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SMALL BALL AND LARGE DEVIATION PROBABILITIES ESTIMATES FOR GAUSSIAN PROCESSES WITH STATIONARY INCREMENTS

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ABSTRACT. In this paper we obtain sharp upper and lower bounds of small ball and large deviation probabilities for the increments of Gaussian processes with stationary increments, whose results are essential to establish Chung type laws of iterated logarithm.

1. Introduction

Recently, the upper and lower bounds of small ball probabilities for Gaussian processes have been studied in several situations by many authors: Shao [9], Kuelbs, Li and Shao [6], Shao and Wang [11], Monrad and Rootzén [8], Talagrand [13], Shao [10], Kuelbs and Li [5] and Li and Shao [7], etc.

Among the above recent results, Shao [9] proved the following fundamental theorem on the upper and lower bounds of small ball probabilities for a Gaussian process:

THEOREM 1.1. *Let $\{X(t), 0 \le t \le 1\}$ be a real-valued Gaussian process on the probability space $(\Omega, \mathcal{S}, P)$ with mean zero, stationary increments and $X(0) = 0$. Put $\sigma^2(h) = E\{X(t+h) - X(t)\}^2$, $0 \le t \le t + h \le 1$, where $\sigma^2(h)$ is nondecreasing and concave on $[0, 1]$. Then we have*

$$(1.1) \qquad P\Big\{\sup_{0\le t\le 1} |X(t)| \le \sigma(x)\Big\} \le 2\exp\Big(-0.17/x\Big),$$

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$$(1.2)\quad P\Big\{\sup_{0\le t\le 1}|X(t)|\le \sigma(x)+6e\int_0^\infty \sigma\big(xe^{-y^2}\big)\,dy\Big\}\ge \exp\Big(-2/x\Big)$$

for every $x\in(0,1)$.

On the other hand, Choi and Lin [1] proved the following large deviation probability theorem which is a version of Fernique lemma [2] for Gaussian processes with stationary increments: Let $\mathbb{D}=\{\mathbf{t}:\mathbf{t}=(t_1,\cdots,t_N), a_i\le t_i\le b_i, i=1,2,\cdots,N\}$ be a real N-dimensional parameter space. We assume that the space $\mathbb{D}$ has the usual Euclidean norm $\|\cdot\|$, that is, $\|\mathbf{t}-\mathbf{s}\|^2=\sum_{i=1}^N(t_i-s_i)^2$. Let $\{X(\mathbf{t}),\mathbf{t}\in\mathbb{D}\}$ be a real-valued separable Gaussian process with $EX(\mathbf{t})=0$. Suppose that

$$0<\sup_{\mathbf{t}\in\mathbb{D}}E\big(X(\mathbf{t})\big)^2=:\Gamma^2<\infty,\qquad \Gamma>0,$$

and

$$E\{X(\mathbf{t})-X(\mathbf{s})\}^2\le\varphi^2(\|\mathbf{t}-\mathbf{s}\|),$$

where $\varphi(\cdot)$ is a nondecreasing continuous function such that

$$\int_0^\infty \varphi(e^{-y^2})dy<\infty.$$

THOEREM 1.2. [1] *Let* $\{X(\mathbf{t}),\ \mathbf{t}\in\mathbb{D}\}$ *with* Γ *and* $\varphi(\cdot)$ *be given as in the above statements. Then, for* $\lambda>0$, $z\ge 1$ and $\mathcal{A}>\sqrt{2N\log 2}$, *we have*

$$P\Big\{\sup_{\mathbf{t}\in\mathbb{D}}X(\mathbf{t})\ge z\{\Gamma+(2\sqrt{2}+2)\mathcal{A}\int_0^\infty\varphi\big(\sqrt{N}\lambda 2^{-y^2}\big)\,dy\}\Big\}$$
$$\le(2^N+\psi)\Big(\prod_{i=1}^N\big(\frac{b_i-a_i}{\lambda}\vee 1\big)\Big)e^{-z^2/2},$$

where $a\vee b=\max\{a,b\}$ *and* $\psi=\sum_{n=1}^\infty\exp\{-2^{n-1}(\mathcal{A}^2-2N\log 2)\}<\infty$.

It is well known that such kinds of small ball and large deviation probabilities are the key steps in establishing Chung type laws of iterated logarithm. The main aim of this paper is to develop the above Theorems 1.1 and 1.2 and apply to another types with two "sups" for the increments of the Gaussian process and to obtain some Chung type laws of iterated logarithm as their applications.

2. Upper and lower bounds of probabilities

Let $\{X(t),\ 0 \le t \le 1\}$ be an almost surely continuous Gaussian process on the probability space $(\Omega, \mathcal{S}, P)$ with mean zero, stationary increments and $X(0) = 0$. Put $\sigma^2(h) = E\{X(t+h) - X(t)\}^2$, $0 \le t \le t+h \le 1$, where $\sigma^2(\cdot)$ is a nondecreasing function on $[0,1]$. Throughout this paper we always assume that $X(\cdot)$ and $\sigma(\cdot)$ are as in the above statements. First we shall consider upper bounds of small ball probabilities for the increments of the Gaussian process $X(\cdot)$. To prove our results, we need the following lemmas:

LEMMA 2.1. [9] *Let $\{X(t), 0 \le t \le 1\}$ be an almost surely continuous, centered Gaussian process with $X(0) = 0$ and stationary increments*

$$\sigma^2(|t-s|) = E\{X(t) - X(s)\}^2, \qquad 0 \le t \ne s \le 1.$$

Assume also that $\sigma^2(\cdot)$ is nondecreasing and concave on $[0,1]$. Then we have

$$P\Big\{\max_{0\le i\le n} |X(t_i)| \le x\Big\} \le \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{2}x/\sigma(t_{i+1}-t_i)} e^{-t^2/2}\, dt$$

for every $0 = t_0 < t_1 < t_2 < \cdots < t_{n+1} \le 1$ and for every $x > 0$.

LEMMA 2.2. [12] *Let $\{\xi_i,\ i = 1, 2, \cdots, n\}$ and $\{\eta_i,\ i = 1, 2, \cdots, n\}$ be sequences of jointly standardized normal random variables with* covariance$(\xi_i,\ \xi_j) \le$ covariance (η_i, η_j), $i \ne j$. *Then for any real numbers $u_1, u_2, \cdots, u_n$,*

$$P\{\xi_j \le u_j,\ j = 1, 2, \cdots, n\} \le P\{\eta_j \le u_j : j = 1, 2, \cdots, n\}.$$

LEMMA 2.3. [4] *Let $\{Y(t), t > 0\}$ be a separable Gaussian process with mean zero and finite variances. Then*

$$P\Big\{\sup_{0<t<\infty} \frac{|Y(t)|}{x(t)} \le 1,\ |Y(t_0)| \le x(t_0)\Big\}$$
$$\ge P\Big\{\sup_{0<t<\infty} \frac{|Y(t)|}{x(t)} \le 1\Big\} P\{|Y(t_0)| \le x(t_0)\}$$

for every $x(t) > 0, x(t_0) > 0, t_0 > 0$.

The following Theorems 2.1 and 2.2 are another versions of (1.1) in Theorem 1.1 which have two "sups" for the increments of a Gaussian process.

THEOREM 2.1. *Let $\{X(t), 0 \le t \le 1\}$ be an almost surely continuous, centered Gaussian process with $X(0) = 0$ and stationary increments $\sigma^2(|t-s|) = E\{X(t) - X(s)\}^2$, $0 \le t \ne s \le 1$. Assume also that $\sigma^2(\cdot)$ is nondecreasing and concave on $[0,1]$. Then*

$$P\Big\{\sup_{0\le s\le 1-h}\sup_{0\le t\le h}|X(t+s)-X(s)| \le \sigma(x)\Big\} \le \exp\Big(-0.17\Big(\frac{1-h}{x}\Big)\Big) \tag{2.1}$$

for every $0 < x \le h$.

Proof. For any $0 < x \le h$, put $U_i = X((i+1)x) - X(ix)$, $i = 0, \cdots, [(1-h)/x]$, where $[\cdot]$ denotes the integer part. It follows from the relation $ab = \big(a^2 + b^2 - (a-b)^2\big)/2$ that, for $l = |i-j| \ge 1$,

$$\begin{aligned} &covariance(U_i, U_j) = E(U_iU_j) \\ &= E\big\{X((i+1)x)X((j+1)x)\big\} - E\big\{X((i+1)x)X(jx)\big\} \\ &\qquad - E\big\{X(ix)X((j+1)x)\big\} + E\big\{X(ix)X(jx)\big\} \\ &= \frac{1}{2}\Big\{(\sigma^2((l+1)x) - \sigma^2(lx)) - (\sigma^2(lx) - \sigma^2((l-1)x))\Big\} \\ &\le 0, \end{aligned} \tag{2.2}$$

because $\sigma^2(h)$ is concave. In order to apply Lemma 2.2, set $\xi_i = U_i/\sigma(x)$ in Lemma 2.2 and let ζ_i be independent standard normal random variables. From (2.2), $covariance(\xi_i, \xi_j) \le 0 = covariance(\zeta_i, \zeta_j)$, $i \ne j$. Applying Lemma 2.2, we have

$$\begin{aligned} &P\Big\{\sup_{0\le s\le 1-h}\sup_{0\le t\le h}|X(t+s)-X(s)| \le \sigma(x)\Big\} \\ &\le P\Big\{\max_{0\le i\le[(1-h)/x]}\sup_{0\le t\le h}|X(t+ix)-X(ix)| \le \sigma(x)\Big\} \\ &\le P\Big\{\max_{0\le i\le[(1-h)/x]}|X(x+ix)-X(ix)| \le \sigma(x)\Big\} \\ &\le P\Big\{\max_{0\le i\le[(1-h)/x]} U_i \le \sigma(x)\Big\} = P\Big\{\max_{0\le i\le[(1-h)/x]} \xi_i \le 1\Big\} \\ &\le P\Big\{\max_{0\le i\le[(1-h)/x]} \zeta_i \le 1\Big\} = P\Big\{\zeta_0 \le 1, \cdots, \zeta_{[(1-h)/x]} \le 1\Big\} \\ &\le \big(\Phi(1)\big)^{(1-h)/x} \le \exp\Big(-0.17\ \Big(\frac{1-h}{x}\Big)\Big), \end{aligned} \tag{2.3}$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-z^2/2}\, dz$. □

For $0 < x \leq 1$, let $\sigma^2(x) = x^{2\alpha}$, $0 < \alpha \leq 1/2$. Then $\sigma^2(x)$ is a nondecreasing and concave function of x, but the converse is not true in general. It is interesting to compare upper bounds of (2.1) and (2.4) below. Moreover, the variance function $\sigma^2(x) = x^{2\alpha}$ $(1/2 < \alpha < 1)$ in (2.5) below, is of the convex type on $(0, 1]$.

THEOREM 2.2. *Let $\{X(t),\ 0 \leq t \leq 1\}$ be an almost surely continuous, centered Gaussian process with $X(0) = 0$ and stationary increments*

$$\sigma^2(|t-s|) := E\{X(t) - X(s)\}^2 = |t-s|^{2\alpha}$$

for $0 < \alpha < 1$, $0 \leq t \neq s \leq 1$. Then

$$\begin{aligned} &P\Big\{\sup_{0\leq s\leq 1-h}\ \sup_{0\leq t\leq h} |X(t+s) - X(s)| \leq x^\alpha\Big\} \\ &\leq \exp\Big\{-2\Big(\frac{1-h}{x}\Big)\Big(1-\Phi\Big(\sqrt{\frac{2}{4-4^\alpha}}\Big)\Big)\Big\} \\ &\leq \exp\Big\{-0.317\Big(\frac{1-h}{x}\Big)\Big\} \end{aligned} \tag{2.4}$$

for every $0 < x \leq h$ and $0 < \alpha \leq 1/2$, and

$$\begin{aligned} &P\Big\{\sup_{0\leq s\leq 1-h}\ \sup_{0\leq t\leq h} |X(t+s) - X(s)| \leq x^\alpha\Big\} \\ &\leq \exp\Big\{-\Big(\frac{1-h-x}{2x}\Big)\Big(1-\Phi\Big(\frac{2}{\sqrt{4-4^\alpha}}\Big)\Big)\Big\} \end{aligned} \tag{2.5}$$

for every $0 < x \leq h$ and $1/2 < \alpha < 1$.

Proof. (2.4): As in (2.3), we have

$$\begin{aligned} &P\Big\{\sup_{0\leq s\leq 1-h}\ \sup_{0\leq t\leq h} |X(t+s) - X(s)| \leq x^\alpha\Big\} \\ &\leq P\Big\{\max_{0\leq i\leq [(1-h)/x]} |X((i+1)x) - X(ix)| \leq x^\alpha\Big\}. \end{aligned} \tag{2.6}$$

For any $0 < x \le h$, set $U_i = X((i+1)x) - X(ix)$, $0 \le i \le [(1-h)/x]$. Then it follows that

$$\begin{aligned}
&E\{U_{i+1} - U_i\}^2 \\
&= 2x^{2\alpha} - 2E\{X((i+2)x)X((i+1)x) \\
&\quad - X((i+2)x)X(ix) - X^2((i+1)x) + X((i+1)x)X(ix)\} \\
&= 2x^{2\alpha} - \{-x^{2\alpha} + (2x)^{2\alpha} - x^{2\alpha}\} = (4-4^\alpha)x^{2\alpha}.
\end{aligned}$$

Applying Lemma 2.1 for

$$\begin{aligned}
&X(t_i) = U_i, \quad i = 0, 1, \cdots, n(= [(1-h)/x]), \\
&\sigma(t_{i+1} - t_i) = \sqrt{E\{X(t_{i+1}) - X(t_i)\}^2} = \sqrt{4-4^\alpha}\, x^\alpha,
\end{aligned}$$

we have

$$\begin{aligned}
&P\Big\{\max_{0 \le i \le [(1-h)/x]} |X((i+1)x) - X(ix)| \le x^\alpha\Big\} \\
&\le \prod_{i=0}^{[(1-h)/x]} \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{2}x^\alpha/\sqrt{4-4^\alpha}x^\alpha} e^{-t^2/2}\, dt \\
&= \Big(2\Phi\Big(\sqrt{\frac{2}{4-4^\alpha}}\Big) - 1\Big)^{[(1-h)/x]+1} \\
&\le \exp\Big\{-2\Big(\frac{1-h}{x}\Big)\Big(1 - \Phi\Big(\sqrt{\frac{2}{4-4^\alpha}}\Big)\Big)\Big\}.
\end{aligned} \tag{2.7}$$

The inequalities (2.6) and (2.7) yield (2.4).

(2.5): Let $U_i = X((i+1)x) - X(ix)$, $0 \le i \le [(1-h)/x]$, and set $\eta_j = U_{2j+1} - U_{2j}$, $0 \le j \le [(1-h-x)/(2x)]$. Then

$$\begin{aligned}
E{\eta_j}^2 &= (4-4^\alpha)x^{2\alpha} \quad \text{and} \\
E\eta_i\eta_j &= -\frac{1}{2}\big[6(2|j-i|)^{2\alpha} + (2|j-i|+2)^{2\alpha} \\
&\quad + (2|j-i|-2)^{2\alpha} - 4(2|j-i|+1)^{2\alpha} \\
&\quad - 4(2|j-i|-1)^{2\alpha}\big]x^{2\alpha} \\
&\le 0, \qquad i \ne j.
\end{aligned}$$

Thus by Lemma 2.2, we obtain

$$\begin{aligned}
&P\{\sup_{0\le s\le 1-h}\sup_{0\le t\le h}|X(t+s)-X(s)|\le x^\alpha\}\\
&\le P\Big\{\max_{0\le i\le[(1-h)/x]}|X((i+1)x)-X(ix)|\le x^\alpha\Big\}\\
&\le P\Big\{\max_{0\le j\le[(1-h-x)/(2x)]}|\eta_j|\le 2x^\alpha\Big\}\\
&\le P\Big\{\max_{0\le j\le[(1-h-x)/(2x)]}\eta_j\le 2x^\alpha\Big\}\\
&\le \Big\{\Phi\Big(\frac{2}{\sqrt{4-4^\alpha}}\Big)\Big\}^{(1-h-x)/(2x)}\\
&\le \exp\Big\{-\Big(\frac{1-h-x}{2x}\Big)\Big(1-\Phi\Big(\frac{2}{\sqrt{4-4^\alpha}}\Big)\Big)\Big\}.
\end{aligned}$$

□

Let us next consider lower bounds of small ball probabilities on the two-parameter Gaussian process. In the proofs of the following Lemma 2.4, we use the same techniques as the proof of Lemma 2.3 in Shao [9] to which one-parameter Gaussian process is referred.

LEMMA 2.4. *Let $\{Y(s,t),\ s\ge 0, t\ge 0\}$ be an almost surely continuous, centered 2-parameter Gaussian process with $Y(0,0)=0$ and stationary increments*

$$E\{Y(s',t')-Y(s'',t'')\}^2\le 4\cdot 2^\alpha\{(s'-s'')^2+(t'-t'')^2\}^\alpha,\quad 0<\alpha<1.$$

Then

$$\begin{aligned}
&P\Big\{\sup_{0\le s\le 1-h}\sup_{0\le t\le h}|Y(s,t)|\le x+2e^{3/2}\Big(\frac{2}{R}\Big)^\alpha\Big(1+\frac{1}{\alpha}\Big)\Big\}\\
&\ge e^{-R^2}\Big\{2\Phi\big(x/(2\{4h^2-4h+2\}^{\alpha/2})\big)-1\Big\}^{(Rh+1)(R(1-h)+1)}
\end{aligned}$$

for every $R\ge 1/h$ and $0<x\le h$.

Proof. Let

$$t_i=\Big[\frac{tRe^{e^i}}{e}\Big]\frac{e}{Re^{e^i}},\quad s_i=\Big[\frac{sRe^{e^i}}{e}\Big]\frac{e}{Re^{e^i}},\quad i=0,1,2,\cdots,$$

where $[\cdot]$ denotes the integer part. Note that

$$
\begin{aligned}
|Y(s,t)| &\le |Y(s_0,t_0)| + |Y(s,t) - Y(s_0,t_0)| \\
&\le |Y(s_0,t_0)| + \sum_{i=0}^{\infty} |Y(s_{i+1},t_{i+1}) - Y(s_i,t_i)|.
\end{aligned} \tag{2.8}
$$

Setting

$$x_i = e^{(i+3)/2}\, 2^{1+\alpha} \left(e^{1-e^i}/R\right)^{\alpha}, \qquad i = 0,1,2,\cdots,$$

we have

$$
\begin{aligned}
\sum_{i=0}^{\infty} x_i &= e^{3/2}\, 2^{1+\alpha} \left(\frac{1}{R}\right)^{\alpha} + e^{3/2}\, 2^{1+\alpha} \left(\frac{e}{R}\right)^{\alpha} \sum_{i=1}^{\infty} e^{-\alpha e^i + \frac{i}{2}} \\
&\le e^{3/2}\, 2^{1+\alpha} \left(\frac{1}{R}\right)^{\alpha} + e^{3/2}\, 2^{1+\alpha} \left(\frac{e}{R}\right)^{\alpha} \int_0^{\infty} e^{-\alpha e^x + \frac{x}{2}}\, dx \\
&\le e^{3/2}\, 2^{1+\alpha} \left(\frac{1}{R}\right)^{\alpha} + e^{3/2}\, 2^{1+\alpha} \left(\frac{e}{R}\right)^{\alpha} \frac{1}{\alpha} e^{-\alpha} \\
&= 2\, e^{3/2} \Big(\frac{2}{R}\Big)^{\alpha} \Big(1 + \frac{1}{\alpha}\Big).
\end{aligned} \tag{2.9}
$$

Note that, for $i = 0,1,2,\cdots$,

$$
\begin{aligned}
&E\{Y(s_{i+1},t_{i+1}) - Y(s_i,t_i)\}^2 \le 4^{1+\alpha} (e^{1-e^i}/R)^{2\alpha}, \\
&\text{Card}\{|Y(s_{i+1},t_{i+1}) - Y(s_i,t_i)| : 0 \le s \le 1-h, 0 \le t \le h\} \\
&\qquad \le \Big((1-h)\frac{Re^{e^{i+1}}}{e} + 1\Big)\Big(h\frac{Re^{e^{i+1}}}{e} + 1\Big), \\
&1 - y \ge e^{-1.5y} \qquad \text{for} \quad 0 \le y \le (2/3)\ln(3/2),
\end{aligned} \tag{2.10}
$$

and for random variables X and Y, we have

$$P\{X + Y \le a + b\} \ge P\{X \le a, Y \le b\}, \quad a > 0, \quad b > 0. \tag{2.11}$$

From Fernique ([3], P. 71), we see that the inequality

$$\Phi(u) \ge 1 - \frac{4}{3} \frac{1}{\sqrt{2\pi}(u+1)} e^{-u^2/2} \tag{2.12}$$

holds for all $u \geq 0$. Let Z denote the standard normal random variable. Then it follows from (2.8) $\sim$ (2.12) and Lemma 2.3 that

$$
\begin{aligned}
&P\Big\{\sup_{0\le s\le 1-h}\sup_{0\le t\le h}|Y(s,t)| \le x + 2\,e^{3/2}\Big(\frac{2}{R}\Big)^{\alpha}\Big(1+\frac{1}{\alpha}\Big)\Big\}\\
&\ge P\Big\{\sup_{0\le s\le 1-h}\sup_{0\le t\le h}|Y(s_0,t_0)| \le x,\\
&\qquad \sup_{0\le s\le 1-h}\sup_{0\le t\le h}|Y(s_{i+1},t_{i+1}) - Y(s_i,t_i)| \le x_i,\ i=0,1,2,\cdots\Big\}\\
&\ge P\{\sup_{0\le s\le 1-h}\sup_{0\le t\le h}|Y(s_0,t_0)| \le x\}\\
&\quad \times \prod_{i=0}^{\infty}\Big(P\Big\{|Z| \le \frac{x_i}{2^{1+\alpha}(e^{1-e^i}/R)^{\alpha}}\Big\}\Big)^{(Rhe^{e^{i+1}-1}+1)(R(1-h)e^{e^{i+1}-1}+1)}\\
&= P\Big\{\max_{0\le j\le[R(1-h)]}\max_{0\le k\le[Rh]}\Big|Y\Big(\frac{j}{R},\frac{k}{R}\Big)\Big| \le x\Big\}\\
&\quad \times \prod_{i=0}^{\infty}\Big(P\Big\{|Z| \le e^{\frac{i+3}{2}}\Big\}\Big)^{(Rhe^{e^{i+1}-1}+1)(R(1-h)e^{e^{i+1}-1}+1)}\\
&\ge P\Big\{\max_{0\le j\le[R(1-h)]}\max_{0\le k\le[Rh]}\Big|Y\Big(\frac{j}{R},\frac{k}{R}\Big)\Big| \le x\Big\}\\
&\quad \times \exp\Big(-1.6\sum_{i=0}^{\infty}\frac{e^{-\frac{1}{2}e^{i+3}}}{1+e^{(i+3)/2}}(Rhe^{e^{i+1}-1}+1)(R(1-h)e^{e^{i+1}-1}+1)\Big)\\
&\ge \Big(P\Big\{|Z| \le x/(2\{4h^2-4h+2\}^{\alpha/2})\Big\}\Big)^{(Rh+1)(R(1-h)+1)}\exp(-R^2)\\
&= e^{-R^2}\{2\,\Phi(x/(2\{4h^2-4h+2\}^{\alpha/2}))-1\}^{(Rh+1)(R(1-h)+1)}. \qquad \square
\end{aligned}
$$

THEOREM 2.3. *Let $\{X(t),\ 0 \le t \le 1\}$ be an almost surely continuous, centered Gaussian process with $X(0)=0$ and stationary increments*

$$E\{X(t)-X(s)\}^2 = |t-s|^{2\alpha}, \quad 0 \le \alpha < 1.$$

Then

$$
\begin{aligned}
&P\Big\{\sup_{0\le s\le 1-h}\sup_{0\le t\le h}|X(t+s)-X(s)|\\
&\qquad \le x^{\alpha}\Big(1+2^{1+\alpha}e^{3/2}\Big(1+\frac{1}{\alpha}\Big)\Big)\Big\}
\end{aligned}
\tag{2.13}
$$

$$\geq e^{-1/x^2}\Big\{2\Phi\Big(\frac{x^\alpha}{2\{4h^2-4h+2\}^{\alpha/2}}\Big)-1\Big\}^{(\frac{h}{x}+1)(\frac{1-h}{x}+1)}$$

for every $0 < x \leq h$.

Proof. Let $Y(s,t) = X(t+s) - X(s),\ 0 \leq s \leq t+s \leq 1$. Then

$$\begin{aligned} &E\big\{Y(s',t') - Y(s'',t'')\big\}^2 \\ &\quad \leq 2E\big\{(X(t'+s') - X(t''+s''))^2 + (X(s') - X(s''))^2\big\} \\ &\quad \leq 4(|t'-t''| + |s'-s''|)^{2\alpha} \leq 4\cdot 2^\alpha\big\{(s'-s'')^2 + (t'-t'')^2\big\}^\alpha. \end{aligned}$$

From Lemma 2.4 with $R = 1/x$, it follows that

$$\begin{aligned} &P\Big\{\sup_{0\leq s\leq 1-h}\sup_{0\leq t\leq h}|X(t+s)-X(s)| \\ &\qquad \leq x^\alpha\Big(1+2^{1+\alpha}e^{3/2}\Big(1+\frac{1}{\alpha}\Big)\Big)\Big\} \\ &\geq e^{-1/x^2}\Big\{2\Phi\Big(\frac{x^\alpha}{2\{4h^2-4h+2\}^{\alpha/2}}\Big)-1\Big\}^{(\frac{h}{x}+1)(\frac{1-h}{x}+1)} \end{aligned} \tag{2.14}$$

for every $0 < x \leq h$. □

EXAMPLE 2.1. Setting $h = 1/2$ in Theorem 2.3, we have

$$\begin{aligned} &P\Big\{\sup_{0\leq s,t\leq 1/2}|X(t+s)-X(s)| \leq x^\alpha\Big(1+2^{1+\alpha}e^{3/2}\Big(1+\frac{1}{\alpha}\Big)\Big)\Big\} \\ &\geq e^{-1/x^2}\Big\{2\Phi\Big(\frac{x^\alpha}{2}\Big)-1\Big\}^{(\frac{1}{2x}+1)^2}, \qquad 0 < x \leq \frac{1}{2}. \end{aligned}$$

COROLLARY 2.1. *Let* $\{X(t),\ 0 \leq t \leq 1\}$ *be a fractional Brownian motion of order* 2α *with* $0 < \alpha < 1$, *that is, let* $\{X(t),\ 0 \leq t \leq 1\}$ *be an almost surely continuous, real-valued Gaussian process with mean zero,* $X(0) = 0$ *and stationary increments* $E\{X(t+h) - X(t)\}^2 = h^{2\alpha}$, $0 \leq t \leq t+h \leq 1$. *Then, for every* $0 < x \leq h$,

$$\begin{aligned} &\exp(-C_\alpha^{2/\alpha}x^{-2/\alpha})\Big\{2\Phi\Big(\frac{x}{2^{1+\alpha/2}C_\alpha}\Big)-1\Big\}^{2h(C_\alpha/x)^{2/\alpha}} \\ &\leq P\Big\{\sup_{0\leq s\leq 1-h}\sup_{0\leq t\leq h}|X(t+s)-X(s)| \leq x\Big\} \\ &\leq \begin{cases} \exp\{-0.317(1-h)x^{-1/\alpha}\}, & 0 < \alpha \leq 1/2, \\ \exp\{-0.5((1-h)x^{-1/\alpha}-1)K_\alpha\}, & 1/2 < \alpha < 1, \end{cases} \end{aligned} \tag{2.15}$$

where $C_\alpha = 1 + 2^{1+\alpha} e^{3/2}\left(1 + \frac{1}{\alpha}\right)$ *and*

$$K_\alpha = 1 - \Phi\Big(\frac{2}{\sqrt{4 - 4^\alpha}}\Big) > 0.$$

Proof. In Theorem 2.3, set $z = C_\alpha x^\alpha,\ 0 < \alpha < 1$. Noting that

$$\Big(\frac{h}{x} + 1\Big)\Big(\frac{1-h}{x} + 1\Big) \le \frac{2h}{x^2}, \qquad 0 < x \le h$$

and $4h^2 - 4h + 2 \le 2$ for $0 < h \le 1$, we get the left hand side of (2.15). The right hand side of (2.15) immediately follows from Theorem 2.2.□

Next we shall estimate upper bounds of large deviation probabilities for the increments of the Gaussian process which applied Theorem 1.2, whose results are also used to obtain the limsup theorems concerning Chung type laws of iterated logarithm:

THEOREM 2.4. *Let* $\{X(\mathbf{t}),\ \mathbf{t} \in [0,1]^N\}$ *be an almost surely continuous, centered* N*-parameter Gaussian process with* $X(\mathbf{0}) = 0$ *and stationary increments* $\sigma^2(\|\mathbf{t}-\mathbf{s}\|) = E\{X(\mathbf{t}) - X(\mathbf{s})\}^2$, $\mathbf{t} \ne \mathbf{s} \in [0,1]^N$, *where* $\sigma(\cdot)$ *is a nondecreasing continuous, regularly varying function on* $(0,1]$ *with exponent* α *for some* $0 < \alpha < 1$ *at zero. Let* $\mathbf{a} = (a_1, \cdots, a_N)$ *and* $\mathbf{a}' = (a'_1, \cdots, a'_N)$ *be two vectors in* $(0,1]^N$. *Then, for any* $\epsilon > 0$ *there exists a positive constant* C_ϵ *depending only on* ϵ *such that*

$$P\Big\{\sup_{\mathbf{0}<\mathbf{s}\le\mathbf{1}-\mathbf{a}'}\ \sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{a}} \frac{|X(\mathbf{t}+\mathbf{s}) - X(\mathbf{s})|}{\sigma(\|\mathbf{a}\|)} \ge u\Big\}$$
$$\le C_\epsilon\Big(\prod_{i=1}^N\Big\{\Big(\frac{1-a'_i}{a_1} \vee 1\Big)\Big(\frac{a_i}{a_1} \vee 1\Big)\Big\}\Big)e^{-u^2/(2+\epsilon)}$$

for all $u > 0$.

Proof. Let $\mathbb{D} = \{(\mathbf{s},\mathbf{t}) : \mathbf{0} < \mathbf{s} \le \mathbf{1} - \mathbf{a}', \mathbf{0} \le \mathbf{t} \le \mathbf{a}\}$ be a $2N$-dimensional space. In order to apply Theorem 1.2, we set

$$Y(\mathbf{s},\mathbf{t}) = \frac{X(\mathbf{t}+\mathbf{s}) - X(\mathbf{s})}{\sigma(\|\mathbf{a}\|)}, \quad (\mathbf{s},\mathbf{t}) \in \mathbb{D},$$

and

$$\varphi(z) = \frac{2\sigma(\sqrt{2}z)}{\sigma(\|\mathbf{a}\|)}, \qquad z > 0.$$

Clearly, $E\{Y(\mathbf{s},\mathbf{t})\} = 0$ and $\Gamma^2 = \sup_{(\mathbf{s},\mathbf{t})\in\mathbb{D}} E\{Y(\mathbf{s},\mathbf{t})\}^2 = 1$. Letting $\mathbf{u} = (\mathbf{s}', \mathbf{t}')$ and $\mathbf{v} = (\mathbf{s}'', \mathbf{t}'')$ in $\mathbb{D}$, it follows that

$$\begin{aligned}
&E\{Y(\mathbf{u}) - Y(\mathbf{v})\}^2 \\
&\quad = \frac{1}{\sigma^2(\|\mathbf{a}\|)} E\Big\{\big(X(\mathbf{t}'+\mathbf{s}') - X(\mathbf{t}''+\mathbf{s}'')\big) - \big(X(\mathbf{s}') - X(\mathbf{s}'')\big)\Big\}^2 \\
&\quad \le \frac{2}{\sigma^2(\|\mathbf{a}\|)} \Big\{E\big\{X(\mathbf{t}'+\mathbf{s}') - X(\mathbf{t}''+\mathbf{s}'')\big\}^2 + E\big\{X(\mathbf{s}') - X(\mathbf{s}'')\big\}^2\Big\} \\
&\quad = \frac{2}{\sigma^2(\|\mathbf{a}\|)} \Big\{\sigma^2\big(\|(\mathbf{t}'+\mathbf{s}') - (\mathbf{t}''+\mathbf{s}'')\|\big) + \sigma^2\big(\|\mathbf{s}'-\mathbf{s}''\|\big)\Big\} \\
&\quad \le \frac{4}{\sigma^2(\|\mathbf{a}\|)} \sigma^2\Big(\sqrt{2}\sqrt{\|\mathbf{t}'-\mathbf{t}''\|^2 + \|\mathbf{s}'-\mathbf{s}''\|^2}\Big) \\
&\quad = \varphi^2(\|\mathbf{u}-\mathbf{v}\|).
\end{aligned}$$

For any $\epsilon > 0$ there exists a small constant $c = c(\epsilon) > 0$ such that

$$(2\sqrt{2}+2)\mathcal{A}\int_0^\infty \varphi(\sqrt{2N}\, ca_1 2^{-y^2})\,dy < \epsilon/8,$$

where $\mathcal{A} > 2\sqrt{N\log 2}$. Indeed, for any $\epsilon > 0$ there exists a small $c > 0$ such that

$$\begin{aligned}
\int_0^\infty \varphi(\sqrt{2N}\, ca_1 2^{-y^2})\,dy &\le 2(2\sqrt{N}\,c)^\alpha \int_0^\infty 2^{-\alpha y^2}\,dy \\
&= (2\sqrt{N}\,c)^\alpha \sqrt{\frac{\pi}{\alpha\log 2}} < (\epsilon/8)/((2\sqrt{2}+2)\mathcal{A}).
\end{aligned}$$

Let $u = v(1 + (\epsilon/8))$, $v \ge 1$. Then it follows from Theorem 1.2 that

$$\begin{aligned}
&P\Big\{\sup_{(\mathbf{s},\mathbf{t})\in\mathbb{D}} |Y(\mathbf{s},\mathbf{t})| \ge u\Big\} \\
&\quad \le 2P\Big\{\sup_{(\mathbf{s},\mathbf{t})\in\mathbb{D}} Y(\mathbf{s},\mathbf{t}) \ge v\Big(1 + (2\sqrt{2}+2)\mathcal{A}\int_0^\infty \varphi(\sqrt{2N}\, ca_1 2^{-y^2})\,dy\Big)\Big\}
\end{aligned}$$

$$\le 2\,(2^{2N}+\psi)\Big(\prod_{i=1}^{N}\Big\{\Big(\frac{1-a_i'}{ca_1}\vee 1\Big)\Big(\frac{a_i}{ca_1}\vee 1\Big)\Big\}\Big)e^{-v^2/2}$$

$$\le C_\epsilon\Big(\prod_{i=1}^{N}\Big\{\Big(\frac{1-a_i'}{a_1}\vee 1\Big)\Big(\frac{a_i}{a_1}\vee 1\Big)\Big\}\Big)e^{-u^2/(2+\epsilon)},$$

where C_ϵ is a positive constant depending only on $\epsilon > 0$. In case $0 < u \le 1$, the result follows immediately if we take C_ϵ large enough.□

The following Corollary 2.2 is immediate from Theorem 2.4.

COROLLARY 2.2. *Let $\{X(t),\ 0 \le t \le 1\}$ be an almost surely continuous, centered Gaussian process with $X(0) = 0$ and stationary increments*

$$\sigma^2(|t-s|) = E\{X(t)-X(s)\}^2, \quad 0 \le t \ne s \le 1,$$

where $\sigma(\cdot)$ is a nondecreasing continuous, regularly varying function on $(0,1]$ with exponent α for some $0<\alpha<1$ at zero. Then, for any $\epsilon > 0$ there exists a positive constant C_ϵ depending only on ϵ such that

$$P\Big\{\sup_{0\le s\le 1-h}\ \sup_{0\le t\le h}|X(t+s)-X(s)| \ge \sigma(x)\Big\}$$
$$\le C_\epsilon\Big(\frac{1-h}{h}\vee 1\Big)\exp\Big(-\frac{1}{2+\epsilon}\Big(\frac{\sigma(x)}{\sigma(h)}\Big)^2\Big)$$

for every $x > 0$.

Using Corollaries 2.1~ 2.2 and the relation $1-e^{-z} \ge ze^{-z},\ z \ge 0$, we have the following

COROLLARY 2.3. *Let $\{X(t),\ 0 \le t \le 1\}$ be a fractional Brownian motion of order 2α with $0<\alpha<1$. Then, for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$C_\epsilon\Big(\frac{1-h}{h}\vee 1\Big)\exp\Big(-\frac{1}{2+\epsilon}\Big(\frac{x}{h}\Big)^2\Big)$$
$$\ge P\Big\{\sup_{0\le s\le 1-h}\ \sup_{0\le t\le h}|X(t+s)-X(s)| \ge x\Big\}$$
$$\ge \begin{cases} 0.317(1-h)x^{-1/\alpha}\exp\{-0.317(1-h)x^{-1/\alpha}\}, & 0<\alpha\le 1/2,\\ 0.5\big((1-h)x^{-1/\alpha}-1\big)K_\alpha\exp\{-0.5\big((1-h)^{-1/\alpha}-1\big)K_\alpha\}, & \\ & 1/2<\alpha<1,\end{cases}$$

for every $0 < x \le h$.

Corollaries 2.1 and 2.3 represent upper and lower bounds of small ball probabilities with converse inequalities each other.

3. Applications

In this section we shall establish some Chung type laws of iterated logarithm as applications of the results in section 2.

The following theorems are referred to the fractional Brownian motion. It is interesting to compare the slight changes of normalizing factors in denominators of the following (3.1)~(3.3):

THEOREM 3.1. *Let* $\{X(t), 0 \le t \le 1\}$ *be a fractional Brownian motion of order* 2α *with* $0 < \alpha < 1$. *Then we have*

$$\liminf_{h\to 0} \sup_{0\le s\le 1-h} \sup_{0\le t\le h} \frac{|X(t+s)-X(s)|}{((0.317-\zeta)h/\log|\log h|)^{\alpha}} \ge 1 \quad \text{a.s.} \tag{3.1}$$

for some $0 < \alpha \le 1/2$, *and*

$$\liminf_{h\to 0} \sup_{0\le s\le 1-h} \sup_{0\le t\le h} \frac{|X(t+s)-X(s)|}{\{(0.5-\zeta)hK_\alpha/((0.5-\zeta)K_\alpha+\log|\log h|)\}^{\alpha}} \ge 1 \quad \text{a.s.} \tag{3.2}$$

for some $1/2 < \alpha < 1$, *where* $\zeta > 0$ *is small enough and*

$$K_\alpha = 1 - \Phi\Big(\frac{2}{\sqrt{4-4^{\alpha}}}\Big) > 0.$$

Proof. (3.1): From Theorem 2.2(2.4), we have, for any $0 < \epsilon < 1$,

$$P\Big\{\sup_{0\le s\le 1-h} \sup_{0\le t\le h} |X(t+s)-X(s)| \le x^{\alpha}(1-\epsilon)\Big\} \le \exp\big(-0.317\, h/x\big), \qquad 0 < x \le h < 1/2.$$

Choose

$$x = \frac{(0.317-\zeta)h}{\log|\log h|} \quad \text{and} \quad h = e^{-n},$$

where $n(\geq 2)$ is an integer. Then we have

$$\sum_n P\Big\{\sup_{0\leq s\leq 1-e^{-n}}\sup_{0\leq t\leq e^{-n}}|X(t+s)-X(s)| \leq \Big(\frac{(0.317-\zeta)e^{-n}}{\log n}\Big)^{\alpha}(1-\epsilon)\Big\}$$
$$\leq \sum_n \exp\Big(-\frac{0.317}{0.317-\zeta}\log n\Big) < \infty.$$

So, the Borel-Cantelli lemma implies that

$$\liminf_{n\to\infty}\sup_{0\leq s\leq 1-e^{-n}}\sup_{0\leq t\leq e^{-n}}\frac{|X(t+s)-X(s)|}{\big((0.317-\zeta)e^{-n}/\log n\big)^{\alpha}} \geq 1-\epsilon \qquad \text{a.s.}$$

This gives the result (3.1).

(3.2): From Theorem 2.2(2.5), we have, for any $0<\epsilon<1$,

$$P\Big\{\sup_{0\leq s\leq 1-h}\sup_{0\leq t\leq h}|X(t+s)-X(s)| \leq x^{\alpha}(1-\epsilon)\Big\}$$
$$\leq \exp\Big(-0.5\Big(\frac{h}{x}-1\Big)K_{\alpha}\Big), \qquad 0<x\leq h<1/2.$$

Choose

$$x=\frac{(0.5-\zeta)hK_{\alpha}}{(0.5-\zeta)K_{\alpha}+\log|\log h|} \quad \text{and} \quad h=e^{-n}, \quad n\geq 1.$$

Then we have

$$\sum_n P\Big\{\sup_{0\leq s\leq 1-e^{-n}}\sup_{0\leq t\leq e^{-n}}|X(t+s)-X(s)| \leq \Big(\frac{(0.5-\zeta)\,e^{-n}K_{\alpha}}{(0.5-\zeta)K_{\alpha}+\log n}\Big)^{\alpha}(1-\epsilon)\Big\}$$
$$\leq \sum_n \exp\Big(-\frac{0.5}{0.5-\zeta}\log n\Big) < \infty.$$

So, the result (3.2) follows from the Borel-Cantelli lemma. □

THEOREM 3.2. *Let* $\{X(t), 0 \leq t \leq 1\}$ *be a fractional Brownian motion of order* 2α *with* $0 < \alpha < 1$. *Then we have*

$$\limsup_{h\to 0} \sup_{0\leq s\leq 1-h} \sup_{0\leq t\leq h} \frac{|X(t+s)-X(s)|}{h^\alpha\{2(\log(1/h)+\log|\log h|)\}^{1/2}} \leq 1 \quad \text{a.s.} \tag{3.3}$$

Proof. From Corollary 2.2, it follows that for any small $\epsilon > 0$ there exists a positive constant C_ϵ such that, for every $x > 0$,

$$P\Big\{\sup_{0\leq s\leq 1-h} \sup_{0\leq t\leq h} \big|X(t+s)-X(s)\big| \geq x^\alpha(1+\epsilon)\Big\} \leq C_\epsilon \frac{1}{h} \exp\Big(-\frac{1}{2+\epsilon}\Big(\frac{x}{h}\Big)^{2\alpha}\Big).$$

Set $x = h\big\{(2+\epsilon)\log(1/h)+(2+2\epsilon)\log|\log h|\big\}^{1/(2\alpha)}$ and $h = e^{-n}$, $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\sum_n P\Big\{&\sup_{0\leq s\leq 1-e^{-n}} \sup_{0\leq t\leq e^{-n}} \big|X(t+s)-X(s)\big| \\ &\geq e^{-\alpha n}\big\{(2+\epsilon)n+(2+2\epsilon)\log n\big\}^{1/2}(1+\epsilon)\Big\} \\ &\leq C_\epsilon \sum_n e^n \exp\Big(-\frac{1}{2+\epsilon}\{(2+\epsilon)n+(2+2\epsilon)\log n\}\Big) < \infty.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the result (3.3) immediately follows from the Borel-Cantelli lemma. □

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