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SOME FORMULAS FOR THE GENERALIZED HARDIE-JANSEN PRODUCT AND ITS DUAL

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ABSTRACT. The generalized Hardie-Jansen product and its dual are defined and the fundamental results on these products are obtained. By studying the adjoint maps, we give proofs to them. Moreover we characterize the generalized Hardie-Jansen product making use of the ΓW-Whitehead product. We also obtain a characterization of the dual of the generalized Hardie-Jansen product using $(\Gamma W)^*$-Whitehead product.

Introduction

We work in the category of compactly generated Hausdorff spaces with nondegenerate base point $*$ [8]. Let $X \wedge Y$ be the smash product of topological spaces X and Y. We denote an element of $X \wedge Y$ by $x \wedge y$ for any $x \in X$ and $y \in Y$. Let Γ be a co-Hopf space. We define $\Gamma X = \Gamma \wedge X$. The space ΓX is called the Γ-suspension space of a space X. Let X^W be the space of base point preserving maps from W to X. We define $\Gamma^* X = X^\Gamma$. The space $\Gamma^* X$ is called the Γ-loop space of a space X. Let $[A, Z]$ be the set of base point preserving homotopy classes of base point preserving maps from A to Z. For any maps $\alpha, \beta : A \to Z$, we define the following maps; if A is a co-Hopf space and $\nu : A \to A \vee A$ is the co-multiplication of A, then we define a map

$$\alpha \dot{+} \beta = \nabla_Z \circ (\alpha \vee \beta) \circ \nu : A \to Z$$

where $\nabla_Z : Z \vee Z \to Z$ is the folding map; or if Z is a Hopf space and $\mu : Z \times Z \to Z$ is the multiplication of Z, then we define a map

$$\alpha \underset{\cdot}{+} \beta = \mu \circ (\alpha \times \beta) \circ \Delta_A : A \to Z$$

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where $\Delta_A : A \to A \times A$ is the diagonal map. We use the same symbol for a homotopy class and a map which represents the homotopy class.

Now let Γ be a co-grouplike space, namely an associative co-Hopf space with an inverse. Throughout this paper, Γ is a co-grouplike space, except otherwise stated explicitly. Then the Γ-Whitehead product is defined in [6]. It is an element $[\alpha, \beta]_\Gamma \in [\Gamma(X \wedge Y), Z]$ for any elements $\alpha \in [\Gamma X, Z]$ and $\beta \in [\Gamma X, Z]$. If $\Gamma = S^1$, then the Γ-Whitehead product $[\alpha, \beta]_\Gamma$ is the generalized Whitehead product $[\alpha, \beta]$ of Arkowitz [1] and Barratt [2]. Let

$$\theta : [\Gamma X \wedge W, Z] \longrightarrow [\Gamma X, Z^W]$$

be the adjoint isomorphism for a fixed space W. The generalized Hardie-Jansen product is defined in [7] as follows, using the Γ-Whitehead product and the adjoint isomorphism; for any elements $\alpha \in [\Gamma X \wedge W, Z]$ and $\beta \in [\Gamma Y \wedge W, Z]$, we define the element $[\alpha, \beta]_\Gamma^W \in [\Gamma(X \wedge Y) \wedge W, Z]$ by $\theta^{-1}([\theta(\alpha), \theta(\beta)]_\Gamma)$. If $\Gamma = S^1$, $X = S^{m-1}$ and $Y = S^{n-1}$, then this product is the Hardie-Jansen product in [3] and [4]. In §1, we prove the following results of the generalized Hardie-Jansen product.

THEOREM 1.5 *Let $\alpha \in [\Gamma X \wedge W, Z]$ and $\beta \in [\Gamma Y \wedge W, Z]$ be any elements. Then we have the following results.*

(i) $\gamma \circ [\alpha, \beta]_\Gamma^W = [\gamma \circ \alpha, \gamma \circ \beta]_\Gamma^W$ *for any* $\gamma : Z \to Z'$.

(ii) $[\alpha, \beta]_\Gamma^W \circ (\Gamma(\delta \wedge \varepsilon) \wedge 1_W) = [\alpha \circ (\Gamma\delta \wedge 1_W), \beta \circ (\Gamma\varepsilon \wedge 1_W)]_\Gamma^W$ *for any* $\delta : X' \to X$ *and* $\varepsilon : Y' \to Y$.

THEOREM 1.6 *Let $T : Y \wedge X \to X \wedge Y$ be the switching map. Let $\alpha \in [\Gamma X \wedge W, Z]$ and $\beta \in [\Gamma Y \wedge W, Z]$ be any elements. Then the product $[\alpha, \beta]_\Gamma^W \in [\Gamma(X \wedge Y) \wedge W, Z]$ satisfies*

$$(\Gamma T \wedge 1_W)^*([\alpha, \beta]_\Gamma^W) = \dot{-}\, [\beta, \alpha]_\Gamma^W.$$

THEOREM 1.7 *If X and Y are co-Hopf spaces, then we have*

(i) $[\alpha \dotplus \beta, \gamma]_\Gamma^W = [\alpha, \gamma]_\Gamma^W \dotplus [\beta, \gamma]_\Gamma^W$ *for any* $\alpha, \beta \in [\Gamma X \wedge W, Z]$ *and* $\gamma \in [\Gamma Y \wedge W, Z]$.

(ii) $[\alpha, \beta \dotplus \gamma]_\Gamma^W = [\alpha, \beta]_\Gamma^W \dotplus [\alpha, \gamma]_\Gamma^W$ *for any* $\alpha \in [\Gamma X \wedge W, Z]$ *and* $\beta, \gamma \in [\Gamma Y \wedge W, Z]$.

THEOREM 1.8 *If X_1, X_2 and X_3 are co-Hopf spaces, then we have*

$$[[\alpha_1,\alpha_2]^W_\Gamma,\alpha_3]^W_\Gamma \dotplus (\Gamma T_{231}\wedge 1_W)^*([[\alpha_2,\alpha_3]^W_\Gamma,\alpha_1]^W_\Gamma) \dotplus (\Gamma T_{312}\wedge 1_W)^*([[\alpha_3,\alpha_1]^W_\Gamma,\alpha_2]^W_\Gamma) = 0$$

for any $\alpha_1 \in [\Gamma X_1 \wedge W, Z]$, $\alpha_2 \in [\Gamma X_2 \wedge W, Z]$ and $\alpha_3 \in [\Gamma X_3 \wedge W, Z]$, where $T_{ijk} : X_1 \wedge X_2 \wedge X_3 \to X_i \wedge X_j \wedge X_k$ is the permutation of factors of the smash products.

THEOREM 1.10 *Let Γ_1 be a co-grouplike space and Γ_2 a co-Hopf space. Then we have $\Gamma_2([\alpha,\beta]^W_{\Gamma_1}) = 0$ for any elements $\alpha \in [\Gamma_1 X \wedge W, Z]$ and $\beta \in [\Gamma_1 Y \wedge W, Z]$.*

In §2, we consider dual results. The Γ^*-Whitehead product is defined in [6]. It is an element $[\alpha,\beta]_{\Gamma^*} \in [A, \Gamma^*(X \flat Y)]$ for any elements $\alpha \in [A, \Gamma^* X]$ and $\beta \in [A, \Gamma^* Y]$. We define an isomorphism

$$\tau : [A, \Gamma^*(X^W)] \xrightarrow{\eta} [A, (\Gamma^* X)^W] \xrightarrow{\theta^{-1}} [A \wedge W, \Gamma^* X].$$

where η is an isomorphism induced by a natural homeomorphism and θ^{-1} is the inverse map of θ. Hardie and Jansen do not formulate the dual product. Using the Γ^*-Whitehead product and the isomorphism τ, we define the dual of the generalized Hardie-Jansen product as follows. For any element $\alpha \in [A, \Gamma^*(X^W)]$ and $\beta \in [A, \Gamma^*(Y^W)]$, it is an element $[\alpha,\beta]^W_{\Gamma^*} = \tau^{-1}([\tau(\alpha),\tau(\beta)]_{\Gamma^*}) \in [A, \Gamma^*((X \flat Y)^W)]$. In this section, we prove the following results of the dual of the generalized Hardie-Jansen product.

THEOREM 2.6 *Let $\alpha \in [A, \Gamma^*(X^W)]$ and $\beta \in [A, \Gamma^*(Y^W)]$ be any elements. Then we have the following results.*

(i) $[\alpha,\beta]^W_{\Gamma^*} \circ \gamma = [\alpha\circ\gamma, \beta\circ\gamma]^W_{\Gamma^*}$ *for any $\gamma : A' \to A$.*

(ii) $(\Gamma^*((\delta \flat \varepsilon)^W)) \circ [\alpha,\beta]^W_{\Gamma^*} = [(\Gamma^*(\delta^W))\circ\alpha, (\Gamma^*(\varepsilon^W))\circ\beta]^W_{\Gamma^*}$ *for any $\delta : X \to X'$ and $\varepsilon : Y \to Y'$.*

THEOREM 2.7 *Let $\widetilde{T} : X \flat Y \to Y \flat X$ be the switching map between induced fibres. Let $\alpha \in [A, \Gamma^*(X^W)]$ and $\beta \in [A, \Gamma^*(Y^W)]$ be any elements. Then the product $[\alpha,\beta]^W_{\Gamma^*} \in [A, \Gamma^*((X \flat Y)^W)]$ satisfies*

$$(\Gamma^*(\widetilde{T}^W))_*([\alpha,\beta]^W_{\Gamma^*}) = \dot{-}\, [\beta,\alpha]^W_{\Gamma^*}.$$

THEOREM 2.9 *Let Γ_1 be a co-grouplike space and Γ_2 a co-Hopf space. Then we have $\Gamma_2^*([\alpha,\beta]_{\Gamma_1^*}^W) = 0$ for any $\alpha \in [A, \Gamma_1^*(X^W)]$ and $\beta \in [A, \Gamma_1^*(Y^W)]$.*

In §3, we consider an isomorphism $\xi_{W\wedge X}$ which is closely connected with the generalized Hardie-Jansen product. We define the isomorphism

$$\xi_{W\wedge X} : [\Gamma X \wedge W, Z] \longrightarrow [\Gamma W \wedge X, Z]$$

induced by the switching map $t_{W\wedge X} : W \wedge X \to X \wedge W$. Then we have a relation between the generalized Hardie-Jansen product and ΓW-Whitehead product in [7]. It is the following relation

$$[\alpha,\beta]_\Gamma^W = \xi_{W\wedge(X\wedge Y)}^{-1}([\xi_{W\wedge X}(\alpha), \xi_{W\wedge Y}(\beta)]_{\Gamma W})$$

for any elements $\alpha \in [\Gamma X \wedge W, Z]$ and $\beta \in [\Gamma Y \wedge W, Z]$. Hence the isomorphism $\xi_{W\wedge X}$ is also useful to investigate the properties of the generalized Hardie-Jansen product. In this section we prove some fundamental properties of the isomorphism $\xi_{W\wedge X}$.

We also consider an isomorphism $\zeta_{W^*X} : [A, \Gamma^*(X^W)] \to [A, (\Gamma W)^*X]$ for the characterization of the dual of the generalized Hardie-Jansen product. We prove some fundamental properties of the isomorphism ζ_{W^*X}. And we have a relation between the dual of the generalized Hardie-Jansen product and $(\Gamma W)^*$-Whitehead product.

THEOREM 3.7 *Let $\alpha \in [A, \Gamma^*(X^W)]$ and $\beta \in [A, \Gamma^*(Y^W)]$ be any elements. Then we have*

$$[\alpha,\beta]_{\Gamma^*}^W = \zeta_{W^*(X\flat Y)}^{-1}([\zeta_{W^*X}(\alpha), \zeta_{W^*Y}(\beta)]_{(\Gamma W)^*}).$$

This isomorphism ζ_{W^*X} is also useful to investigate the properties of the dual of the generalized Hardie-Jansen product.

1. Formulas for generalized Hardie-Jansen product

Let W be a fixed space. We define an adjoint map

$$\theta : [X \wedge W, Z] \longrightarrow [X, Z^W]$$

by $(\theta(\alpha)(x))(w) = \alpha(x \wedge w)$ for any map $\alpha : X \wedge W \to Z$ and any elements $x \in X$ and $w \in W$. In the following lemmas, we obtain some formulas for θ.

LEMMA 1.1. *Let $f : X \to Y$ be any map. Then there is a commutative diagram*

$$\begin{array}{ccc} [Y \wedge W, Z] & \xrightarrow{(f\wedge 1_W)^*} & [X \wedge W, Z] \\ \theta \downarrow & & \downarrow \theta \\ [Y, Z^W] & \xrightarrow[f^*]{} & [X, Z^W]. \end{array}$$

Therefore we have $\theta(\alpha)\circ f = \theta(\alpha\circ(f\wedge 1_W))$ for any map $\alpha : Y\wedge W \to Z$.

Proof. By the definition of θ, we have

$$\begin{aligned} (\theta(\alpha \circ (f \wedge 1_W))(x))(w) &= (\alpha \circ (f \wedge 1_W))(x \wedge w) \\ &= \alpha(f(x) \wedge w) \\ &= ((\theta(\alpha) \circ f)(x))(w) \end{aligned}$$

for any elements $x \in X$ and $w \in W$.

LEMMA 1.2. *Let $f : X \to Y$ be any map. Then there is a commutative diagram*

$$\begin{array}{ccc} [A \wedge W, X] & \xrightarrow{f_*} & [A \wedge W, Y] \\ \theta \downarrow & & \downarrow \theta \\ [A, X^W] & \xrightarrow[(f^W)_*]{} & [A, Y^W]. \end{array}$$

Therefore we have $f^W \circ \theta(\alpha) = \theta(f \circ \alpha)$ for any $\alpha : A \wedge W \to X$.

Proof. By the definition of θ, we have

$$\begin{aligned} ((f^W \circ \theta(\alpha))(x))(w) &= (f \circ \theta(\alpha)(x))(w) \\ &= f((\theta(\alpha)(x))(w)) \\ &= f(\alpha(x \wedge w)) \\ &= (f \circ \alpha)(x \wedge w) = (\theta(f \circ \alpha)(x))(w) \end{aligned}$$

for any elements $x \in A$ and $w \in W$.

LEMMA 1.3. (i) *Let X be a co-Hopf space. For any elements α and β* in $[X \wedge W, Z]$, we have

$$\theta(\alpha \dotplus \beta) = \theta(\alpha) \dotplus \theta(\beta) \in [X, Z^W].$$

(ii) *Let Z be a Hopf space. For any elements α and β in $[X \wedge W, Z]$, we have*

$$\theta(\alpha \dotplus \beta) = \theta(\alpha) \dotplus \theta(\beta) \in [X, Z^W].$$

Proof. (i) Let ν be a co-multiplication of X. Then the co-multiplication of $X \wedge W$ is obtained by

$$\nu \wedge 1_W : X \wedge W \longrightarrow (X \vee X) \wedge W \approx (X \wedge W) \vee (X \wedge W).$$

We have

$$\begin{aligned}\theta(\alpha \dotplus \beta) &= \theta(\nabla_Z \circ (\alpha \vee \beta) \circ (\nu \wedge 1_W)) \\ &= \theta(\nabla_Z \circ (\alpha \vee \beta)) \circ \nu.\end{aligned}$$

Here, we see $\theta(\nabla_Z \circ (\alpha \vee \beta)) \circ j_1 = \theta(\nabla_Z \circ (\alpha \vee \beta) \circ (j_1 \wedge 1_W)) = \theta(\alpha)$ and $\theta(\nabla_Z \circ (\alpha \vee \beta)) \circ j_2 = \theta(\beta)$ where j_1, $j_2 : X \to X \vee X$ is inclusions defined by $j_1(x) = (x, *)$ and $j_2(x) = (*, x)$ for any element $x \in X$. Hence we have

$$\theta(\alpha \dotplus \beta) = \nabla_{Z^W} \circ (\theta(\alpha) \vee \theta(\beta)) \circ \nu = \theta(\alpha) \dotplus \theta(\beta).$$

(ii) Let μ be a multiplication of Z. Then the multiplication of Z^W is obtained by

$$\mu^W : Z^W \times Z^W \approx (Z \times Z)^W \longrightarrow Z^W.$$

We have

$$\begin{aligned}\theta(\alpha \dotplus \beta) &= \theta(\mu \circ (\alpha \times \beta) \circ \Delta_{X \wedge W}) \\ &= \mu^W \circ \theta((\alpha \times \beta) \circ \Delta_{X \wedge W}).\end{aligned}$$

Here, we see $p_1^W \circ \theta((\alpha \times \beta) \circ \Delta_{X \wedge W}) = \theta(p_1 \circ (\alpha \times \beta) \circ \Delta_{X \wedge W}) = \theta(\alpha)$ and $p_2^W \circ \theta((\alpha \times \beta) \circ \Delta_{X \wedge W}) = \theta(\beta)$ where p_1, $p_2 : X \times X \to X$ are projections to the first and the second factors respectively. Hence we have

$$\theta(\alpha \dotplus \beta) = \mu^W \circ (\theta(\alpha) \times \theta(\beta)) \circ \Delta_X = \theta(\alpha) \dotplus \theta(\beta).$$

We notice that θ is an isomorphism of groups when the domain and the codomain of θ have the group structure. (cf. Theorem 6.3.28 Maunder [5])

We recall the definition of the generalized Hardie-Jansen product in [7]. It defines a pairing

$$[\Gamma X \wedge W, Z] \times [\Gamma Y \wedge W, Z] \longrightarrow [\Gamma(X \wedge Y) \wedge W, Z]$$

by $[\alpha,\beta]_\Gamma^W = \theta^{-1}([\theta(\alpha),\theta(\beta)]_\Gamma) \in [\Gamma(X\wedge Y)\wedge W, Z]$ for any elements $\alpha \in [\Gamma X\wedge W, Z]$ and $\beta \in [\Gamma Y\wedge W, Z]$, where $[\,\cdot\,,\,\cdot\,]_\Gamma$ is the Γ-Whitehead product. Now we prove the following results by making use of the lemmas mentioned above. These results are quoted in [7] without proof.

THEOREM 1.4. *If Γ is a commutative co-Hopf space or if W is a co-Hopf space or if Z is a Hopf space, then $[\alpha,\beta]_\Gamma^W = 0$ for any $\alpha \in [\Gamma X\wedge W, Z]$ and $\beta \in [\Gamma Y\wedge W, Z]$.*

Proof. Using Propositions 1.2 and 1.3 of [6], we have the result.

THEOREM 1.5. *Let $\alpha \in [\Gamma X\wedge W, Z]$ and $\beta \in [\Gamma Y\wedge W, Z]$ be any elements. Then we have the following results.*

(i) $\gamma\circ[\alpha,\beta]_\Gamma^W = [\gamma\circ\alpha,\gamma\circ\beta]_\Gamma^W$ *for any* $\gamma : Z \to Z'$.

(ii) $[\alpha,\beta]_\Gamma^W\circ(\Gamma(\delta\wedge\varepsilon)\wedge 1_W) = [\alpha\circ(\Gamma\delta\wedge 1_W),\beta\circ(\Gamma\varepsilon\wedge 1_W)]_\Gamma^W$ *for any* $\delta : X'\to X$ *and* $\varepsilon : Y'\to Y$.

Proof. (i) Using Lemma 1.2 and Proposition 1.1 (i) of [6], we have

$$\begin{aligned}\theta(\gamma\circ[\alpha,\beta]_\Gamma^W) &= \gamma^W\circ\theta([\alpha,\beta]_\Gamma^W)\\ &= \gamma^W\circ[\theta(\alpha),\theta(\beta)]_\Gamma\\ &= [\gamma^W\circ\theta(\alpha),\gamma^W\circ\theta(\beta)]_\Gamma\\ &= [\theta(\gamma\circ\alpha),\theta(\gamma\circ\beta)]_\Gamma\\ &= \theta([\gamma\circ\alpha,\gamma\circ\beta]_\Gamma^W).\end{aligned}$$

(ii) Using Lemma 1.1 and Proposition 1.1 (ii) of [6], we have

$$\begin{aligned}\theta([\alpha,\beta]_\Gamma^W\circ(\Gamma(\delta\wedge\varepsilon)\wedge 1_W)) &= \theta([\alpha,\beta]_\Gamma^W)\circ\Gamma(\delta\wedge\varepsilon)\\ &= [\theta(\alpha),\theta(\beta)]_\Gamma\circ\Gamma(\delta\wedge\varepsilon)\\ &= [\theta(\alpha)\circ\Gamma\delta,\theta(\beta)\circ\Gamma\varepsilon]_\Gamma\\ &= [\theta(\alpha\circ(\Gamma\delta\wedge 1_W)),\theta(\beta\circ(\Gamma\varepsilon\wedge 1_W))]_\Gamma\\ &= \theta([\alpha\circ(\Gamma\delta\wedge 1_W),\beta\circ(\Gamma\varepsilon\wedge 1_W)]_\Gamma^W).\end{aligned}$$

THEOREM 1.6. *Let $T : Y\wedge X\to X\wedge Y$ be the switching map. Let $\alpha\in[\Gamma X\wedge W, Z]$ and $\beta\in[\Gamma Y\wedge W, Z]$ be any elements. Then the product $[\alpha,\beta]_\Gamma^W \in [\Gamma(X\wedge Y)\wedge W, Z]$ satisfies*

$$(\Gamma T\wedge 1_W)^*([\alpha,\beta]_\Gamma^W) = \dot{-}\,[\beta,\alpha]_\Gamma^W.$$

Proof. Using Lemma 1.1 and Proposition 1.5 of [6], we have

$$\begin{aligned}\theta([\alpha,\beta]_\Gamma^W\circ(\Gamma T\wedge 1_W)) &= \theta([\alpha,\beta]_\Gamma^W)\circ\Gamma T\\ &= [\theta(\alpha),\theta(\beta)]_\Gamma\circ\Gamma T\\ &= \dot{-}\,[\theta(\beta),\theta(\alpha)]_\Gamma\\ &= \dot{-}\,\theta([\beta,\alpha]_\Gamma^W)\\ &= \theta(\dot{-}\,[\beta,\alpha]_\Gamma^W).\end{aligned}$$

THEOREM 1.7. (Bilinearity) *If X and Y are co-Hopf spaces, then we have*

(i) $[\alpha\dot{+}\beta,\gamma]_\Gamma^W=[\alpha,\gamma]_\Gamma^W\dot{+}[\beta,\gamma]_\Gamma^W$ *for any* $\alpha,\beta\in[\Gamma X\wedge W,Z]$ *and* $\gamma\in[\Gamma Y\wedge W,Z]$.

(ii) $[\alpha,\beta\dot{+}\gamma]_\Gamma^W=[\alpha,\beta]_\Gamma^W\dot{+}[\alpha,\gamma]_\Gamma^W$ *for any* $\alpha\in[\Gamma X\wedge W,Z]$ *and* $\beta,\gamma\in[\Gamma Y\wedge W,Z]$.

Proof. We only prove (i). Using Lemma 1.3 and Proposition 1.9 of [6], we have

$$\begin{aligned}\theta([\alpha\dot{+}\beta,\gamma]_\Gamma^W) &= [\theta(\alpha\dot{+}\beta),\theta(\gamma)]_\Gamma\\ &= [\theta(\alpha)\dot{+}\theta(\beta),\theta(\gamma)]_\Gamma\\ &= [\theta(\alpha),\theta(\gamma)]_\Gamma\dot{+}[\theta(\beta),\theta(\gamma)]_\Gamma\\ &= \theta([\alpha,\gamma]_\Gamma^W)\dot{+}\theta([\beta,\gamma]_\Gamma^W)\\ &= \theta([\alpha,\gamma]_\Gamma^W\dot{+}[\beta,\gamma]_\Gamma^W).\end{aligned}$$

THEOREM 1.8. (Jacobi identity) *If X_1, X_2 and X_3 are co-Hopf spaces, then we have*

$$\begin{aligned}&[[\alpha_1,\alpha_2]_\Gamma^W,\alpha_3]_\Gamma^W\dot{+}(\Gamma T_{231}\wedge 1_W)^*([[\alpha_2,\alpha_3]_\Gamma^W,\alpha_1]_\Gamma^W)\\ &\dot{+}(\Gamma T_{312}\wedge 1_W)^*([[\alpha_3,\alpha_1]_\Gamma^W,\alpha_2]_\Gamma^W)=0\end{aligned}$$

for any $\alpha_1\in[\Gamma X_1\wedge W,Z]$, $\alpha_2\in[\Gamma X_2\wedge W,Z]$ *and* $\alpha_3\in[\Gamma X_3\wedge W,Z]$, *where* $T_{ijk}:X_1\wedge X_2\wedge X_3\to X_i\wedge X_j\wedge X_k$ *is the permutation of factors of the smash products.*

Proof. Using Lemma 1.1 and Proposition 1.10 of [6], we have

$$\begin{aligned}
&\theta\left([[\alpha_1,\alpha_2]^W_\Gamma,\alpha_3]^W_\Gamma \dotplus (\Gamma T_{231}\wedge 1_W)^*([[\alpha_2,\alpha_3]^W_\Gamma,\alpha_1]^W_\Gamma)\right.\\
&\qquad \left.\dotplus (\Gamma T_{312}\wedge 1_W)^*([[\alpha_3,\alpha_1]^W_\Gamma,\alpha_2]^W_\Gamma)\right)\\
&= [\theta([\alpha_1,\alpha_2]^W_\Gamma),\theta(\alpha_3)]_\Gamma\\
&\qquad \dotplus [\theta([\alpha_2,\alpha_3]^W_\Gamma),\theta(\alpha_1)]_\Gamma\circ\Gamma T_{231}\\
&\qquad \dotplus [\theta([\alpha_3,\alpha_1]^W_\Gamma),\theta(\alpha_2)]_\Gamma\circ\Gamma T_{312}\\
&= [[\theta(\alpha_1),\theta(\alpha_2)]_\Gamma,\theta(\alpha_3)]_\Gamma \dotplus [[\theta(\alpha_2),\theta(\alpha_3)]_\Gamma,\theta(\alpha_1)]_\Gamma\circ\Gamma T_{231}\\
&\qquad \dotplus [[\theta(\alpha_3),\theta(\alpha_1)]_\Gamma,\theta(\alpha_2)]_\Gamma\circ\Gamma T_{312}\\
&= 0.
\end{aligned}$$

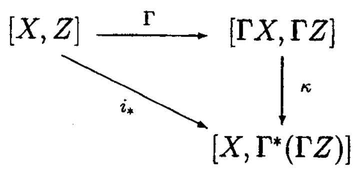
This completes the proof.

We define an inclusion map $i: Z \to \Gamma^*(\Gamma Z)$ by $i(z)(t) = t\wedge z$ for any $z\in Z$ and $t\in\Gamma$. Let

$$\Gamma : [X,Z] \longrightarrow [\Gamma X,\Gamma Z]$$

be the Γ-suspension map. The following lemma says that the inclusion map i and the Γ-suspension map are essentially equivalent.

Lemma 1.9. *Let $\kappa : [\Gamma X,\Gamma Z] \to [X,\Gamma^*(\Gamma Z)]$ be the adjoint isomorphism of Γ. Then the following diagram is commutative.*



Proof. For any map $\alpha : X \to Z$ and any elements $t\in\Gamma$, $x\in X$, we have $(\kappa(\Gamma\alpha)(x))(t) = \Gamma\alpha(t\wedge x) = t\wedge\alpha(x)$ and $(i_*(\alpha)(x))(t) = ((i\circ\alpha)(x))(t) = t\wedge\alpha(x)$. Hence we have the result.

Theorem 1.10. *Let Γ_1 be a co-grouplike space and Γ_2 a co-Hopf space. Then we have $\Gamma_2([\alpha,\beta]^W_{\Gamma_1}) = 0$ for any elements $\alpha\in[\Gamma_1X\wedge W, Z]$ and $\beta\in[\Gamma_1Y\wedge W, Z]$.*

Proof. Let $i : Z \to \Gamma_2^*(\Gamma_2 Z)$ be the inclusion map. By Lemma 1.9, we have

$$\begin{aligned} \kappa(\Gamma_2[\alpha,\beta]_{\Gamma_1}^W) &= i \circ [\alpha,\beta]_{\Gamma_1}^W \\ &= [i\circ\alpha, i\circ\beta]_{\Gamma_1}^W \text{ in } [\Gamma_1(X\wedge Y)\wedge W, \Gamma_2^*(\Gamma_2 Z)]. \end{aligned}$$

Since $\Gamma_2^*(\Gamma_2 Z)$ is a Hopf space, the element $[i\circ\alpha, i\circ\beta]_{\Gamma_1}^W$ vanishes by Theorem 1.4. Hence, we have the result.

2. Dual product

Let $t_{W\wedge\Gamma} : W\wedge\Gamma \to \Gamma\wedge W$ be the switching map. We define a natural homeomorphism $\tilde{t}_{W\wedge\Gamma} : \Gamma^*(X^W) \approx X^{\Gamma\wedge W} \to X^{W\wedge\Gamma} \approx (\Gamma^* X)^W$ induced by $t_{W\wedge\Gamma}$. And we define a map

$$\eta : [A, \Gamma^*(X^W)] \longrightarrow [A, (\Gamma^* X)^W]$$

by $\eta(\alpha) = \tilde{t}_{W\wedge\Gamma}\circ\alpha$ for any map $\alpha : A \to \Gamma^*(X^W)$. We notice that η is a bijection of sets. We now show some formulas for the bijection η.

LEMMA 2.1. *Let α and β be any elements in $[A, \Gamma^*(X^W)]$. Then we have*

$$\eta(\alpha \dot{+} \beta) = \eta(\alpha) \dot{+} \eta(\beta)$$

Proof. We see that the map $\tilde{t}_{W\wedge\Gamma}$ is a Hopf map. Then we have

$$\begin{aligned} \eta(\alpha \dot{+} \beta) &= \tilde{t}_{W\wedge\Gamma}\circ(\alpha \dot{+} \beta) \\ &= \tilde{t}_{W\wedge\Gamma}\circ\alpha \dot{+} \tilde{t}_{W\wedge\Gamma}\circ\beta \\ &= \eta(\alpha) \dot{+} \eta(\beta). \end{aligned}$$

PROPOSITION 2.2. *Let $\alpha : A \to \Gamma^*(X^W)$ be any map. Then we have*

(i) $\eta(\Gamma^*(f^W)\circ\alpha) = (\Gamma^* f)^W \circ \eta(\alpha)$ *for any map* $f : X \to Y$.

(ii) $\eta(\alpha\circ g) = \eta(\alpha)\circ g$ *for any map* $g : A' \to A$.

Proof. (i) The following diagram is commutative.

$$\begin{array}{ccc} \Gamma^*(X^W) & \xrightarrow{\tilde{t}_{W\wedge\Gamma}} & (\Gamma^* X)^W \\ {\scriptstyle \Gamma^*(f^W)}\downarrow & & \downarrow{\scriptstyle (\Gamma^* f)^W} \\ \Gamma^*(Y^W) & \xrightarrow{\tilde{t}_{W\wedge\Gamma}} & (\Gamma^* Y)^W \end{array}$$

Then we have $\eta(\Gamma^*(f^W)\circ\alpha) = \tilde{t}_{W\wedge\Gamma}\circ\Gamma^*(f^W)\circ\alpha = (\Gamma^* f)^W \circ \tilde{t}_{W\wedge\Gamma}\circ\alpha = (\Gamma^* f)^W \circ \eta(\alpha)$.
(ii) From the definition of η, we see $\eta(\alpha\circ g) = \tilde{t}_{W\wedge\Gamma}\circ\alpha\circ g = \eta(\alpha)\circ g$.

Let $\tau : [A, \Gamma^*(X^W)] \to [A\wedge W, \Gamma^* X]$ be the composition map of η and θ^{-1}, namely $\tau = \theta^{-1}\circ\eta$.

PROPOSITION 2.3. *Let α and β be any elements in $[A, \Gamma^*(X^W)]$. Then we have*

$$\tau(\alpha \dotplus \beta) = \tau(\alpha) \dotplus \tau(\beta).$$

Proof. From Lemmas 1.3 and 2.1, we have

$$\begin{aligned} \tau(\alpha \dotplus \beta) &= (\theta^{-1}\circ\eta)(\alpha \dotplus \beta) \\ &= \theta^{-1}(\eta(\alpha) \dotplus \eta(\beta)) \\ &= \theta^{-1}(\eta(\alpha)) \dotplus \theta^{-1}(\eta(\beta)) \\ &= \tau(\alpha) \dotplus \tau(\beta). \end{aligned}$$

The homomorphism τ is an isomorphism since η and θ are bijections. We obtain the following formulas for τ.

LEMMA 2.4. (i) *Let $f : X \to Y$ be any map. Then there is a commutative diagram*

$$\begin{array}{ccc} [A, \Gamma^*(X^W)] & \xrightarrow{(\Gamma^*(f^W))_*} & [A, \Gamma^*(Y^W)] \\ \tau \downarrow & & \downarrow \tau \\ [A\wedge W, \Gamma^* X] & \xrightarrow[(\Gamma^* f)_*]{} & [A\wedge W, \Gamma^* Y]. \end{array}$$

Therefore we have $\Gamma^* f\circ\tau(\alpha) = \tau(\Gamma^*(f^W)\circ\alpha)$ for any $\alpha : A \to \Gamma^*(X^W)$.
(ii) *Let $g : A' \to A$ be any map. Then there is a commutative diagram*

$$\begin{array}{ccc} [A, \Gamma^*(X^W)] & \xrightarrow{g^*} & [A', \Gamma^*(X^W)] \\ \tau \downarrow & & \downarrow \tau \\ [A\wedge W, \Gamma^* X] & \xrightarrow[(g\wedge 1_W)^*]{} & [A'\wedge W, \Gamma^* X]. \end{array}$$

Therefore we have $\tau(\alpha)\circ(g\wedge 1_W) = \tau(\alpha\circ g)$ for any $\alpha : A \to \Gamma^*(X^W)$.

Proof. (i) From Lemma 1.2 and (i) of Proposition 2.2, the result follows. (ii) From Lemma 1.1 and (ii) of Proposition 2.2, the result follows.

We recall the dual of the Hardie-Jansen product in [7]. It defines a pairing

$$[A,\Gamma^*(X^W)]\times[A,\Gamma^*(Y^W)]\longrightarrow[A,\Gamma^*((X\flat Y)^W)]$$

by $[\alpha,\beta]^W_{\Gamma^*}=\tau^{-1}([\tau(\alpha),\tau(\beta)]_{\Gamma^*})\in[A,\Gamma^*((X\flat Y)^W)]$ for any elements $\alpha\in[A,\Gamma^*(X^W)]$ and $\beta\in[A,\Gamma^*(Y^W)]$, where $[\,\cdot\,,\,\cdot\,]_{\Gamma^*}$ is the Γ^*-Whitehead product and $X\flat Y$ is the homotopy fibre of the inclusion map $j:X\vee Y\to X\times Y$. Now we give proofs to the following propositions. These results are quoted in [7] without proof.

THEOREM 2.5. *If Γ is a commutative co-Hopf space or if W or A is a co-Hopf space, then $[\alpha,\beta]^W_{\Gamma^*}=0$ for any $\alpha\in[A,\Gamma^*(X^W)]$ and $\beta\in[A,\Gamma^*(Y^W)]$.*

Proof. Using Proposition 2.2 of [6], we have the result.

THEOREM 2.6. *Let $\alpha\in[A,\Gamma^*(X^W)]$ and $\beta\in[A,\Gamma^*(Y^W)]$ be any elements. Then we have the following results.*

(i) $[\alpha,\beta]^W_{\Gamma^*}\circ\gamma=[\alpha\circ\gamma,\beta\circ\gamma]^W_{\Gamma^*}$ *for any* $\gamma:A'\to A$.

(ii) $(\Gamma^*((\delta\flat\varepsilon)^W))\circ[\alpha,\beta]^W_{\Gamma^*}=[(\Gamma^*(\delta^W))\circ\alpha,(\Gamma^*(\varepsilon^W))\circ\beta]^W_{\Gamma^*}$ *for any* $\delta:X\to X'$ *and* $\varepsilon:Y\to Y'$.

Proof. (i) Using Lemma 2.4 (ii) and Proposition 2.1 (i) of [6], we have

$$\begin{aligned}\tau([\alpha,\beta]^W_{\Gamma^*}\circ\gamma) &= [\tau(\alpha),\tau(\beta)]_{\Gamma^*}\circ(\gamma\wedge 1_W)\\ &= [\tau(\alpha)\circ(\gamma\wedge 1_W),\tau(\beta)\circ(\gamma\wedge 1_W)]_{\Gamma^*}\\ &= [\tau(\alpha\circ\gamma),\tau(\beta\circ\gamma)]_{\Gamma^*}\\ &= \tau([\alpha\circ\gamma,\beta\circ\gamma]^W_{\Gamma^*}).\end{aligned}$$

(ii) Using Lemma 2.4 (i) and Proposition 2.1 (ii) of [6], we have

$$\begin{aligned}\tau((\Gamma^*((\delta\flat\varepsilon)^W))\circ[\alpha,\beta]^W_{\Gamma^*}) &= \Gamma^*(\delta\flat\varepsilon)\circ\tau([\alpha,\beta]^W_{\Gamma^*})\\ &= \Gamma^*(\delta\flat\varepsilon)\circ[\tau(\alpha),\tau(\beta)]_{\Gamma^*}\\ &= [\Gamma^*\delta\circ\tau(\alpha),\Gamma^*\varepsilon\circ\tau(\beta)]_{\Gamma^*}\\ &= [\tau((\Gamma^*(\delta^W))\circ\alpha),\tau((\Gamma^*(\varepsilon^W))\circ\beta)]_{\Gamma^*}\\ &= \tau([(\Gamma^*(\delta^W))\circ\alpha,(\Gamma^*(\varepsilon^W))\circ\beta]^W_{\Gamma^*}).\end{aligned}$$

THEOREM 2.7. *Let $\widetilde{T} : X \flat Y \to Y \flat X$ be the switching map between induced fibres. Let $\alpha \in [A, \Gamma^*(X^W)]$ and $\beta \in [A, \Gamma^*(Y^W)]$ be any elements. Then the product $[\alpha, \beta]^W_{\Gamma^*} \in [A, \Gamma^*((X \flat Y)^W)]$ satisfies*

$$(\Gamma^*(\widetilde{T}^W))_*([\alpha, \beta]^W_{\Gamma^*}) = \underset{\cdot}{-} [\beta, \alpha]^W_{\Gamma^*} .$$

Proof. Using Lemma 2.4 (i) and Proposition 2.4 of [6], we have

$$\begin{aligned} \tau((\Gamma^*(\widetilde{T}^W)) \circ [\alpha, \beta]^W_{\Gamma^*}) &= \Gamma^* \widetilde{T} \circ \tau([\alpha, \beta]^W_{\Gamma^*}) \\ &= \Gamma^* \widetilde{T} \circ [\tau(\alpha), \tau(\beta)]_{\Gamma^*} \\ &= \underset{\cdot}{-} [\tau(\beta), \tau(\alpha)]_{\Gamma^*} \\ &= \tau(\underset{\cdot}{-} [\beta, \alpha]^W_{\Gamma^*}). \end{aligned}$$

We define the evaluation map $e : \Gamma(\Gamma^* A) \to A$ by $e(t \wedge \lambda) = \lambda(t)$ for any elements $t \in \Gamma$ and $\lambda \in \Gamma^* A$. Let

$$\Gamma^* : [A, X] \longrightarrow [\Gamma^* A, \Gamma^* X]$$

be the Γ-loop map. Then the following lemma says that the evaluation map e and Γ-loop map are essentially equivalent.

LEMMA 2.8. *Let $\kappa : [\Gamma(\Gamma^* A), X] \to [\Gamma^* A, \Gamma^* X]$ be the adjoint isomorphism of the space Γ. Then the following diagram is commutative.*

$$\begin{array}{ccc} [A, X] & \xrightarrow{\Gamma^*} & [\Gamma^* A, \Gamma^* X] \\ & \searrow^{e^*} & \uparrow \kappa \\ & & [\Gamma(\Gamma^* A), X] \end{array}$$

Proof. For any map $\alpha : A \to X$ and any elements $t \in \Gamma$, $\lambda \in \Gamma^* A$, we have

$$\begin{aligned} (\kappa(e^*(\alpha))(\lambda))(t) &= e^*(\alpha)(t \wedge \lambda) \\ &= (\alpha \circ e)(t \wedge \lambda) \\ &= \alpha(e(t \wedge \lambda)) \\ &= \alpha(\lambda(t)) \\ &= ((\Gamma^* \alpha)(\lambda))(t). \end{aligned}$$

THEOREM 2.9. *Let Γ_1 be a co-grouplike space and Γ_2 a co-Hopf space. Then we have $\Gamma_2^*([\alpha,\beta]^W_{\Gamma_1^*}) = 0$ for any $\alpha \in [A, \Gamma_1^*(X^W)]$ and $\beta \in [A, \Gamma_1^*(Y^W)]$.*

Proof. Let $\kappa^{-1} : [\Gamma_2^* A, \Gamma_2^*(\Gamma_1^*((X \flat Y)^W))] \to [\Gamma_2(\Gamma_2^* A), \Gamma_1^*((X \flat Y)^W)]$ be the inverse map of the adjoint isomorphism κ. By Lemma 2.8,

$$\kappa^{-1}(\Gamma_2^*([\alpha,\beta]^W_{\Gamma_1^*})) = [\alpha,\beta]^W_{\Gamma_1^*} \circ e = [\alpha \circ e, \beta \circ e]^W_{\Gamma_1^*}$$

in $[\Gamma_2(\Gamma_2^* A), \Gamma_1^*((X \flat Y)^W)]$. The element $[\alpha \circ e, \beta \circ e]^W_{\Gamma_1^*}$ vanishes by Theorem 2.5, since $\Gamma_2(\Gamma_2^* A)$ is a co-Hopf space. This completes the proof.

3. Related isomorphisms $\xi_{W\wedge X}$ and ζ_{W*X}

Let $t_{W\wedge X} : W \wedge X \to X \wedge W$ be the switching map defined by $t_{W\wedge X}(w \wedge x) = x \wedge w$ for any elements $x \in X$ and $w \in W$, where $x \wedge w$ is an elements in $X \wedge W$. It is a natural homeomorphism. We define

$$\xi_{W\wedge X} : [\Gamma X \wedge W, Z] \longrightarrow [\Gamma W \wedge X, Z]$$

by $\xi_{W\wedge X}(\alpha) = \alpha \circ \Gamma t_{W\wedge X}$ for $\Gamma t_{W\wedge X} : \Gamma W \wedge X = \Gamma \wedge W \wedge X \to \Gamma \wedge X \wedge W = \Gamma X \wedge W$. Then we can define the ΓW-Whitehead product $[\xi_{W\wedge X}(\alpha), \xi_{W\wedge Y}(\beta)]_{\Gamma W} \in [\Gamma W \wedge (X \wedge Y), Z]$ for any elements $\alpha \in [\Gamma X \wedge W, Z]$ and $\beta \in [\Gamma Y \wedge W, Z]$. Then we obtain the following equality in Theorem 1.3 [7];

$$[\alpha,\beta]^W_\Gamma = \xi^{-1}_{W\wedge(X\wedge Y)}([\xi_{W\wedge X}(\alpha), \xi_{W\wedge Y}(\beta)]_{\Gamma W}).$$

Therefore we can use both isomorphisms θ and ξ to study the properties of the product $[\alpha,\beta]^W_\Gamma$. Now we consider the properties of $\xi_{W\wedge X}$ in this section as is shown below.

PROPOSITION 3.1. *Let α and β be any elements in $[\Gamma X \wedge W, Z]$. Then we have*

$$\xi_{W\wedge X}(\alpha \dotplus \beta) = \xi_{W\wedge X}(\alpha) \dotplus \xi_{W\wedge X}(\beta).$$

Proof. By the definition of $\xi_{W\wedge X}$, we have

$$\begin{aligned} \xi_{W\wedge X}(\alpha \dotplus \beta) &= (\alpha \dotplus \beta) \circ \Gamma t_{W\wedge X} \\ &= \alpha \circ \Gamma t_{W\wedge X} \dotplus \beta \circ \Gamma t_{W\wedge X} \\ &= \xi_{W\wedge X}(\alpha) \dotplus \xi_{W\wedge X}(\beta). \end{aligned}$$

We notice that $\xi_{W\wedge X}$ is an isomorphism of groups. We have the following formula for the isomorphism $\xi_{W\wedge X}$.

PROPOSITION 3.2. *Let $f : X \to Y$ and $g : V \to W$ be any maps. Then the following diagram is commutative.*

$$\begin{array}{ccc} [\Gamma Y \wedge W, Z] & \xrightarrow{(\Gamma(f\wedge g))^*} & [\Gamma X \wedge V, Z] \\ \downarrow{\scriptstyle \xi_{W\wedge Y}} & & \downarrow{\scriptstyle \xi_{V\wedge X}} \\ [\Gamma W \wedge Y, Z] & \xrightarrow[(\Gamma(g\wedge f))^*]{} & [\Gamma V \wedge X, Z]. \end{array}$$

Therefore we have $\xi_{W\wedge Y}(\alpha)\circ\Gamma(g\wedge f) = \xi_{V\wedge X}(\alpha\circ\Gamma(f\wedge g))$ for any map $\alpha : \Gamma Y \wedge W \to Z$.

Proof. We have

$$\begin{aligned} \xi_{V\wedge X}(\alpha\circ\Gamma(f\wedge g)) &= \alpha\circ\Gamma(f\wedge g)\circ\Gamma t_{V\wedge X} \\ &= \alpha\circ\Gamma t_{W\wedge Y}\circ\Gamma(g\wedge f) \\ &= \xi_{W\wedge Y}(\alpha)\circ\Gamma(g\wedge f). \end{aligned}$$

COROLLARY 3.3. *If $g = 1_W : W \to W$ in* Proposition 3.2, *then we have*

$$\xi_{W\wedge Y}(\alpha)\circ\Gamma Wf = \xi_{W\wedge X}(\alpha\circ(\Gamma f\wedge 1_W))$$

for any map $\alpha : \Gamma Y \wedge W \to Z$.

From the results above, we can give other proofs to Theorems 1.5, 1.6, 1.7, 1.8 and 1.10, using the isomorphism ξ instead of θ. But we omit them here.

Let $\Phi_X : \Gamma^*(X^W) \to (\Gamma W)^*X$ be the homeomorphism which is defined in Definition 6.2.35 of [5] or (2.9) of Chapter III of [8]. Now we define a map

$$\zeta_{W^*X} : [A, \Gamma^*(X^W)] \longrightarrow [A, (\Gamma W)^*X]$$

by $\zeta_{W^*X}(\alpha) = \Phi_X \circ \alpha$ for any element $\alpha \in [A, \Gamma^*(X^W)]$.

PROPOSITION 3.4. *Let α and β be any elements in $[A, \Gamma^*(X^W)]$. Then we have*

$$\zeta_{W^*X}(\alpha \dotplus \beta) = \zeta_{W^*X}(\alpha) \dotplus \zeta_{W^*X}(\beta).$$

Proof. We see that Φ_X is a Hopf map. Thus we have

$$\begin{aligned}\zeta_{W*X}(\alpha \dotplus \beta) &= \Phi_X \circ (\alpha \dotplus \beta)\\ &= \Phi_X \circ \alpha \dotplus \Phi_X \circ \beta\\ &= \zeta_{W*X}(\alpha) \dotplus \zeta_{W*X}(\beta).\end{aligned}$$

If we are given any two maps $f : X \to Y$ and $g : V \to W$, then we define two maps $f^g : X^W \to Y^V$ by $f^g(\lambda) = f \circ \lambda \circ g$ for any element $\lambda \in X^W$, and $(\Gamma g)^{(*)}f : (\Gamma W)^*X \to (\Gamma V)^*Y$ by $((\Gamma g)^{(*)}f)(\lambda') = f \circ \lambda' \circ \Gamma g$ for any element $\lambda' \in (\Gamma W)^*X$. Then we have the following result.

PROPOSITION 3.5. *Let $f : X \to Y$ and $g : V \to W$ be any maps. Then the following diagram is commutative.*

$$\begin{array}{ccc}[A, \Gamma^*(X^W)] & \xrightarrow{(\Gamma^*(f^g))_*} & [A, \Gamma^*(Y^V)]\\ \downarrow{\scriptstyle \zeta_{W*X}} & & \downarrow{\scriptstyle \zeta_{V*Y}}\\ [A, (\Gamma W)^*X] & \xrightarrow[((\Gamma g)^{(*)}f)_*]{} & [A, (\Gamma V)^*Y]\end{array}$$

Therefore we have $((\Gamma g)^{(*)}f) \circ \zeta_{W*X}(\alpha) = \zeta_{V*Y}(\Gamma^*(f^g) \circ \alpha)$ for any map $\alpha : A \to \Gamma^*(X^W)$.

Proof. The following diagram is commutative.

$$\begin{array}{ccc}\Gamma^*(X^W) & \xrightarrow{\Phi_X} & (\Gamma W)^*X\\ \downarrow{\scriptstyle \Gamma^*(f^g)} & & \downarrow{\scriptstyle (\Gamma g)^{(*)}f}\\ \Gamma^*(Y^V) & \xrightarrow[\Phi_Y]{} & (\Gamma V)^*Y\end{array}$$

Hence we have

$$\begin{aligned}((\Gamma g)^{(*)}f) \circ \zeta_{W*X}(\alpha) &= ((\Gamma g)^{(*)}f) \circ \Phi_X \circ \alpha\\ &= \Phi_Y \circ \Gamma^*(f^g) \circ \alpha\\ &= \zeta_{V*Y}(\Gamma^*(f^g) \circ \alpha).\end{aligned}$$

COROLLARY 3.6. *If $g = 1_W : W \to W$ in* Proposition 3.5, *then we have*

$$(\Gamma W)^*f \circ \zeta_{W*X}(\alpha) = \zeta_{W*Y}(\Gamma^*(f^W) \circ \alpha)$$

for any map $\alpha : A \to \Gamma^*(X^W)$.

THEOREM 3.7. *Let $\alpha \in [A, \Gamma^*(X^W)]$ and $\beta \in [A, \Gamma^*(Y^W)]$ be any elements. Then we have*

$$[\alpha, \beta]^W_{\Gamma^*} = \zeta^{-1}_{W^*(X \flat Y)}([\zeta_{W^*X}(\alpha), \zeta_{W^*Y}(\beta)]_{(\Gamma W)^*}).$$

Proof. Let $X \flat Y \xrightarrow{i} X \vee Y \xrightarrow{j} X \times Y$ be the fibration. Let $j_1 : X \to X \vee Y$ and $j_2 : Y \to X \vee Y$ be the inclusions. By Corollary 2.10 of [7], we have

$$\begin{aligned}
&\Gamma^*(i^W) \circ [\alpha, \beta]^W_{\Gamma^*} \\
&= [\![\Gamma^*(j_1^W) \circ \alpha, \Gamma^*(j_2^W) \circ \beta]\!] \\
&= \Gamma^*(j_1^W) \circ \alpha \dotplus \Gamma^*(j_2^W) \circ \beta \dot{-} \Gamma^*(j_1^W) \circ \alpha \dot{-} \Gamma^*(j_2^W) \circ \beta
\end{aligned}$$

where $[\![x, y]\!]$ denotes the commutator of x and y. Then using Corollary 3.6, we have

$$\begin{aligned}
&(\Gamma^*(i^W))_*([\alpha, \beta]^W_{\Gamma^*}) \\
&= \Gamma^*(j_1^W) \circ \alpha \dotplus \Gamma^*(j_2^W) \circ \beta \dot{-} \Gamma^*(j_1^W) \circ \alpha \dot{-} \Gamma^*(j_2^W) \circ \beta \\
&= [\![\zeta^{-1}_{W^*(X \vee Y)}((\Gamma W)^* j_1 \circ \zeta_{W^*X}(\alpha)), \zeta^{-1}_{W^*(X \vee Y)}((\Gamma W)^* j_2 \circ \zeta_{W^*Y}(\alpha))]\!] \\
&= \zeta^{-1}_{W^*(X \vee Y)}((\Gamma W)^* i \circ [\zeta_{W^*X}(\alpha), \zeta_{W^*Y}(\beta)]_{(\Gamma W)^*}) \\
&= (\Gamma^*(i^W))_*(\zeta^{-1}_{W^*(X \flat Y)}([\zeta_{W^*X}(\alpha), \zeta_{W^*Y}(\beta)]_{(\Gamma W)^*})).
\end{aligned}$$

Since $(\Gamma^*(i^W))_*$ is a monomorphism, we have the result.

From the results above, we can give other proofs to Theorems 2.6 and 2.7, using the isomorphism ζ instead of τ. But we omit them here.

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