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NORMAL QUINTIC ENRIQUES SURFACES

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ABSTRACT. In this paper we describe normal quintic surfaces in $\mathbf{P}^3$ which are birationally isomorphic to Enriques surfaces. Especially we characterize the sublinear systems which give rise to one of two Stagnaro's normal quintic surfaces in $\mathbf{P}^3$.

1. Introduction

1.1. Let S be an Enriques surface over the complex number field $\mathbb{C}$, that is, a non-singular projective surface with $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$, $K_S \neq 0$ and $2K_S = 0$.

This paper is concerned with the problem of describing normal quintic surfaces in $\mathbf{P}^3$ which are birationally isomorphic to Enriques surfaces. E. Stagnaro constructed two families of normal quintic surfaces in $\mathbf{P}^3$ as special birational models of Enriques surfaces employing classical methods [8]. Projective models of an Enriques surface by complete linear systems have even degrees. Since Stagnaro's models are quintic surfaces, it is likely that they are birational projective models by certain sublinear systems of complete linear systems, which may have base points.

The motivation for this work is to characterize the sublinear systems which give rise to Stagnaro's normal quintic surfaces in $\mathbf{P}^3$. We consider one of his two models, say a Stagnaro's first birational model of Enriques surfaces or simply a Stagnaro's first model, and reproduce his proof using cohomologies (Theorem 2.1). We characterize this model by a special type of divisors as follows (Proposition 3.3, Theorem 3.4):

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Main Theorem. *Let S be an Enriques surface. Then the following statements are equivalent:*

1. *S is birationally isomorphic to a normal quintic surface F_5 in $\mathbf{P}^3$ which has two tacnodes and two triple points in general position, where each of two tacnodal planes to F_5 at two tacnodes passes through two triple points (For the definition of tacnodes and triple points, see* Definition 1.5).
2. *S has a divisor $D = e_1 + e_2 + e_3 + e_4$, where $e_1, \cdots, e_3$ are half-pencils of S with $e_i \cdot e_j = 1$ for $i \neq j$ and $e_4 = e_2' = e_2 + K_S$, the adjoint of e_2. Furthermore if $p_1, \cdots, p_5$ are the intersection points of $e_1, \cdots, e_4$, then $p_1, \cdots, p_5$ are distinct (cf. Figure 1).*

If an Enriques surface S has a divisor D of the type given at (2), then we show that the sublinear system of $|D + K_S|$ which has $p_1, \cdots, p_5$ as its base points induces a birational morphism to a normal quintic surface F_5 in $\mathbf{P}^3$ with singularities described at (1) (Theorem 3.4).

E. Stagnaro noticed that the normal quintic surface described at the above theorem is originally due to G. Castelnuovo [1]. G. Castelnuovo found that the birational quadratic transformation $x_1 : x_2 : x_3 : x_4 = y_3y_4 : y_1y_2 : y_1y_3 : y_1y_4$ maps a classic Enriques sextic to the above normal quintic surface.

F. Cossec proved that every Enriques surface S has three half-pencils e_1, e_2, e_3 with $e_i \cdot e_j = 1$ for $i \neq j$, [3]. By applying this result, we note that every Enriques surface S has a divisor $D = e_1 + e_2 + e_3 + e_4$, where $e_1, \cdots, e_3$ are half-pencils satisfying $e_i \cdot e_j = 1$ for $i \neq j$ and $e_4 = e_2'$. If for every Enriques surface S, the five intersection points $p_1, \cdots, p_5$ of $e_1, \cdots, e_4$ are distinct, then every Enriques surface S would be birationally isomorphic to a normal quintic surface in $\mathbf{P}^3$. The author once claimed that they are distinct, but later a gap in its proof was found by Y. Umezu. It is obvious that e_2 and e_4 both can not pass through the intersection point of e_1 and e_3 since $e_2 \cdot e_4 = 0$.

Let $C = e_1 + e_2 + e_3$, where e_1, e_2 and e_3 are half-pencils on an Enriques surface S with $e_i \cdot e_j = 1$ for $i \neq j$. The linear system $|C|$ on S is said to be *superelliptic* if $e_1 + e_2 - e_3$ is effective. We show that if both of linear systems $|C|$ and $|C + K_S|$ are not superelliptic, then the Enriques surface S has a divisor D of the type described at the above theorem, which implies that S is birationally isomorphic to a normal quintic surface in $\mathbf{P}^3$ (Proposition 3.5). On the other hand, F. Cossec showed that $|C|$ and $|C + K_S|$ are not simultaneously superelliptic [3].

Y. Umezu also independently obtained the above results for the Stagnaro's first model. Furthermore she shows that if $d(S) \geq 4$, then S has a divisor $D = e_1 + e_2 + e_3 + e_4$ with the distinct intersection points $p_1, \cdots, p_5$ [9], [10], where $d(S) = \max\{r : S$ has half-pencils $e_1, \cdots, e_r$ such that $e_i \cdot e_j = 1$ for $1 \leq i, j \leq r\}$, called the *non-degeneracy invariant* of S. Thus an Enriques surface S with $d(S) \geq 4$ is birationally isomorphic to a normal quintic surface of the Stagnaro's first kind. On the other hand, it is known that $d(S) \geq 3$ for all Enriques surfaces S [3], [4], but there is no known Enriques surface S with $d(S) = 3$. In the last section, we also give the similar characterization for the Stagnaro's second model.

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1.2. We will always understand an elliptic curve to be an effective divisor of arithmetic genus one. By an elliptic fibration of a surface X over an algebraic curve B, we mean a proper morphism with connected fibres $f : X \to B$ such that a general fibre X_b ($b \in B$) is a non-singular elliptic curve. We call the elliptic fibration f (relatively) minimal if fibres of f do not contain exceptional curves of the first kind. Unless otherwise stated we assume that the elliptic fibration f is always minimal.

Every Enriques surface S admits an elliptic fibration over $\mathbf{P}^1$. Furthermore, every elliptic fibration $f : S \to \mathbf{P}^1$ has exactly two multiple fibres $2e$ and $2e'$, and $K_S = \mathcal{O}_S(e' - e) = \mathcal{O}_S(e - e')$. Each of the linear systems $|e|, |e'|$ contains a single curve. For this reason, e and e' are called *isolated elliptic curves* or *half-pencils.* It is also known that isolated elliptic curves e and e' are of type I_n in Kodaira's notation ($0 \leq n \leq 9$) [2], [4].

LEMMA 1.1 (Dolgachev [5]). *If X is a minimal elliptic surface with $q(X) = 0$, then the plurigenus of X are given by*

$$(1.1)\quad P_n(X) = h^0(X, nK_X) = n(p_g(X) - 1) + \sum_{i=1}^{k} [n(m_i - 1)/m_i] + 1,$$

where the integer m_i are multiplicity of multiple fibers of an elliptic fibration of X and $[x]$ denotes an integral part of x. Furthermore if

$p_g(X) = 0$, $P_2(X) = 1$ and $P_3(X) = 0$, then $k = 2$, $m_1 = m_2 = 2$. In particular, X is an Enriques surface.

Proof. Let $f : X \to B$ be an elliptic fibration of X over an algebraic curve B. We note that $B = \mathbf{P}^1$ since $q(X) = 0$. In particular, all fibres of the elliptic fibration $f : X \to \mathbf{P}^1$ are linearly equivalent. Let F be a general fibre of f and $X_i = m_i F_i$ $(i = 1, \cdots, k)$ the multiple fibres of f with multiplicity m_i. From the canonical bundle formula for elliptic fibrations, it follows that

$$K_X \sim (p_g(X) - 1)F + \sum_{i=1}^{k} (m_i - 1) F_i .$$

Thus we have

$$\begin{aligned} n K_X &\sim n\,(p_g(X) - 1)\,F + \sum_{i=1}^{k} n\,(m_i - 1)\,F_i \\ &= (n\,(p_g(X) - 1) + \sum_{i=1}^{k} [n\,(m_i - 1)/m_i])\,F + \sum_{i=1}^{k} a_i\,F_i , \end{aligned}$$

where $0 \leq a_i < m_i$. Note that the divisor $\sum_{i=1}^{k} a_i F_i$ is the fixed part of the linear system $|nK_X|$. Hence we get the desired plurigenus formula.

Now we assume that $p_g(X) = 0$, $P_2(X) = 1$ and $P_3(X) = 0$. Then from the formula (1),

$$\begin{aligned} 0 &= p_g(X) = \sum_{i=1}^{k} [(m_i - 1)/m_i] \\ 1 &= P_2(X) = -1 + \sum_{i=1}^{k} [2\,(m_i - 1)/m_i] \\ 0 &= P_3(X) = -2 + \sum_{i=1}^{k} [3\,(m_i - 1)/m_i] \end{aligned}$$

We notice that $k = 2$ and $m_1 = m_2 = 2$ is the unique integral solution of the above system of equations. Therefore X is an Enriques surface. □

1.3. Let $\pi : M \to V$ be the minimal resolution of a Stein normal surface V with a unique isolated singular point p. Then the *geometric genus* $h(p)$ of V at p is the dimension of the complex vector space

$H^0(V, R^1\pi_*(\mathcal{O}_M))$. We state the following well known lemma without proof.

LEMMA 1.2. *If a surface F in $\mathbf{P}^3$ has finite number of isolated singular points and $\tilde{F}$ its minimal desingularization, then $\chi(\mathcal{O}_F) = \chi(\mathcal{O}_{\tilde{F}}) + \sum_{p\ sing} h(p)$. In particular, if F is a normal quintic surface in $\mathbf{P}^3$, then*

$$5 = \chi(\mathcal{O}_{\tilde{F}}) + \sum_{p\ sing} h(p).$$

COROLLARY 1.3. *If a normal quintic surface F in $\mathbf{P}^3$ is birationally isomorphic to an Enriques surface S, then*

$$\sum_{i=1}^{k} h(p_i) = 4, \tag{1.2}$$

where p_i $(i = 1, \cdots, k)$ are the isolated singular points of F with $h(p_i) > 0$.

REMARK 1.4. The identity (2) is the necessary condition for a normal quintic surface F in $\mathbf{P}^3$ to be birationally isomorphic to an Enriques surface S. Hence it could be used as a criterion for the classification of normal quintic Enriques surfaces in $\mathbf{P}^3$. We have the following four possibilities:

1. $k = 1$ and $h(p_1) = 4$.
2. $k = 2$ and $h(p_1) = 2, h(p_2) = 2$ or $h(p_1) = 1, h(p_2) = 3$.
3. $k = 3$ and $h(p_1) = 1, h(p_2) = 1, h(p_3) = 2$.
4. $k = 4$ and $h(p_i) = 1, i = 1, \cdots, 4$.

DEFINITION 1.5. Let p be an isolated singularity on a hypersurface V with a minimal resolution of V, $\pi : M \to V$. Then p is *minimally elliptic* if $h(p) = 1$. And *tacnodes* (*triple points*) are minimally elliptic double points with $Z^2 = -2$ (minimally elliptic triple points), where Z is the fundamental cycle on the exceptional set $\pi^{-1}(p)$, [7].

Throughout this paper, we will say simply tacnodes or triple points of type Γ ignoring self-intersection numbers of irreducible components of Γ, the dual graph of the singularity. Most of tacnodes we will treat in this paper are tacnodes of type I_n, $0 \leq n \leq 9$. In particular, tacnodes and triple points of type I_0 are simple elliptic singularities, whose exceptional sets are non-singular elliptic curves. Their equations are given by

$$T_{2,4,4} : z^2 + x^4 + y^4 + ax^2y^2 \, , \, a^4 \neq 4 \quad \text{and}$$
$$T_{3,3,3} : x^3 + y^3 + z^3 + axyz \, , \, a^3 + 27 \neq 0 \, .$$

In general, the equation of tacnodes is given by

$$z^2 + f(x, y) = 0 \, ,$$

where $f(x, y)$ is a polynomial in x, y whose smallest total degree among non-zero monomial terms is four or five (see Table 2 in [7]). Then a *tacnodal plane* is the plane given by the equation $z = 0$ in the above equation.

DEFINITION 1.6. Let (S, p) be a normal complex surface singularity and $\pi : (\tilde{S}, A) \to (S, p)$ be the minimal resolution of (S, p). Then (S, p) is a *cusp singularity* if and only if the exceptional set $A = \cup A_i$ is an irreducible rational curve with a node or a cycle of nonsingular rational curves A_i. Furthermore if (S, p) is a hypersurface singularity, then its defining equation is given by $T_{p,q,r} : x^p + y^q + z^r = xyz$, where $1/p + 1/q + 1/r < 1$.

2. Stagnaro's first birational model of Enriques surfaces

2.1. In this section, we reproduce one of the two families of normal quintic surfaces which were given by E. Stagnaro as birational models of Enriques surfaces.

THEOREM 2.1 (Stagnaro [8]). *Let F_5 be a normal quintic surface in $\mathbf{P}^3$ with the following property $\mathcal{P}$:*

> *The normal quintic surface F_5 has two tacnodes at A_2, A_4 and two triple points at A_1, A_3, where A_1, A_2, A_3, A_4 are the vertices of a tetrahedron T. Two tacnodal planes to F_5, α_1 at A_2 and α_2 at A_4, pass through two triple points A_1 and A_3. Any other singular points of F_5 are rational double points.*

If S is a minimal non-singular model of F_5, then S is an Enriques surface.

Proof. Let $\tilde{S}$ be the minimal desingularization of the surface F_5. We compute the surface invariants p_g, q, P_2 and P_3 of $\tilde{S}$. At the end of the proof, we show that $p_g(\tilde{S}) = 0$, $q(\tilde{S}) = 0$, $P_2(\tilde{S}) = 1$ and $P_3(\tilde{S}) = 0$. After observing that $\varkappa(\tilde{S}) = 0$, we conclude that from the classification of surfaces with $\varkappa = 0$, the minimal model S of $\tilde{S}$ is an Enriques surface.

To get a minimal non-singular model of F_5, we first resolve the singularities of F_5 in an affine neighborhood V of one of two tacnodal singular points A_2 and A_4. Let us take one of them, say A_2. By choosing a small enough neighborhood of the point A_2, we may assume that V has the only singular point A_2 and is a hypersurface in $\mathbf{A}^3$. The surface V may be considered as the surface defined by the equation : $z^2 + f(x,y) = 0$, where $f(x,y)$ is a polynomial in x, y whose the first non-zero monomial term is of total degree 4 or 5 and A_2 corresponds to the origin. Let H_0 be the hyperplane of $\mathbf{A}^3$ given by the equation: $z = 0$, which is the tacnodal plane to V at the origin.

Let $\sigma_1 : \mathbf{M} \longrightarrow \mathbf{A}^3$ be the blow-up of $\mathbf{A}^3$ at the origin, V' the proper transform of V and E' the exceptional divisor. Then V' meets E' along a double line L, and $H_0' \cap E' = L$, where H_0' is the proper transform of the tacnodal plane H_0 in $\mathbf{M}$.

Next we blow up $\mathbf{M}$ along a line L, and let $\sigma_2 : \mathbf{N} \longrightarrow \mathbf{M}$ be the blow-up of $\mathbf{M}$ along a double line L of V', $\tilde{V}$ the proper transform of V' and $\tilde{E} \simeq \mathbf{P}^1 \times \mathbf{P}^1$ the exceptional divisor.

$$\begin{array}{ccccccc}
\tilde{V} & \subset & \mathbf{N} & & & \supset & \tilde{E} \\
\downarrow & & \downarrow{\scriptstyle \sigma_2} & & & & \downarrow \\
V' & \subset & \mathbf{M} & \supset & E' & \supset & L \\
\downarrow & & \downarrow{\scriptstyle \sigma_1} & & \downarrow & & \\
V & \subset & \mathbf{A}^3 & \ni & \mathbf{0} = (0,0,0) & &
\end{array}$$

The surface $\tilde{V}$ may have rational double points which do not change the surface invariants, hence we may assume that $\tilde{V}$ is non-singular. Let $\sigma = \sigma_1 \circ \sigma_2$.

The following two lemmas are essential in understanding the reason why we have to impose a special condition on two tacnodal planes α_1 and α_2 at 2.1. It is a fairly easy exercise to prove the following lemmas from the above two blow-ups computation, so we omit their proofs.

LEMMA 2.2. *Any hypersurface H_1 of $\mathbf{A}_3$ which is tangent to the tacnodal plane at the origin is the unique hypersurface of $\mathbf{A}^3$, including the tacnodal plane H_0, for which the total transform is given by*

$$\begin{aligned}\sigma^*(H_1) &= {\sigma_2}^* \circ {\sigma_1}^*(H_1)\\ &= {\sigma_2}^*(H_1' + E')\\ &= {\sigma_2}^*(H_1') + {\sigma_2}^*(E')\\ &= \tilde{H}_1 + \tilde{E} + {\sigma_2}^*(E')\,;\end{aligned}$$

and for all other hypersurfaces H_2 of $\mathbf{A}_3$ passing through the origin as a smooth point,

$$\sigma^*(H_2) = \tilde{H}_2 + {\sigma_2}^*(E')\,,$$

where H_1' is the proper transform of H_1 in $\mathbf{M}$ and $\tilde{H}_1$ and $\tilde{H}_2$ are the proper transforms of H_1, H_2 in $\mathbf{N}$.

Abusing notations, let H_1 and H_2 be divisors on V cut out by hypersurfaces H_1 and H_2 of $\mathbf{A}_3$.

LEMMA 2.3. *Let $\tilde{e} \subset \tilde{V}$ be the exceptional set of the minimal desingularization $\sigma : \tilde{V} \to V$. Then $\sigma^*(H_1) = \tilde{H}_1 + 2\tilde{e}$ and $\sigma^*(H_2) = \tilde{H}_2 + \tilde{e}$.*

Next we examine two triple points A_1 and A_3. To resolve singularities at these two points, we need to blow up points alone because triple points in our definition are absolutely isolated, which can be resolved by blowing up points alone.

We now return to the normal quintic surface F_5. After blowing up F_5 at infinitely near points or infinitely near double lines over $A_1, \cdots, A_4$, we may assume that we resolved singularities of F_5.

Let $\sigma : \tilde{S} \to F_5$ be the minimal desingularization and $\tilde{e}_1, \cdots, \tilde{e}_4$ the exceptional sets over $A_1, \cdots, A_4$. Then it is easy to check that

$$K_{\tilde{S}} = \sigma^*(H) - \tilde{e}_1 - \tilde{e}_2 - \tilde{e}_3 - \tilde{e}_4\,, \tag{2.1}$$

where H is a hyperplane section of F_5.

The adjunction formula implies that

$$K_{\tilde{S}} \cdot \tilde{e}_i + \tilde{e}_i \cdot \tilde{e}_i = -\tilde{e}_i \cdot \tilde{e}_i + \tilde{e}_i \cdot \tilde{e}_i = 0,\ 1 \leq i \leq 4\,.$$

Thus we see that the exceptional sets $\tilde{e}_1, \cdots, \tilde{e}_4$ are elliptic curves on $\tilde{S}$, and note that $\tilde{e}_1^2 = \tilde{e}_3^2 = -3$ and $\tilde{e}_2^2 = \tilde{e}_4^2 = -2$.

If there is any effective divisor in $|K_{\tilde{S}}| = \mathbf{P}H^0(\tilde{S}, \mathcal{O}(K_{\tilde{S}}))$, it would correspond to a hyperplane of $\mathbf{P}^3$ passing through $A_1, \cdots, A_4$. However, this is clearly impossible since four points $A_1, \cdots, A_4$ are in general position in $\mathbf{P}^3$, and thus $p_g(\tilde{S}) = 0$.

From the formula (3), we have

$$2K_{\tilde{S}} = \sigma^*(2H) - 2\tilde{e}_1 - 2\tilde{e}_2 - 2\tilde{e}_3 - 2\tilde{e}_4 .$$

This formula and Lemma 2.3 shows that bicanonical divisors of $\tilde{S}$ correspond to quadrics in $\mathbf{P}^3$ whose total transforms under the map $\sigma : \tilde{S} \to F_5$ contain $2\tilde{e}_1 + 2\tilde{e}_2 + 2\tilde{e}_3 + 2\tilde{e}_4$. Hence the quadrics in $\mathbf{P}^3$ either:

1. have four double points at vertices of the tetrahedron which are the only non-rational singular points of the normal quintic surface F_5 or
2. are the singular quadric $\alpha_1 + \alpha_2$, the union of two tangent planes α_1, α_2.

Let Q be a quadric in $\mathbf{P}^3$ satisfying the first condition. We note that the quadric Q must contain four planes coming from planes through three out of four vertices of the tetrahedron, which is not true. Hence our conclusion is that there is no quadric in $\mathbf{P}^3$ satisfying the first case.

The second case corresponds to the quadrics which are not singular at two tacnodes of F_5. It is the unique effective divisor of $|2K_{\tilde{S}}| = \mathbf{P}H^0(\tilde{S}, \mathcal{O}(2K_{\tilde{S}}))$. Hence $P_2(\tilde{S}) = 1$.

Next we consider the formula

$$3K_{\tilde{S}} = \sigma^*(3H) - 3\tilde{e}_1 - 3\tilde{e}_2 - 3\tilde{e}_3 - 3\tilde{e}_4 .$$

Tricanonical divisors of $\tilde{S}$ correspond to cubic surfaces in $\mathbf{P}^3$ which either:

1. have four triple points at vertices of the tetrahedron or
2. are the union of two tacnodal planes and a plane through the four vertices of the tetrahadron.

Similarly to the calculation of the bicanonical genus, it is easy to show that cubics with the first case must contain four planes coming from planes through each three out of four vertices of the tetrahedron, which is impossible. The second case corresponds to the cubics which do not have triple points, but double points at two tacnodes of F_5. Cubics with the second case do not exist since there is no plane through the four vertices of the tetrahedron. Hence our conclusion is that $P_3(\tilde{S}) = 0$.

Since tacnodal points A_2, A_4 and triple points A_1, A_3 are defined to be minimally elliptic double points and triple points respectively, their geometric genera, $h_i = h(A_i) = 1$ $(i = 1, \cdots, 4)$. Then from Lemma 1.2, we have

$$\begin{aligned} 5 = \chi(F_5) &= \chi(\tilde{S}) + \sum_{i=1}^{4} h_i \\ &= 1 - q(\tilde{S}) + p_g(\tilde{S}) + 4\,. \end{aligned}$$

From this equation, we see that $q(\tilde{S}) = p_g(\tilde{S}) = 0$.

Summarizing, we have shown that the minimal non-singular model S of the normal quintic surface F_5 with the property $\mathcal{P}$ has the surface invariants: $p_g(S) = 0$, $P_2(S) = 1$, $P_3(S) = 0$ and $q(S) = 0$.

By the Riemann-Roch formula,

$$\begin{aligned} 1 = P_2(S) &\geq 1 - q(S) + p_g(S) + \big((2K_S)^2 - 2K_S \cdot K_S\big)/2 \\ &= 1 + K_S^2\,, \end{aligned}$$

from which we induce that $K_S^2 \leq 0$.

One can easily check that $K_S^2 \not< 0$. Thus $K_S^2 = 0$, and $\varkappa(S) = 0$ or 1. Suppose that $\varkappa(S) = 1$, which implies that S is a properly elliptic surface. Then from Lemma 1.1, S is an Enriques surface, therefore $\varkappa(S) = 0$, a contradiction to our assumption. Hence $\varkappa(S) = 0$. From the classification of surfaces with $\varkappa = 0$, we now come to a conclusion that the surface S is an Enriques surface. □

COROLLARY 2.4. *Let X be a minimal non-singular model of a normal quintic surface F_5 which has two tacnodes and two triple points in general position, and does not satisfy the property $\mathcal{P}$. Then X is a rational surface.*

Proof. Similarly to the proof of Theorem 2.1, it is easy to see that $p_g(X) = 0$, $q(X) = 0$ and $P_2(X) = 0$, which implies that $(K_X)^2 \leq -1$. Hence $\varkappa(X) = -\infty$, and so X is a rational surface. □

2.2. We now fix four points of a tetrahedron T, say $A_1 = (1,0,0,0)$, $A_2 = (0,0,1,0)$, $A_3 = (0,1,0,0)$, $A_4 = (0,0,0,1)$. Suppose that the equations of two tacnodal planes to F_5 at A_2 and A_4 are given as follows:

$$\alpha_1 \,:\, x_4 = 0 \text{ and } \alpha_2 \,:\, x_3 = 0\,.$$

PROPOSITION 2.5. *Let F_5 be a normal quintic surface in $\mathbf{P}^3$ satisfying the property $\mathcal{P}$ of Theorem 2.1. Then F_5 contains five lines $L_1, \cdots, L_5$,* *where*

$$L_1 = \overline{A_1A_3}\ ,\ L_2 = \overline{A_1A_2}\ ,\ L_3 = \overline{A_1A_4}\ ,\ L_4 = \overline{A_3A_4}\ ,\ L_5 = \overline{A_2A_3}.$$

The tacnodal plane α_1 cuts out the quintic surface F_5 a hyperplane section $L_1 + 2L_2 + 2L_5$. Similarly, the tacnodal plane α_2 cuts out the quintic surface F_5 a hyperplane section $L_1 + 2L_3 + 2L_4$. Furthermore, the normal quintic surface F_5 has a defining equation :

$$\begin{aligned} F_5 :\ & a_1x_3^3x_4^2 + a_2x_3^2x_4^3 \\ & +a_3x_1^2x_2^2x_3 + a_4x_1^2x_2^2x_4 + a_5x_1^2x_3^2x_4 + a_6x_1^2x_3x_4^2 \\ & +a_7x_1x_3^2x_4^2 + a_8x_2^2x_3^2x_4 + a_9x_2^2x_3x_4^2 + a_{10}x_2x_3^2x_4^2 \\ & +a_{11}x_1^2x_2x_3x_4 + a_{12}x_1x_2^2x_3x_4 \\ & +a_{13}x_1x_2x_3^2x_4 + a_{14}x_1x_2x_3x_4^2, \end{aligned}$$

where $a_1, \cdots, a_6$ and a_8, a_9 are non-zero.

Proof. The first part is easy to check. To get the equation of the quintic surface F_5, which was originally given by Stagnaro [8], first consider all monomials of degree 5. Next we discard those monomials which do not satisfy the conditions on F_5. Then the remaining monomials are the ones appearing at the above equation (see [6]). □

We note that the equation of F_5 has fourteen coefficients with the action by the torus group, hence ten parameters correspond to the space of Enriques surfaces obtained from normal quintic surfaces F_5 with the property $\mathcal{P}$. Since the dimension of the moduli space of Enriques surfaces is ten, one infers that generic Enriques surfaces could be obtained from normal quintic surfaces in $\mathbf{P}^3$ with the property $\mathcal{P}$.

3. Characterization of Stagnaro's first model

3.1. In this section we will investigate Enriques surfaces obtained from normal quintic surfaces in $\mathbf{P}^3$ with the property $\mathcal{P}$.

Let $\sigma : \tilde{S} \to F_5$ be the minimal desingularization of F_5. Let $\tilde{L}_1, \cdots, \tilde{L}_5$ be the proper transforms of lines $L_1, \cdots, L_5$ of F_5 which was described at Proposition 2.5 and $\tilde{e}_1, \cdots, \tilde{e}_4$ the exceptional sets over $A_1, \cdots, A_4$. Then $K_{\tilde{S}} = \sigma^*(H) - \tilde{e}_1 - \tilde{e}_2 - \tilde{e}_3 - \tilde{e}_4$, where H is a hyperplane section of F_5. It is easy to check that $\sigma^*(H) \cdot \tilde{L}_1 = 1$, $\tilde{L}_1 \cdot \tilde{e}_1 = \tilde{L}_1 \cdot \tilde{e}_3 = 1$, and $\tilde{L}_1 \cdot \tilde{e}_2 = \tilde{L}_1 \cdot \tilde{e}_4 = 0$. Hence $K_{\tilde{S}} \cdot \tilde{L}_1 = -1$, which implies that

$\tilde{L}_1^2 = 2\ g(\tilde{L}_1) - 2 - K_{\tilde{S}} \cdot \tilde{L}_1 = -1$, so $\tilde{L}_1$ is an exceptional curve of the first kind. Similarly we see that $\tilde{L}_2, \cdots, \tilde{L}_5$ are exceptional curves of the first kind.

Let $\tau : \tilde{S} \to S$ be the blow-down of $\tilde{L}_1, \cdots, \tilde{L}_5$. Then $K_S^2 = 0$ since $K_{\tilde{S}}^2 = -5$. Since we showed that a minimal non-singular model of F_5 is an Enriques surface and thus $K_S^2 = 0$, we deduce that the surface S is a minimal surface, that is, the surface S is free from exceptional curves of the first kind.

Let E be a divisor on S corresponding to $\sigma^*(H)$. The goal of this section is to characterize the divisor E.

Before going further, we present a lemma which also shows the above claim that the surface S obtained from $\tilde{S}$ after blowing down $\tilde{L}_1, \cdots, \tilde{L}_5$ is a minimal surface. This lemma gives us a formula which could be used as a criterion on Enriques surfaces which are birationally isomorphic to a normal quintic surface in $\mathbf{P}^3$.

As before let $\sigma : \tilde{S} \to F_5$ be a minimal desingularization of F_5, and $\sigma^*(H)$ the total transform of a hyperplane section H of F_5. Let $L_1, \cdots, L_n, L'_1, \cdots, L'_l$ be all exceptional curves of the first kind on $\tilde{S}$ which are blown down to regular points of S, and $\tau : \tilde{S} \to S$ the corresponding blow-down, where

$$\begin{aligned} &\sigma^*(H) \cdot L_i = m_i > 0 \qquad && \text{for } i = 1, \cdots, n \\ &\sigma^*(H) \cdot L'_j = 0 && \text{for } j = 1, \cdots, l\,. \end{aligned}$$

LEMMA 3.1. *Let $K_{\tilde{S}} = \tau^*(K_S) + \sum_{i=1}^n c_i L_i + \sum_{j=1}^l d_j L'_j$, where c_i and d_j are positive integers. Then $\sum_{i=1}^n c_i\, m_i = 5$.*

Proof. We observe that $\sigma^*(H)^2 = 5$, and the arithmetic genus of $\sigma^*(H)$, $p_a(\sigma^*(H)) = 6$ since $\sigma^*(H)$ is the total transform of a hyperplane section of the normal quintic surface $F_5 \subset \mathbf{P}^3$. Then by the adjunction formula, we have

$$\begin{aligned} 10 &= 2\ p_a(\sigma^*(H)) - 2 \\ &= \sigma^*(H) \cdot \sigma^*(H) + K_{\tilde{S}} \cdot \sigma^*(H) \\ &= 5 + \sum_{i=1}^n c_i\, m_i\,. \end{aligned}$$

Thus $\sum_{i=1}^n c_i\, m_i = 5$, and completes the proof. □

REMARK 3.2. Lemma 3.1 is originally due to A. Verra [11] without the constants c_i. The equation $\sum_{i=1}^n c_i\, m_i = 5$ could be used when we

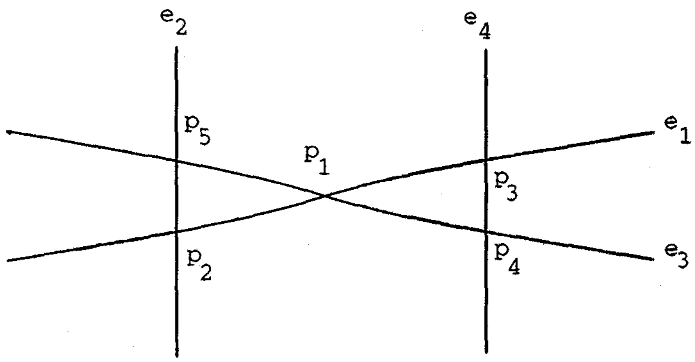

FIGURE 1

check whether an Enriques surface S with a divisor E is birationally isomorphic to a normal quintic surface F in $\mathbf{P}^3$ while E is corresponding to $\sigma^*(H)$, and this will be tested at Theorem 3.4. This remark is comparable to Remark 1.4 which describes the necessary condition for a normal quintic surface F in $\mathbf{P}^3$ to be birationally isomorphic to an Enriques surface S.

Now by setting $c_i = 1, m_i = 1$ $(i = 1, \cdots, 5)$ and applying Lemma 3.1, we confirm that $\tilde{L}_i$ $(i = 1, \cdots, 5)$ are the only exceptional curves of the first kind which intersect $\sigma^*(H)$. Hence after blowing down the exceptional curves of the first kind $\tilde{L}_i$ $(i = 1, \cdots, 5)$, we get the same divisor E.

Let $e_1, \cdots, e_4$ be the images of elliptic curves $\tilde{e}_1, \cdots, \tilde{e}_4$ by the map $\tau : \tilde{S} \to S$. Then we see that $e_2 \cdot e_4 = 0$, and $e_i \cdot e_j = 1, i \neq j$ except $i = 2$, $j = 4$. Consider an invertible sheaf $\mathcal{O}_S(e_i)$, $i = 1, \cdots, 4$. If the linear system $|\mathcal{O}_S(e_i)|$ has no fixed components for some i, then [2], Theorem 1.5.1 says that $\mathcal{O}_S(e_i) \simeq \mathcal{O}_S(k\,P)$ for an elliptic pencil $|P|$, where $k \geq 1$. Then we would not have the equality, $e_i \cdot e_j = 1, i \neq j$ except $i = 2, j = 4$ because $P \simeq 2\,e$ for some isolated elliptic curve e and $e_i \cdot e_j$ would be an even number. It follows that $|\mathcal{O}_S(e_i)|$ has a fixed component for each i, $1 \leq i \leq 4$. Thus we conclude that each elliptic curve e_i, $1 \leq i \leq 4$, is an isolated elliptic curve, which is an indecomposable divisor of canonical type (for the definition, see [2], [4].

We defined E as the image of $\sigma^*(H)$ by the map τ. Then $E = D + K_S$, where $D = e_1 + e_2 + e_3 + e_4$ is a divisor with the configuration in Figure

1, and $e_1, \cdots, e_4$ are isolated elliptic curves on S. We note that $e_4 = e_2' = e_2 + K_S$ since e_2, e_4 are the isolated elliptic curves and $e_2 \cdot e_4 = 0$.

We recall that the points $p_1, \cdots, p_5$ of S are the contractions of $\tilde{L}_1, \cdots, \tilde{L}_5$, and the isolated elliptic curves $e_1, \cdots, e_4$ on S are from the points $A_1, \cdots, A_4$ of a normal quintic surface F_5, hence especially five points $p_1, \cdots, p_5$ are mutually distinct.

By summarizing what we have observed, we get the following proposition.

PROPOSITION 3.3. *Let S be an Enriques surface obtained from the normal quintic surface F_5 satisfying the property $\mathcal{P}$ of* Theorem 2.1, *then S has a divisor $D = e_1 + e_2 + e_3 + e_4$ with the configuration in the Figure 1, where $e_1, \cdots, e_4$ are isolated elliptic curves on S, and the intersection points $p_1, \cdots, p_5$ are distinct.*

3.2. We will now show that an Enriques surface S with a divisor D with the configuration in Figure 1 is birationally isomorphic to a normal quintic surface F_5 satisfying the property $\mathcal{P}$ of Theorem 2.1. Thus Enriques surfaces with such a divisor D are exactly those Enriques surfaces which are minimal non-singular models of normal quintic surfaces in $\mathbf{P}^3$ with the property $\mathcal{P}$. In particular, the type of divisors D with the configuration in Figure 1 gives a criterion for determining which Enriques surfaces might be birationally isomorphic to normal quintic surfaces in $\mathbf{P}^3$.

THEOREM 3.4. *Let S be an Enriques surface with a divisor $D = e_1 + e_2 + e_3 + e_4$ with the configuration in Figure 1, where $p_1, \cdots, p_5$ are distinct points. Then S is birationally isomorphic to a normal quintic surface F_5 satisfying the property $\mathcal{P}$ of* Theorem 2.1 *with possibly finitely many rational double points. In particular, two triple points and two tacnodes are cusp singularities*

Proof. Let $\phi : \tilde{S} \to S$ be the blow-up of S at $p_1, \cdots, p_5$. Let $\tilde{L}_1, \cdots, \tilde{L}_5$ be the exceptional divisors of the first kind and $\tilde{e}_1, \cdots, \tilde{e}_4$ the proper transforms of $e_1, \cdots, e_4$ by the map ϕ. Then we set

$$\begin{aligned}\tilde{D} &= \phi^*(K_S) + \tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4 + \tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_4 + \tilde{L}_5 \\ &= \phi^*(K_S + e_1 + e_2 + e_3 + e_4) - \tilde{L}_1 - \tilde{L}_2 - \tilde{L}_3 - \tilde{L}_4 - \tilde{L}_5.\end{aligned}$$

Let $\mathcal{L}$ be the sublinear system of $|D + K_S|$ which consists of divisors of $|D + K_S|$ with base points $p_1, \cdots, p_5$. Then the sublinear system $\mathcal{L}$ on

S may be identified with the complete linear system $|\tilde{D}|$ on $\tilde{S}$. We note that $\tilde{L}_1, \cdots, \tilde{L}_5$ are the only exceptional divisors of the first kind on $\tilde{S}$ satisfying $\tilde{D} \cdot \tilde{L}_i = 1$ for $i = 1, \cdots, 5$ and $K_{\tilde{S}} = \phi^*(K_S) + \sum_{i=1}^{5} L_i$. If we set $\sigma^*(H)$ to be $\tilde{D}$ at Lemma 3.1, then $\sum_{i=1}^{n} c_i m_i = 5$ because $c_i = 1$ and $m_i = 1$ for $i = 1, \cdots, 5$. Hence we could expect that the divisor $\tilde{D}$ may induce a morphism from $\tilde{S}$ to a surface in $\mathbf{P}^3$.

The sublinear system $\mathcal{L}$ has the following four generators:

$$\begin{aligned} D_1 &= e_1' + e_2 + e_3 + e_4 \\ D_2 &= e_1 + e_2' + e_3 + e_4 \\ D_3 &= e_1 + e_2 + e_3' + e_4 \\ D_4 &= e_1 + e_2 + e_3 + e_4', \end{aligned}$$

where $e_i' = e_i + K_S$.

Thus we could expect that $\dim|\tilde{D}| = 3$. Indeed, it is easy to compute the cohomology $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{D}))$ as follows.

Since $\tilde{D} = K_{\tilde{S}} + \tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4$, we get $K_{\tilde{S}} - \tilde{D} = -\tilde{e}_1 - \tilde{e}_2 - \tilde{e}_3 - \tilde{e}_4$. Now we compute the cohomologies $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{D}))$ and $H^2(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{D}))$:

$$\begin{aligned} H^2(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{D})) &= H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} - \tilde{D})) \\ &= H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{e}_1 - \tilde{e}_2 - \tilde{e}_3 - \tilde{e}_4)) \\ &= H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{D})) = 0 \\ H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{D})) &= H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} - \tilde{D})) \\ &= H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{e}_1 - \tilde{e}_2 - \tilde{e}_3 - \tilde{e}_4)) \\ &= H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{D})), \end{aligned}$$

where $\tilde{D} = \tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4$.

To compute the cohomology $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{D}))$, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{S}}(-\tilde{D}) \longrightarrow \mathcal{O}_{\tilde{S}} \longrightarrow \mathcal{O}_{\tilde{D}} \longrightarrow 0.$$

Taking cohomology, we get the long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{D})) &\longrightarrow H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) \longrightarrow H^0(\tilde{D}, \mathcal{O}_{\tilde{D}}) \\ &\longrightarrow H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{D})) \longrightarrow H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) \longrightarrow . \end{aligned}$$

Since $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = H^1(S, \mathcal{O}_S) = 0$, from the above exact sequence of cohomologies,

$$\begin{aligned} h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{D})) &= h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{D})) - h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}) + h^0(\tilde{D}, \mathcal{O}_{\tilde{D}}) \\ &= 0 - 1 + 4 = 3, \end{aligned}$$

where $h^0(\tilde{D}, \mathcal{O}_{\tilde{D}}) = 4$ because $\tilde{D}$ is composed of four mutually disjoint elliptic curves (i.e. degenerate elliptic fibres of type I_n) $\tilde{e}_1, \cdots, \tilde{e}_4$ and Ramanujam's Lemma (Lemma 1.2.4, [2]), which says that $H^0(D, \mathcal{O}_D) = \mathbb{C}$ for numerically 1-connected effective divisor D. We note that degenerate elliptic fibres I_n are numerically 1-connected.

Hence

$$h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{D})) = h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-\tilde{D})) = 3.$$

By applying Riemann-Roch formula,

$$\begin{aligned} h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{D})) &= h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{D})) - h^2(\tilde{S}, \mathcal{O}_{\tilde{S}}(\tilde{D})) + \frac{\tilde{D}^2 - \tilde{D} \cdot K_{\tilde{S}}}{2} + \chi(\mathcal{O}_{\tilde{S}}) \\ &= 3 - 0 + \frac{5-5}{2} + \chi(\mathcal{O}_{\tilde{S}}) \\ &= 3 + 1 = 4. \end{aligned}$$

Therefore $\dim|\tilde{D}| = 3$.

If $\psi = \psi|_{\tilde{D}}$ is the map induced by the complete linear system $|\tilde{D}|$, then the map ψ maps $\tilde{S}$ into $\mathbf{P}^3$, that is,

$$\psi : \tilde{S} \to F \subset \mathbf{P}^3,$$

where F is a surface in $\mathbf{P}^3$. Since $(D + K_S)^2 = 16$ and the linear system $|\tilde{D}|$ can be identified with the sublinear system $\mathcal{L}$ of $|D + K_S|$, the above map $\psi : \tilde{S} \to F$ corresponds to the projection

$$\pi_{\mathcal{L}} : S \subset \mathbf{P}^8 \longrightarrow F \subset \mathbf{P}^3$$

which is induced by the sublinear system $\mathcal{L}$. Next we prove that the map ψ is well defined. To do so, we have to show that the linear system $|\tilde{D}|$ is without fixed components and base-points-free.

From the above generators of $\mathcal{L}$, we see that the linear system $|\tilde{D}|$ does not have fixed components. Furthermore, the linear system $|\tilde{D}|$ is base-point-free since $p_1, \cdots, p_5$ are the only common points of $e_1, \cdots, e_4$. Then, the map induced by the complete linear system $|\tilde{D}|$, $\psi = \psi|_{\tilde{D}} : \tilde{S} \to F \subset \mathbf{P}^3$ is well defined, and is of degree one since $\tilde{D}^2 = 5$. So the surface F is a quintic surface in $\mathbf{P}^3$. Let us set $F = F_5$.

Next we claim that the quintic surface F_5 is normal, that is, F_5 has only isolated singularities. By the adjunction formula, $p_a(\tilde{D}) = (\tilde{D} \cdot \tilde{D} + \tilde{D} \cdot K_{\tilde{S}})/2 + 1 = 6$. Obviously, the geometric genus of generic curves on $\tilde{S}$ is invariant under the map $\psi : \tilde{S} \to F_5$ since it is generically one-to-one. Thus the geometric genus of a generic hyperplane section of F_5 is also 6 since by Bertini's theorem, generic divisors of $|\tilde{D}|$ are non-singular curves with the geometric genus 6. While the maximum possible geometric genus of a hyperplane section of F_5 is 6 since it is a plane curve of degree five. Hence the quintic surface F_5 would have to have only isolated singularities, that is, F_5 is a normal quintic surface in $\mathbf{P}^3$.

Since $\tilde{D} \cdot \tilde{e}_i = 0$, $i = 1, \cdots, 4$ and $\tilde{D} \cdot \tilde{L}_j = 1$, $j = 1, \cdots, 5$, the map ψ maps elliptic curves $\tilde{e}_1, \cdots, \tilde{e}_4$ to points $A_1, \cdots, A_4$ of F_5, and $\tilde{L}_1, \cdots, \tilde{L}_5$ to lines $L_1, \cdots, L_5$ of F_5. The normal quintic surface F_5 may have finitely many rational double points. These rational double points are contractions of the proper transforms of non-singular rational curves R on S with $R^2 = -2$ by the map $\psi : \tilde{S} \to F_5$. Let us show that F_5 has no more singular points. Suppose that $\tilde{C}$ is a curve on $\tilde{S}$ other than $\tilde{e}_1, \cdots, \tilde{e}_4$, which is mapped to a point by the map ψ. Then the Hodge index theorem implies that $\tilde{C}^2 < 0$. Since $\tilde{C} \cdot K_{\tilde{S}} = 0$, the adjunction formula implies that $\chi(\mathcal{O}_{\tilde{C}}) > 0$. Hence the curve $\tilde{C}$ is a non-singular rational curve and $\tilde{C}^2 = -2$.

Since isolated elliptic curves $e_1, \cdots, e_4$ are divisors of type I_n, $\tilde{e}_1, \cdots, \tilde{e}_4$ are exceptional sets over $A_1, \cdots, A_4$ of the same type I_n, $0 \le n \le 9$. In particular, the geometric genus of singular points $A_1, \cdots, A_4$ can not be zero. Then from the equation (2) of Corollary 1.3, we obtain that $h(A_i) = 1$ for all $i = 1, \cdots, 4$. Thus singular points $A_1, \cdots, A_4$ are minimally elliptic singular points of F_5, which are also cusp singularities. Obviously, the fundamental cycle on e_i is e_i itself for $1 \le i \le 4$. Furthermore, since $\tilde{e}_2^2 = \tilde{e}_4^2 = -2$ and $\tilde{e}_1^2 = \tilde{e}_3^2 = -3$, A_1 and A_3 (A_2 and A_4) are triple points (tacnodal points) of type I_n, $0 \le n \le 9$. We also note that since $p_g(\tilde{S}) = 0$, four points $A_1, \cdots, A_4$ are in general position.

Now it remains to prove that the normal quintic surface F_5 satisfies the property $\mathcal{P}$ of Theorem 2.1. Let us set a divisor $\tilde{D}_1$ as follows:

$$\begin{aligned}\tilde{D}_1 &= \tilde{L}_1 + 2\tilde{L}_2 + 2\tilde{L}_5 + \tilde{e}_1 + 2\tilde{e}_2 + \tilde{e}_3 \\ &= \phi^*(e_1 + 2e_2 + e_3) - \tilde{L}_1 - \tilde{L}_2 - \tilde{L}_3 - \tilde{L}_4 - \tilde{L}_5.\end{aligned}$$

Then $\tilde{D}_1 \in |\tilde{D}|$ since

$$\begin{aligned}&e_1 + 2e_2 + e_3 \\ \sim &e_1 + e_2 + e_3 + e_2 + K_S + K_S \\ \sim &e_1 + e_2 + e_3 + e_2' + K_S \\ = &e_1 + e_2 + e_3 + e_4 + K_S.\end{aligned}$$

It says that the divisor $\tilde{D}_1$ on $\tilde{S}$ corresponds to a plane α_1 of $\mathbf{P}^3$ which cuts the divisor $L_1 + 2L_2 + 2L_5$ out of the normal quintic surface F_5. Hence it is the tacnodal plane to F_5 at A_2 which contains two triple points A_1 and A_3. Similarly there exists a tacnodal plane α_2 to F_5 at A_4 containing two triple points A_1 and A_3. □

Let $C = e_1 + e_2 + e_3$, where e_1, e_2, e_3 are half-pencils on an Enriques surface S and $e_i \cdot e_j = 1$ for $i \neq j$. We said that the linear system $|C|$ on S is superelliptic if $e_1 + e_2 - e_3$ is effective. This means that $|C|$ is superelliptic if there are half-pencils e_1, e_2, e_3 on S with $e_i \cdot e_j = 1$ for $i \neq j$ such that $C \sim e_1 + e_2 + e_3$ and $e_1 + e_2 - e_3$ is effective. Particularly the effectiveness condition on $e_1 + e_2 - e_3$ is independent of the order of e_1, e_2, e_3, which means that the linear system $|C|$ is superelliptic if one of divisors $e_1 + e_2 - e_3, e_1 + e_3 - e_2, e_2 + e_3 - e_1$ is effective. Hence the linear system $|C|$ is not superelliptic if any of divisors $e_1 + e_2 - e_3, e_1 + e_3 - e_2, e_2 + e_3 - e_1$ is not effective. The following proposition is a revised version of the author's lemma at his thesis after Professor Igor Dolgachev's remark.

PROPOSITION 3.5. *If both $|C|$ and $|C + K_S|$ are not superelliptic linear systems on S, then either e_2 or e_4 of the divisor D does not pass through the point p_1, the intersection point of e_1 and e_3, which implies that the Enriques surface S is birationally isomorphic to a normal quintic surface in $\mathbf{P}^3$ with the property $\mathcal{P}$.*

Proof. Contrary let us assume that e_2 passes through the point p_1. Then from the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(e_1 - e_2 - e_3) \longrightarrow \mathcal{O}_S(e_1 - e_2) \longrightarrow \mathcal{O}_{e_3}(e_1 - e_2) \longrightarrow 0,$$

we have a long exact sequence of cohomologies

$$\begin{aligned} 0 &\longrightarrow H^0(S, \mathcal{O}_S(e_1 - e_2 - e_3)) \longrightarrow H^0(S, \mathcal{O}_S(e_1 - e_2) \\ &\longrightarrow H^0(e_3, \mathcal{O}_{e_3}(e_1 - e_2)) \longrightarrow H^1(S, \mathcal{O}_S(e_1 - e_2 - e_3)) \longrightarrow . \end{aligned}$$

We note that $H^0(S, \mathcal{O}_S(e_1 - e_2)) = 0$, otherwise there is an effective divisor E which is linear equivalent to $e_1 - e_2$. Then $e_1 \cdot E = e_1 \cdot (e_1 - e_2) = -1$, which is impossible. Now $H^0\big(e_3, \mathcal{O}_{e_3}(e_1 - e_2)\big) = \mathbf{C}$ since $(e_1 - e_2)|_{e_3}$ is a zero divisor on e_3. This in turn implies that $H^0(S, \mathcal{O}_S(e_1 - e_2 - e_3)) = 0$. Next we claim that $H^1\big(S, \mathcal{O}_S(e_1 - e_2 - e_3)\big) = 0$, which induces a contradiction to the exactness of the above long exact sequence at $H^0(e_3, \mathcal{O}_{e_3}(e_1 - e_2))$. From Riemann-Roch formula,

$$h^0(S, \mathcal{O}_S(e_1 - e_2 - e_3)) - h^1(S, \mathcal{O}_S(e_1 - e_2 - e_3)) \\ +h^2(S, \mathcal{O}_S(e_1 - e_2 - e_3)) = 1 + 1/2\ (e_1 - e_2 - e_3)^2 = 0.$$

Since $h^0(S, \mathcal{O}_S(e_1 - e_2 - e_3)) = 0$,

$$h^1(S, \mathcal{O}_S(e_1 - e_2 - e_3)) = h^2(S, \mathcal{O}_S(e_1 - e_2 - e_3)) \\ = h^0(S, \mathcal{O}_S(K_S + e_2 + e_3 - e_1)) = 0.$$

The last identity follows from the assumption that $|C + K_S|$ is not a superelliptic linear system on S. Similarly we conclude that e_4 does not pass through the point p_1 from the assumption that $|C|$ is not superelliptic. Theorem 3.4 then implies that the Enriques surface S is birationally isomorphic to a normal quintic surface in $\mathbf{P}^3$ with the property $\mathcal{P}$. □

It is known that a generic Enriques surface S is birationally isomorphic to the classic Enriques sextic with the defining equation:

$$X_0X_1X_2X_3Q(X) + \lambda_1 X_1^2X_2^2X_3^2 + \lambda_2 X_0^2X_2^2X_3^2 \\ +\lambda_3 X_0^2X_1^2X_3^2 + \lambda_4 X_0^2X_1^2X_2^2 = 0,$$

where $Q(X)$ is a homogeneous polynomial of degree two and $\lambda_i \in \mathbb{C}$ for $i = 1, \cdots, 4$ if S has a divisor $C = e_1+e_2+e_3$ such that $|C|$ and $|C+K_S|$ both are not superelliptic, where e_1, e_2, e_3 are half-pencils with $e_i \cdot e_j = 1$ for $i \neq j$ [3]. This also follows from Proposition 3.5 since G. Castelnuovo showed that a classic Enriques sextic is birationally isomorphic to the Stagnaro's first model.

Let $D' = D + K_S = e_1 + e_2 + e_3 + e_4 + K_S \sim e_1 + 2e_2 + e_3$, where $e_4 = e_2'$. Then the divisor D' on S induces a map $\pi_{D'} : S \to \bar{S} \subset \mathbf{P}^5$, where $\bar{S}$ is a non-normal surface of degree 10, [4]. $\bar{S}$ has two double lines ℓ_1, ℓ_2 which are the images of e_2, e_4 by the map $\pi_{D'}$. Thus the normal quintic surface F_5 with the property $\mathcal{P}$ is the projection of $\bar{S}$ from a very special line ℓ^* of $\mathbf{P}^5$. If we set $\mathbf{P}^5 = |D'| = |D + K_S|$, then the line ℓ^* is the complement of the sublinear system $\mathcal{L}$ of $|D + K_S|$ in $\mathbf{P}^5$, which was introduced at the proof of Theorem 3.4. The projection of $\bar{S}$ from either the line ℓ_1 or the line ℓ_2 is the well known Enriques sextic model, that is, a sextic surface with six edges of a tetrahedron as double lines when $|C|$ and $|C + K_S|$ are not superelliptic for $C = e_1 + e_2 + e_3$.

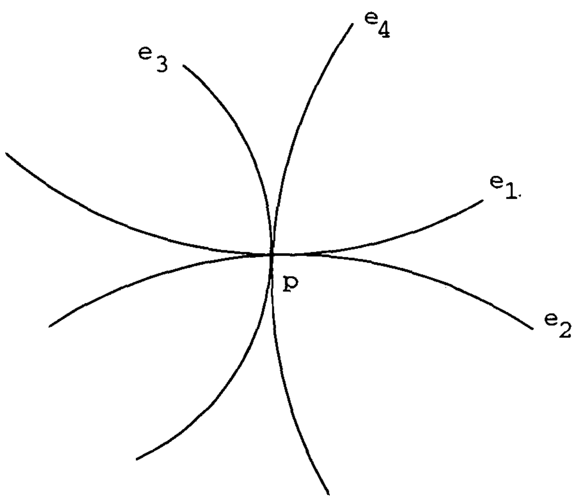

FIGURE 2

4. Stagnaro's second birational model of Enriques surfaces

We summarize similar results for Stagnaro's second model without proofs. Proofs are almost identical to the corresponding statements for the first model, and the details are in [6].

THEOREM 4.1 (E. Stagnaro [8]). *Let F_5 be a normal quintic surface in $\mathbf{P}^3$ with the following condition $\mathcal{Q}$:*

> *F_5 has four tacnodal points at the vertices A_1, A_2, A_3, A_4 of a tetrahedron T such that tacnodal planes to F_5 at A_1, A_2 and A_3, A_4 are identical.*

If S is a minimal non-singular model of F_5, then S is an Enriques surface.

As for the first model, we fix four points of a tetrahedron T, say $A_1 = (1,0,0,0), A_2 = (0,0,1,0), A_3 = (0,1,0,0), A_4 = (0,0,0,1)$.

PROPOSITION 4.2. *Let F_5 be the normal quintic surface in $\mathbf{P}^3$ satisfying the property $\mathcal{Q}$ of* Theorem 4.1. *Then F_5 contains three lines L_1, L_1' and L_2 ; the lines $L_1 = \overline{A_1A_2}$ and $L_1' = \overline{A_3A_4}$ are lines joining two vertices of the tetrahedron T and L_2 is the intersection of two tacnodal planes α_1 and α_2. Furthermore, the normal quintic surface F_5 has the following equation as its defining equation:*

$$\begin{aligned} F\ :\ & (x_2^3 + x_4^3)(x_1 + x_3)^2 \\ & +(x_1^3 + x_3^3)(x_2 + x_4)^2 \\ & +\big(a_1x_1x_2x_3 + a_2x_1x_2x_4 + a_3x_1x_3x_4 + a_4x_2x_3x_4\big)(x_1 + x_3)(x_2 + x_4) \\ & +a_5x_2^2x_4^2(x_1 + x_3) + a_6x_1^2x_3^2(x_2 + x_4) = 0\,;\ a_5 \neq 0\,,\ a_6 \neq 0. \end{aligned}$$

PROPOSITION 4.3. *If S is the Enriques surface obtained from the normal quintic surface F_5 satisfying the condition $\mathcal{Q}$ of* Theorem 4.1, *then S has a divisor $D = e_1 + e_2 + e_3 + e_4$ with the configuration in the Figure 2, where $e_1, \cdots, e_4$ are isolated elliptic curves.*

THEOREM 4.4. *Let S be an Enriques surface with a divisor $D = e_1 + e_2 + e_3 + e_4$ with the configuration in Figure 2, that is, e_1, e_2, e_3, e_4 are isolated elliptic curves and $e_1 \cdot e_3 = e_1 \cdot e_4 = e_2 \cdot e_3 = e_2 \cdot e_4 = 1$, and $e_1 \cdot e_2 = e_3 \cdot e_4 = 2$, where e_1, e_2 and e_3, e_4 meet tangentially at a point p.*

Then the following statements are true:

1. *If the adjoints e_1', e_2', e_3' and e_4' do not have a common point, then S is birationally isomorphic to a normal quintic surface F_5 in $\mathbf{P}^3$ satisfying the property $\mathcal{Q}$ of* Theorem 4.1 *with possibly finitely many rational double points, where four tacnodes are cusp singularities.*
2. *If the adjoints e_1', e_2', e_3' and e_4' have a common point, then S is birational to a surface which is mapped two to one onto a quadric surface Q in $\mathbf{P}^3$.*

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