeled by $a_1, \dots, a_k$ and bottom vertices labeled by $c_1, \dots, c_k$. Then

$$\phi : B_k(1-2m) \longrightarrow End_{\mathcal{G}}(\otimes^k V) \tag{3}$$

where $\phi(d) = \phi_d$ is a homomorphism of algebras ([1] Theorem 2.16).

We note that $\mathbf{C}[S_k]$, the group algebra of the symmetric group S_k, is naturally imbedded in $B_k(1-2m)$, and the action of $B_k(1-2m)$ on T is an extension of the action of $\mathbf{C}[S_k]$ on T in [2].

PROPOSITION 3.1. *Assume that $m \geq k$. Then the map ϕ in (3) is injective.*

Proof. Let d be a k-diagram. Suppose that d has $2p$ horizontal edges (p edges on the top row and p-edges on the bottom row of d) where $0 \leq p \leq [k/2]$. Label the vertices of d as follows: For each of these $2p$ horizontal edges, assign v_i and v_i^* for $i = 1, \dots, 2p$ in any order. For each of the remaining $k-2p$ vertical edges, assign v_{2p+j} for the top vertex and bottom vertex for $j = 1, \dots, k-2p$. Since $m \geq k$, we can label the vertices this way without running out of the basis elements in B. Let $\underline{a} = (a_1, \dots, a_k)$ and $\underline{c} = (c_1, \dots, c_k)$ be labels for top and bottom row of d (from left to right), respectively. By the way the weights are defined, it is clear that $d(\underline{a}, \underline{c}) \neq 0$.

Now consider the two rows of k vertices labeled with $\underline{a}$ and $\underline{c}$. From the way $\underline{a}$ and $\underline{c}$ are chosen, for each of the $2k$ vertices, there is a unique way to connect the given vertex with an edge so that the edge weight is not 0, namely the edge present in d. Thus, it follows that $d'(\underline{a}, \underline{c}) = 0$ for all k-diagram d' such that $d' \neq d$.

Now let d be an arbitrary element of $B_k(1-2m)$. Then $d = q_1d_1 + \cdots + q_rd_r$ where $q_i \in \mathbf{C}$ and d_i are k-diagrams. Let $\underline{a}$ and $\underline{c}$ be the labels chosen so that $d_1(\underline{a},\underline{c}) \neq 0$ and $d_i(\underline{a},\underline{c}) = 0$ for all $d_i \neq d_1$. Then $d(\underline{a},\underline{c}) = q_1d_1(\underline{a},\underline{c}) + q_2d_2(\underline{a},\underline{c}) + \cdots + q_rd_r(\underline{a},\underline{c}) = q_1d_1(\underline{a},\underline{c}) \neq q0$. Then

$$\phi(d)(a_1 \otimes \cdots \otimes a_k) = d(\underline{a},\underline{c})c_1 \otimes \cdots \otimes c_k + \sum_{e_1,\dots,e_k \in B} d(\underline{a},\underline{e})e_1 \otimes \cdots \otimes e_k \neq 0$$

where the sum is over all basis tensors $e_1 \otimes \cdots \otimes e_k$ which are not equal to $c_1 \otimes \cdots \otimes c_k$ since $d(\underline{a},\underline{c}) \neq 0$. This shows that $\phi(d)$ is not the 0-map. Since d is an arbitrary element of $B_k(1-2m)$, $\ker\phi = 0$. Thus, ϕ is injective. □

4. Irreducible modules and characters

Using the commuting actions of $\mathcal{G}$ and $B_k(1-2m)$ on T, a collection of maximal vectors of T were constructed in [1]. In this section, we show that each of these maximal vectors generates an irreducible submodule of T, and describe the irreducible characters in terms of tableaux.

First, we need to describe the maximal vectors of T. Let $c_{p,q}$ denote the transformation on $\otimes^k V$ determined by the diagram in $B_k(1-2m)$ with a horizontal edge connecting the pth and qth vertices on both top and bottom, and with every other top vertex connected to the one directly below it. with $p \neq q$. We refer to the transformation $c_{p,q}$ as a *contraction mapping.* If $\underline{p} = \{p_1, \dots, p_a\}$ and $\underline{q} = \{q_1, \dots, q_a\}$ are disjoint ordered subsets of $\{1, 2, \dots, k\}$, we set $(\underline{p},\underline{q}) = \{(p_1,q_1), \dots, (p_a,q_a)\}$, and denote by $\mathcal{P}(a)$ the set of all such $(\underline{p},\underline{q})$. We set $\mathcal{P} = \cup\mathcal{P}(a)$ where the union is for all $a = 0, \dots, [k/2]$. We also set $c_{\underline{p},\underline{q}} = c_{p_1,q_1} \cdots c_{p_a,q_a}$ and assume that $c_{\emptyset,\emptyset}$ is the identity map.

A sequence of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a *partition* if $\lambda_1 \geq \cdots \geq \lambda_\ell$. The *length* $\ell(\lambda)$ of a partition λ is the number ℓ of nonzero parts. If $|\lambda| = \lambda_1 + \cdots + \lambda_\ell = f$, then λ is a partition of f, denoted by $\lambda \vdash f$. The *Young frame* or *Ferrers diagram* of a partition λ, $F(\lambda)$, is an left-justified array of $|\lambda|$ boxes with λ_i boxes in the ith row. The *conjugate partition* of λ is the partition λ' whose frame is obtained by reflecting that of λ about the main diagonal. Throughout this paper we identify a partition with its frame. Suppose that λ is a partition of f and consider a tableau obtained by filling in the boxes of $F(\lambda)$ with elements in $\{1, 2, \dots, f\}$. A *standard tableau* of shape λ is a filling of the frame of λ where each element in $\{1, 2, \dots, f\}$ is used exactly once and the entries in each row is increasing from left to right and in each column from top to bottom. For a standard tableau τ, let R_τ be the *row group* of permutations in S_f which permutes the entries within a given row of τ, and let C_τ be the corresponding *column group* of permutations moving only the elements in each column. Let $y_\tau = (\sum_{\gamma \in C_\tau} sgn(\gamma)\gamma)(\sum_{\sigma \in R_\tau} \sigma)$. Then y_τ is an essential idempotent (i.e., there is some $k \in \mathbf{Z}^+$ such that $y_\tau^2 = k y_\tau$). The y_τ is called a *Young symmetrizer.*

THEOREM 4.1. ([1] Theorem 3.9, Corollary 3.10, Theorem 3.11) *Suppose λ is a partition such that $\ell(\lambda) \leq m$ and $|\lambda| = f$ where $k - f = 2a$ and $0 \leq a \leq [k/2]$. Assume $(\underline{p}, \underline{q}) \in \mathcal{P}(a)$ and fix $\tau \in ST_\lambda(\underline{p} \cup \underline{q})^c$. Then $\theta = y_\tau c_{\underline{p},\underline{q}} \beta_{\tau,\underline{p},\underline{q}}$ is a maximal vector in T of weight $\lambda_1\epsilon_1 + \cdots + \lambda_\ell\epsilon_\ell$ where $\beta_{\tau,\underline{p},\underline{q}} = w_1 \otimes \cdots \otimes w_k$ is the simple tensor defined by*

$$w_i = \begin{cases} v_1 & \text{if } i \in \underline{p} \\ v_1^* & \text{if } i \in \underline{q} \\ v_j & \text{if } i \in (\underline{p} \cup \underline{q})^c \text{ and } i \text{ is in } j\text{th row of } \tau. \end{cases}$$

Suppose that $m \geq k$. Then

$$\{y_\tau c_{\underline{p},\underline{q}} \beta_{\tau,\underline{p},\underline{q}} \mid (\underline{p}, \underline{q}) \in \mathcal{P}, \tau \in ST((\underline{p} \cup \underline{q})^c)\}$$

is a linearly independent set of maximal vectors.

The only family of the classical Lie superalgebras where the complete reducibility for finite dimensional modules holds is $spo(2m, 1)$ ([3]). Since a highest weight module is indecomposable, the fact that each of the maximal vectors in $\otimes^k V$ generates an irreducible module immediately follows from the complete reducibility. In the following, we prove that the maximal vectors generate irreducible modules without using the complete reducibility, which provides a general argument for other families of Lie superalgebras.

THEOREM 4.2. *Let θ_λ be a maximal vector of weight λ in* Theorem 4.1. *Then $\mathcal{U}(\mathcal{G})\theta_\lambda$ is an irreducible $\mathcal{G}$-module where $\mathcal{U}(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$. Thus $\mathcal{U}(\mathcal{G})\theta_\lambda \cong V(\lambda)$.*

Proof. Let $\theta_\lambda = y_\tau c_{\underline{p},\underline{q}} \beta_{\tau,\underline{p},\underline{q}}$ and $\beta = \beta_{\tau,\underline{p},\underline{q}}$. Let $\widetilde{V}(\lambda)$ denote the Verma module of $\mathcal{G}$ with the highest weight λ. Then $V(\lambda) \cong \widetilde{V}(\lambda)/I(\lambda)$ where

$I(\lambda)$ is the maximal submodule of $\widetilde{V}(\lambda)$. Let v_λ be a maximal vector of $\widetilde{V}(\lambda)$. Then by [4] Theorem 1,

$$I(\lambda) = \left(\sum_{i=1}^{m-1} U(\mathcal{G})f_i^{a_i+1} + U(\mathcal{G})z^{b+1}\right) v_\lambda$$

where f_i is a basis vector for $\mathcal{G}_{-\alpha_i}$ where α_i are simple roots, z is a basis vector for $\mathcal{G}_{-2\epsilon_m}$, $a_i = \lambda_i - \lambda_{i+1}$ for $i = 1, \ldots, m-1$, and $b = \lambda_m$. Consider the canonical homomorphism $\phi : \widetilde{V}(\lambda) = U(\mathcal{G})v_\lambda \to U(\mathcal{G})\theta_\lambda$. We show that $I(\lambda) = \ker\phi$. We may choose f_i as the transpose of e_i where e_i's are as described in (2). It is easy to show that $c_{1,2}(\otimes^2 V)$ generates a 1-dimensional trivial module. Then for each $x \in \mathcal{G}$, the simple tensors in $xc_{\underline{p},\underline{q}}\beta$ with x acting on the positions in $(\underline{p} \cup \underline{q})$ sum up to 0. Then, for $i = 1, \ldots, m-1$, $f_i c_{\underline{p},\underline{q}}\beta$ is the sum of the simple tensors where one v_i of β which appears in a position in $(\underline{p} \cup \underline{q})^c$ is replaced by v_{i+1}. Thus $f_i^{a_i+1}c_{\underline{p},\underline{q}}\beta$ is a sum of simple tensors where $a_i + 1 = \lambda_i - \lambda_{i+1} + 1$ factors of β in $(\underline{p} \cup \underline{q})^c$ which are equal to v_i have been replaced by v_{i+1}. For each such simple tensor u in $f_i^{a_i+1}c_{\underline{p},\underline{q}}\beta$, we argue that $y_\tau u = 0$. Note that from the way β is defined, β has precisely λ_i factors in $(\underline{p} \cup \underline{q})^c$ which are equal to v_i. Thus for each $\sigma \in R_\tau$, there is $(a\ b) \in C_\tau$ such that $(a\ b)\sigma u = \sigma u$. Then $\sum_{\gamma \in C_\tau} sgn(\gamma)\gamma(\sigma u) = \sum_{\gamma \in C_\tau} sgn(\gamma(a\ b))\gamma(a\ b)(\sigma u) = -\sum_{\gamma \in C_\tau} sgn(\gamma)\gamma(\sigma u)$. Thus $\sum_{\gamma \in C_\tau} sgn(\gamma)\gamma(\sigma u) = 0$ for each $\sigma \in R_\tau$. Then $y_\tau u = 0$. Thus $y_\tau f_i^{a_i+1}c_{\underline{p},\underline{q}}\beta = 0$. Then $f_i^{a_i+1}y_\tau c_{\underline{p},\underline{q}}\beta = 0$ since f_i and y_τ commute.

A basis vector for $\mathcal{G}_{-2\epsilon_m}$ can be chosen as $z = E_{2m,m}$. Then $zc_{\underline{p},\underline{q}}\beta$ is the sum of the simple tensors where one v_m in β which appears in a position in $(\underline{p} \cup \underline{q})^c$ is replaced by v_{2m}. Since β has precisely λ_m factors in $(\underline{p} \cup \underline{q})^c$ which are equal to v_m, $z^{\lambda_m+1}c_{\underline{p},\underline{q}}\beta = 0$. Hence $z^{\lambda_m+1}y_\tau c_{\underline{p},\underline{q}}\beta = y_\tau z^{\lambda_m+1}c_{\underline{p},\underline{q}}\beta = 0$.

Thus, $I(\lambda) \subseteq \ker\phi$. Then since $\phi \neq 0$ and $I(\lambda)$ is maximal, $I(\lambda) = \ker(\phi)$. Hence $V(\lambda) \cong \tilde{V}(\lambda)/I(\lambda) \cong U(\mathcal{G})\theta_\lambda$. □

We now describe irreducible characters in terms of tableaux. Let's order the basis B of V as

$$B = \{v_1 < v_1^* < v_2 < v_2^* < \cdots < v_m < v_m^* < v_{2m+1}\}.$$

A *$spo(2m,1)$-tableau* τ of shape λ is a filling of boxes in the Ferrers diagram of λ with entries from B such that (spo.1) the subtableau η of τ obtained by taking all the boxes with entries from $\{v_1, v_1^*, \ldots, v_m, v_m^*\}$ is a *symplectic*-tableau, i.e. it is a column-strict tableau of partition shape (its entries are weakly increasing in each row from left to right and strictly increasing in each column from top to bottom), and the entries in each row i are $\geq v_i$ in η, and

(spo.2) the skew tableau τ/η is a *vertical strip* (no two boxes in a row) filled with v_{2m+1}.

EXAMPLE 4.3. *Let*

$$\tau = \begin{array}{|c|c|c|c|}\hline v_1 & v_1^* & v_2 & v_7 \\ \hline v_2^* & v_2^* & v_7 & \multicolumn{1}{c}{} \\ \cline{1-3} v_3 & v_3 & v_7 & \multicolumn{1}{c}{} \\ \cline{1-3} v_7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Then τ is a spo$(2 \cdot 3, 1)$-tableau where

$$\eta = \begin{array}{|c|c|c|}\hline v_1 & v_1^* & v_2 \\ \hline v_2^* & v_2^* & \multicolumn{1}{c}{} \\ \cline{1-2} v_3 & v_3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad \text{and} \qquad \tau/\eta = \begin{array}{|c|c|c|c|}\hline & & & v_7 \\ \hline & & v_7 & \multicolumn{1}{c}{} \\ \cline{1-3} & & v_7 & \multicolumn{1}{c}{} \\ \cline{1-3} v_7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

We now give a combinatorial description of the irreducible characters in terms of the *spo*$(2m, 1)$-tableaux. This combinatorial description can be obtained from [4] Proposition 2.11, which describes the decomposition of the $\mathcal{G}$-module $V(\lambda)$ into the irreducible $\mathcal{G}_0$-modules in terms of Kostant's functions. The proof presented here is more self-contained and elementary.

THEOREM 4.4. *Let τ be a spo$(2m, 1)$-tableau. For each $i \in \{1, \ldots, m\}$, let $a_i(\tau)$ denote the number of occurrences of v_i in τ, and $b_i(\tau)$ denote the number of occurrences of v_i^* in τ. Define*

$$x^\tau = x_1^{a_1(\tau)-b_1(\tau)} x_2^{a_2(\tau)-b_2(\tau)} \cdots x_m^{a_m(\tau)-b_m(\tau)}.$$

Then

$$chV(\lambda) = \sum_\tau x^\tau$$

where the sum is over all spo$(2m, 1)$-tableaux τ of shape λ.

Proof. The set of positive even roots of $\mathcal{G}$, Δ_0^+, and the set of positive odd roots of $\mathcal{G}$, Δ_1^+, are described as follows:

$$\begin{aligned} \Delta_0^+ &= \{\epsilon_i \pm \epsilon_j,\ 2\epsilon_k \mid 1 \le i < j \le m, 1 \le k \le m\} \quad \text{and} \\ \Delta_1^+ &= \{\epsilon_i \mid 1 \le i \le m\}. \end{aligned}$$

By Proposition 2.8 in [4],

$$\text{(4)} \qquad \mathrm{ch}V(\lambda) = \frac{L_1}{L_0} \sum_{w \in W} \epsilon(w) e^{w(\lambda+\rho_0-\rho_1)},$$

where W is the Weyl group of $\mathcal{G}_0 \cong sp(2m)$,

$$L_0 = \prod_{\alpha \in \Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2}), \quad L_1 = \prod_{\alpha \in \Delta_1^+} (e^{\alpha/2} + e^{-\alpha/2}),$$

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha = \sum_{i=1}^{m} (m-i+1)\epsilon_i, \quad \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha = \sum_{i=1}^{m} \frac{1}{2}\epsilon_i.$$

Let $e^{\epsilon_i} = x_i$ for all $i = 1, \dots, m$. Then (4) can be written as:

$$\mathrm{ch}V(\lambda) = \frac{1}{L_0} \prod_{i=1}^{m} (x_i^{1/2} + x_i^{-1/2}) \sum_{w \in W} \epsilon(w) w \prod_{i=1}^{m} x_i^{\lambda_i+(m-i+1)-(1/2)}.$$

Since $\prod_{i=1}^{m}(x_i^{1/2} + x_i^{-1/2})$ is invariant under each $w \in W$,

$$\begin{aligned} \mathrm{ch}V(\lambda) &= \frac{1}{L_0} \sum_{w \in W} \epsilon(w) w \prod_{i=1}^{m} (x_i^{\lambda_i+(m-i+1)-(1/2)} (x_i^{1/2} + x_i^{-1/2})) \\ &= \frac{1}{L_0} \sum_{w \in W} \epsilon(w) w \prod_{i=1}^{m} (x_i^{\lambda_i+(m-i+1)} + x_i^{\lambda_i-1+(m-i+1)}) \\ &= \frac{1}{L_0} \sum_{\mu} \sum_{w \in W} \epsilon(w) w \prod_{i=1}^{m} x_i^{\mu_i+(m-i+1)} \end{aligned}$$

where the the sum is over all $\mu = \{\mu_1, \dots, \mu_m\}$ such that $\mu_i = \lambda_i$ or $\mu_i = \lambda_i - 1$ for $i = 1, \dots, m$. If μ is a partition, then $\mu \subseteq \lambda$ in the sense that $\mu_i \le \lambda_i$ for $i = 1, \dots, m$ and λ/μ is a vertical strip. If μ is not a partition, then $\mu_i < \mu_{i+1}$ for some i where $1 \le i \le m$, or $\mu_s = \cdots = \mu_m = -1$ for some s where $s > \ell(\lambda)$. We will show that when μ is not a partition, the summand in $\mathrm{ch}V(\lambda)$ corresponding to μ is 0. Suppose that $\mu_i < \mu_{i+1}$ for some i. Since $\mu_i = \lambda_i$ or $\mu_i = \lambda_i - 1$ for each

i, we have $\mu_i < \mu_{i+1}$ only if $\mu_i = \lambda_i - 1$, $\mu_{i+1} = \lambda_{i+1}$, and $\lambda_i = \lambda_{i+1}$. In this case, $\mu_{i+1} = \mu_i + 1$. Let $b_i = \mu_i + (m-i+1)$ for $i = 1, \ldots, m$. Then

$$\sum_{w\in W} \epsilon(w) w \prod_{i=1}^{m} x_i^{b_i} = \sum_{\sigma\in S_m} \epsilon(\sigma) \prod_{i=1}^{m} (x_{\sigma(i)}^{b_i} - x_{\sigma(i)}^{-b_i}) = \sum_{\sigma\in S_m} \epsilon(\sigma) \prod_{i=1}^{m} a_{\sigma(i),i} = \det(A)$$

where $A = (a_{i,j})$ is the $m \times m$ matrix with $a_{i,j} = x_i^{b_j} - x_i^{-b_j}$. Since $\mu_{i+1} = \mu_i + 1$, $\mu_{i+1} + m - (i+1) + 1 = \mu_i + m - i + 1$. Then $\det(A)$=0 since the ith column and the $(i+1)$th column of the matrix A are the same.

Now suppose that $\mu_s = \cdots = \mu_m = -1$ for some s. In this case, we also get $\det(A) = 0$ since each of the entries in the last column of A is 0. Hence

$$\mathrm{ch}V(\lambda) = \frac{1}{L_0} \sum_{\mu} \sum_{w\in W} \epsilon(w) w \prod_{i=1}^{m} x_i^{\mu_i+(m-i+1)} = \sum_{\mu} \frac{1}{L_0} \sum_{w\in W} \epsilon(w) e^{w(\mu+\rho_0)}$$

where μ runs over all partitions such that $\mu \subseteq \lambda$ and λ/μ is a vertical strip. It is well known that the character of the irreducible module for the symplectic Lie algebra $sp(2m)$ labeled by a partition μ is

$$\frac{1}{L_0} \sum_{w\in W} \epsilon(w) e^{w(\mu+\rho_0)} = \sum_{\eta} x^{\eta}$$

where η runs over all symplectic tableaux of shape μ, and x^{η} is the monomial defined in the same way as in the statement of the theorem ([5]). Let τ be a $spo(2m,1)$-tableau of shape λ. Then the subtableau η of τ consisting of boxes filled with elements in $\{v_1, v_1^*, \ldots, v_m, v_m^*\}$ is a symplectic tableau of a partition shape μ, and the skew tableau τ/η is a vertical strip of shape λ/μ filled with v_{2m+1}. By the way they are defined, $x^{\tau} = x^{\eta}$. Thus

$$\sum_{\tau} x^{\tau} = \sum_{\mu} \sum_{\eta} x^{\eta} = \mathrm{ch}V(\lambda)$$

where τ runs over all $spo(2m,1)$-tableaux of shape λ, μ runs over all partitions such that $\mu \subseteq \lambda$ and λ/μ is a vertical strip, and η runs over all symplectic tableaux of shape μ. □

5. Decomposition and centralizer algebra

In this section we obtain a decomposition of $T = \otimes^k V$ into its irreducible $\mathcal{G}$-submodules, and prove that the centralizer algebra, $\mathcal{C} = End_{\mathcal{G}}(T)$, is isomorphic to the Brauer algebra $B_k(1-2m)$ for $m \geq k$.

We begin by recalling some definitions. An *up-down tableau* of *length* k and *shape* λ is a sequence of partitions $\Lambda = (\lambda^0 = \emptyset, \lambda^1, \ldots, \lambda^k = \lambda)$ such that λ^i is obtained from λ^{i-1} by either adding or removing a box for each $i = 1, \ldots, k$. Note that $|\lambda| = k - 2a$ for some $a = 0, 1, \ldots, [k/2]$. Let ud_λ denote the set of all up-down tableau of length k and shape λ. An *up-down m-tableau* of length k and shape λ is an up-down tableau of length k and shape λ such that $\ell(\lambda^i) \leq m$ for each $i = 1, \ldots, k$. Let $ud_\lambda(m)$ denote the set of all up-down m-tableaux of length k and shape λ. When $m \geq k$, note that $ud_\lambda = ud_\lambda(m)$ since all the partitions in ud_λ have length less than or equal to k and $k \leq m$.

THEOREM 5.1. *Assume that* $m \geq k$. *Then*

$$T = \otimes^k V = \oplus \sum_{a=0}^{[k/2]} \sum_{\lambda \vdash k-2a} |ud_\lambda| V(\lambda).$$

Proof. In Theorem 4.2, we constructed irreducible $\mathcal{G}$-submodules of T using maximal vectors of T, which are linearly independent for $m \geq k$ by Theorem 4.1. Then the corresponding sum of the irreducible submodules is direct, and

$$\otimes^k V \supseteq \oplus \sum_{a=0}^{[k/2]} \sum_{\lambda \vdash k-2a} \binom{k}{2a} (2a)!! f^\lambda V(\lambda) \tag{5}$$

where $\binom{k}{2a}$ counts the number of ways to choose $2a$ positions out of k positions to apply contractions, $(2a)!! = (2a-1)(2a-3)\cdots 1$ counts the number of ways to arrange contractions within $2a$ positions, and f^λ is the number of standard tableaux of shape λ.

For $m \geq k$, it is known that $\binom{k}{2a}(2a)!! f^\lambda = |ud_\lambda|$ by [8] Lemma 2.2. By Theorem 4.4, dim$V(\lambda)$ is the number of $spo(2m,1)$-tableaux of shape λ. Now let $\mathcal{W}_k$ be the set of words of length k in the alphabet $B = \{v_1, v_1^*, \ldots, v_m, v_m^*, v_{2m+1}\}$, and $\mathcal{P}_k$ be the set of pairs consisting of a $spo(2m,1)$-tableau of shape λ and an up-down m-tableau of length k and shape λ. Then there is a Schensted-type bijection between $\mathcal{W}_k$ and

$\mathcal{P}_k$ ([1] Theorem 5.5). Thus, the dimensions of the left hand side and right hand side of (5) are the same. Hence the result follows. □

COROLLARY 5.2. *Assume that $m \geq k$. Then*

$$\mathcal{C} = End_{\mathcal{G}}(\otimes^k V) \cong B_k(1-2m).$$

Proof. By the work of [9] and [10], $B_k(\eta)$ is semisimple whenever $|\eta| > k$. When $|\eta| > k$, the irreducible $B_k(\eta)$-modules are indexed by the partitions in $\hat{B} = \{\lambda \vdash k-2a \mid a = 0, \dots, [k/2]\}$, and the dimension of the irreducible $B_k(\eta)$-module labeled by λ is $|ud_\lambda|$. Since we assume that $m \geq k$, $|1-2m| > k$. Thus, by the Wedderburn-Artin Theorem,

$$B_k(1-2m) \cong \sum_{\lambda \in \hat{B}} M_{|ud_\lambda|}(\mathbf{C})$$

where $M_{|ud(\lambda)|}(\mathbf{C})$ is the algebra of $|ud_\lambda| \times |ud_\lambda|$ matrices with entries in $\mathbf{C}$.

On the other hand, from Theorem 5.1,

$$\otimes^k V = \oplus_{\lambda \in \hat{B}} |ud_\lambda| V(\lambda).$$

Then, it is well-known that

$$\mathcal{C} \cong \sum_{\lambda \in \hat{B}} M_{|ud(\lambda)|}(\mathbf{C}).$$

Hence $B_k(1-2m) \cong \mathcal{C}$. □

COROLLARY 5.3. *Assume that $m \geq k$. Then the map ϕ in (3) is an isomorphism.*

Proof. It immediately follows from Proposition 3.1 and Corollary 5.2. □

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