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ON THE EXTREME ZEROS OF ORTHOGONAL POLYNOMIALS

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Abstract. We investigate the asymptotic behavior of the extreme zeros of orthogonal polynomials with respect to a positive measure $d\alpha(x)$ in terms of the three term recurrence coefficients. We then show that the asymptotic behavior of extreme zeros of orthogonal polynomials with respect to $g^2(x)d\alpha(x)$ is the same as that of extreme zeros of orthogonal polynomials with respect to $d\alpha(x)$ when $g(x)$ is a polynomial with all zeros in a certain interval determined by $d\alpha(x)$. Several illustrating examples are also given.

1. Introduction

Let $d\alpha(x)$ be a positive Borel measure on $\mathbb{R}$ with infinite support for which

$$\left|\int_{-\infty}^{\infty} x^n d\alpha(x)\right| < \infty, \quad n = 0, 1, 2, \cdots .$$

We call such $\alpha(x)$ a distribution function. We let $\{P_n(d\alpha; x)\}_{n=0}^{\infty}$ be the orthonormal polynomials with respect to $d\alpha(x)$ such that

$$P_n(d\alpha; x) = \gamma_n(d\alpha)x^n + \text{ lower degree terms}, \qquad \gamma_n(d\alpha) > 0$$

and

$$\int_{-\infty}^{\infty} P_m(d\alpha; x)P_n(d\alpha; x)d\alpha(x) = \delta_{mn}.$$

Then $P_n(d\alpha; x)$ satisfies the three term recurrence relation

$$\begin{aligned}(1.1)\ xP_n(d\alpha; x) \ &= \ a_{n+1}(d\alpha)P_{n+1}(d\alpha; x) \\ &\quad +b_n(d\alpha)P_n(d\alpha; x) + a_n(d\alpha)P_{n-1}(d\alpha; x), \quad n \geq 0,\end{aligned}$$

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where $P_{-1}(x) \equiv 0$, $P_0(d\alpha; x) = (\int_{-\infty}^{\infty} d\alpha)^{-1/2}$, $a_0(d\alpha) = 0$, and

$$a_n(d\alpha) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} = \int_{-\infty}^{\infty} x P_n(d\alpha; x) P_{n-1}(d\alpha; x)\ d\alpha(x)\ (> 0), \quad n \geq 1,$$

$$b_n(d\alpha) = \int_{-\infty}^{\infty} x P_n^2(d\alpha; x)\ d\alpha(x), \quad n \geq 0.$$

If $\mathcal{P}_n$ denotes the space of real polynomials of degree at most n, then the n-th Christoffel function for $d\alpha(x)$ is

$$\lambda_n(d\alpha; x) := \inf_{\pi \in \mathcal{P}_{n-1} \setminus \{0\}} \frac{1}{\pi^2(x)} \int_{-\infty}^{\infty} \pi^2(t) d\alpha(t), \quad x \in \mathbb{R}.$$

Then, it is well (cf. [1]) known that

$$\lambda_n^{-1}(d\alpha; x) = K_n(d\alpha; x, x) := \sum_{k=0}^{n-1} P_k^2(d\alpha; x)$$

and

$$\begin{aligned} (1.2) \qquad & K_n(d\alpha; x, x) \\ & = a_n(d\alpha)[P_n'(d\alpha; x) P_{n-1}(d\alpha; x) - P_n(d\alpha; x) P_{n-1}'(d\alpha; x)]. \end{aligned}$$

For $n \geq 1$, $P_n(d\alpha; x)$ has n real simple zeros $x_{nn}(d\alpha) < x_{n-1,n}(d\alpha) < \cdots < x_{1n}(d\alpha)$. When $d\alpha(x)$ is symmetric, that is, $\int_{-\infty}^{\infty} x^{2n} d\alpha(x) = 0$, $n = 0, 1, \cdots$, Freud [2] (see also [8]) studied the asymptotic behavior of the greatest zero $x_{1n}(d\alpha)$ of $P_n(d\alpha; x)$ in terms of $a_n(d\alpha)$. Note that when $d\alpha(x)$ is symmetric, $b_n(d\alpha) = 0$, $n = 0, 1, \cdots$ and $x_{nn}(d\alpha) = -x_{1n}(d\alpha)$, $n = 1, 2, \cdots$.

In this paper we first generalize results in [2] to the case of arbitrary nonsymmetric $d\alpha(x)$. We then consider another distribution function $d\beta(x) = g^2(x) d\alpha(x)$, where $g(x)$ is any polynomial with only real zeros and show that the asymptotic behavior of the extreme zeros of $P_n(d\beta; x)$ is the same as that of $P_n(d\alpha; x)$ when zeros of $g(x)$ are contained in a certain interval determined by $d\alpha(x)$. We give several illustrating examples, including a partial answer to Freud's conjecture on the asymptotic behavior of extreme zeros of orthogonal polynomials with respect to weights of exponential growth at infinity.

2. Zero behavior for nonsymmetric weights

Throughout this work we use the notation $c_n \sim d_n$ for sequences $\{c_n\}$ and $\{d_n\}$ of real numbers if there exist two positive constants C_1 and C_2, independent of n, such that

$$C_1 d_n \leq c_n \leq C_2 d_n.$$

By the Gauss-Jacobi quadrature formula, we have (due to Chebyshev)

$$x_{1n}(d\alpha) = \max_{\pi \in \mathcal{P}_{n-1} \setminus \{0\}} \frac{\int_{-\infty}^{\infty} x\pi^2(x)\, d\alpha(x)}{\int_{-\infty}^{\infty} \pi^2(x)\, d\alpha(x)} \tag{2.1}$$

and

$$x_{nn}(d\alpha) = \min_{\pi \in \mathcal{P}_{n-1} \setminus \{0\}} \frac{\int_{-\infty}^{\infty} x\pi^2(x) d\alpha(x)}{\int_{-\infty}^{\infty} \pi^2(x) d\alpha(x)}. \tag{2.2}$$

The maximum in (2.1) and the minimum in (2.2) are attained if and only if $\pi(x) = c\frac{P_n(d\alpha;x)}{x - x_{1n}(d\alpha)}$ and $\pi(x) = c\frac{P_n(d\alpha;x)}{x - x_{nn}(d\alpha)}$, respectively, where c is any non-zero constant. We may also express $x_{1n}(d\alpha)$ and $x_{nn}(d\alpha)$ via the coefficients $\{a_n(d\alpha)\}_{n=0}^{\infty}$ and $\{b_n(d\alpha)\}_{n=0}^{\infty}$ of the three term recurrence relation (1.1). We simply write a_n and b_n for $a_n(d\alpha)$ and $b_n(d\alpha)$, respectively if there is no confusion.

PROPOSITION 2.1. *We have for* $n \geq 2$

$$x_{1n}(d\alpha) = \max_J \left\{ \frac{\sum_{k=0}^{n-1} b_k J_k^2 + 2\sum_{k=0}^{n-2} a_{k+1} J_k J_{k+1}}{\sum_{k=0}^{n-1} J_k^2} \right\} \tag{2.3}$$

and

$$x_{nn}(d\alpha) = \min_J \left\{ \frac{\sum_{k=0}^{n-1} b_k J_k^2 + 2\sum_{k=0}^{n-2} a_{k+1} J_k J_{k+1}}{\sum_{k=0}^{n-1} J_k^2} \right\}, \tag{2.4}$$

where $J = (J_0, J_1, \cdots, J_{n-1})$ *is varying in* $\mathbb{R}^n \setminus \{0\}$.

Proof. Write any $\pi(x)$ in $\mathcal{P}_{n-1}$ as

$$\pi(x) = \sum_{k=0}^{n-1} J_k P_k(d\alpha; x).$$

Then (2.3) and (2.4) follow immediately from (2.1) and (2.2) since

$$\int_{-\infty}^{\infty} \pi^2(x)\, d\alpha(x) = \sum_{k=0}^{n-1} J_k^2$$

and

$$\int_{-\infty}^{\infty} x\pi^2(x)\,d\alpha(x) = 2\sum\nolimits_{k=0}^{n-2} a_{k+1}J_kJ_{k+1} + \sum\nolimits_{k=0}^{n-1} b_kJ_k^2.$$

□

In passing, we note that we may let J vary in $\{J = (J_0, \cdots, J_{n-1}) \in \mathbb{R}^n \setminus \{0\} \mid J_k \geq 0,\ 0 \leq k \leq n-1\}$ in (2.3) and in $\{J = (J_0, \cdots, J_{n-1}) \in \mathbb{R}^n \setminus \{0\} \mid J_kJ_{k+1} \leq 0,\ 0 \leq k \leq n-2\}$ in (2.4). Proposition 2.1 was first obtained by Freud [2] in case $d\alpha(x)$ is symmetric so that $b_k = 0$, $k \geq 0$.

THEOREM 2.2. *We have for $n \geq 2$ and $1 \leq m \leq n$,*

$$\max_{0\leq j\leq n-m} A_j^+(m) \leq x_{1n}(d\alpha) \leq \max_{0\leq k\leq n-1} b_k + 2\cos\frac{\pi}{n+1}\max_{0\leq k\leq n-1} a_k\,; \tag{2.5}$$

$$\min_{0\leq k\leq n-1} b_k - 2\cos\frac{\pi}{n+1}\max_{0\leq k\leq n-1} a_k \leq x_{nn}(d\alpha) \leq \min_{0\leq j\leq n-m} A_j^-(m), \tag{2.6}$$

where

$$A_j^{\pm}(m) := \frac{1}{m}\left(\sum_{k=j}^{j+m-1} b_k \pm 2\sum_{k=j+1}^{j+m-1} a_k\right)$$

$$\text{(here } \sum_{k=j+1}^{j} a_k = 0 \quad \text{when} \quad m = 1).$$

Proof. Choose $J_0 = J_1 = \cdots = J_{j-1} = 0$, $J_j = J_{j+1} = \cdots = J_{j+m-1} = 1$, $J_{j+m} = \cdots = J_{n-1} = 0$. Then from the equation (2.3), we have

$$A_j^+(m) \leq x_{1n}(d\alpha).$$

Since j is arbitrary with $0 \leq j \leq n-m$, the left hand side of (2.5) is proved. The right hand side of (2.6) can be proved similarly by choosing $J_0 = J_1 = \cdots = J_{j-1} = 0$, $J_{j+k} = (-1)^k$, $0 \leq k \leq m-1$, and $J_{j+m} = \cdots = J_{n-1} = 0$. We also have from (2.3)

$$\begin{aligned} x_{1n}(d\alpha) &\leq \max_{0\leq k\leq n-1} b_k + \max_J \frac{2\sum_{k=0}^{n-2} a_{k+1}J_kJ_{k+1}}{\sum_{k=0}^{n-1} J_k^2} \\ &\leq \max_{0\leq k\leq n-1} b_k + 2\max_{0\leq k\leq n-1} a_k \max_J \frac{2\sum_{k=0}^{n-2}\frac{1}{2}J_kJ_{k+1}}{\sum_{k=0}^{n-1} J_k^2} \\ &= \max_{0\leq k\leq n-1} b_k + 2\cos\frac{\pi}{n+1}\max_{0\leq k\leq n-1} a_k \end{aligned}$$

since $\max_J \frac{2\sum_{k=0}^{n-1}\frac{1}{2}J_kJ_{k+1}}{\sum_{k=0}^{n-1}J_k^2}$ is the greatest zero of $U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$ $(x = \cos\theta)$, the n-th Chebyshev polynomial of the second kind. On the other hand, $\min_J \frac{2\sum_{k=0}^{n-2}a_{k+1}J_kJ_{k+1}}{\sum_{k=0}^{n-1}J_k^2}$ is the smallest zero of the orthonormal polynomials $\{Q_n(x)\}_{n=0}^{\infty}$, which satisfy the three term recurrence relation:

$$xQ_n(x) = a_{n+1}Q_{n+1}(x) + a_nQ_{n-1}(x), \quad n = 0, 1, \cdots,$$
$$(Q_{-1}(x) \equiv 0, \ Q_0(x) = 1).$$

Since $Q_n(-x) = (-1)^nQ_n(x)$, $n \geq 0$,

$$\min_J \frac{2\sum_{k=0}^{n-2}a_{k+1}J_kJ_{k+1}}{\sum_{k=0}^{n-1}J_k^2} = -\max_J \frac{2\sum_{k=0}^{n-2}a_{k+1}J_kJ_{k+1}}{\sum_{k=0}^{n-1}J_k^2}$$

so that we have from (2.4)

$$\begin{aligned} x_{nn}(d\alpha) &\geq \min_{0\leq k\leq n-1} b_k + \min_J \frac{2\sum_{k=0}^{n-2}a_{k+1}J_kJ_{k+1}}{\sum_{k=0}^{n-1}J_k^2} \\ &= \min_{0\leq k\leq n-1} b_k - \max_J \frac{2\sum_{k=0}^{n-2}a_{k+1}J_kJ_{k+1}}{\sum_{k=0}^{n-1}J_k^2} \\ &\geq \min_{0\leq k\leq n-1} b_k - 2\cos\frac{\pi}{n+1}\max_{0\leq k\leq n-1} a_k. \end{aligned}$$

□

Theorem 2.2 was first proved by Freud [2] when $d\alpha(x)$ is symmetric and $m = 2$.

THEOREM 2.3. *Assume that*

$$\lim_{n\to\infty}\frac{a_{n+1}(d\alpha)}{a_n(d\alpha)} = \frac{1}{A}\,(>0) \tag{2.7}$$

and let

(2.8)

$$b := \liminf_{n\to\infty}\frac{b_{n+1}(d\alpha)}{a_n(d\alpha)}, \quad B := \limsup_{n\to\infty}\frac{b_{n+1}(d\alpha)}{a_n(d\alpha)} \qquad (-\infty \leq b \leq B \leq \infty).$$

Then for any integer $m \geq 1$,

$$\liminf_{n\to\infty}\frac{x_{1n}(d\alpha)}{a_{n-1}} \geq \min_{0\leq k\leq m-1} bA^k + 2\cos\frac{\pi}{m+1}\min_{0\leq k\leq m-2} A^k; \tag{2.9}$$

$$\limsup_{n\to\infty}\frac{x_{nn}(d\alpha)}{a_{n-1}} \leq \max_{0\leq k\leq m-1} BA^k - 2\cos\frac{\pi}{m+1}\min_{0\leq k\leq m-2} A^k. \tag{2.10}$$

Proof. Fix any integer $m \geq 1$. Then, for any ϵ with $0 < \epsilon < \min_{0 \leq r \leq m-2} A^r$, there exists an integer $N = N(\epsilon, m)$ $(\geq m+1)$ such that

$$(A^r - \epsilon)a_{n-1} < a_{n-r-1} < (A^r + \epsilon)a_{n-1} \quad n \geq N,\ 0 \leq r \leq m$$

and

$$(b - \epsilon)a_{n-r-1} < b_{n-r-1} < (B + \epsilon)a_{n-r-1}, \quad n \geq N,\ 0 \leq r \leq m.$$

Then there are three cases:

Case 1. $0 < b \leq B$**:** Choose ϵ with $0 < \epsilon < b$. Then

$$(b - \epsilon)(A^r - \epsilon)a_{n-1} < b_{n-r-1} < (B + \epsilon)(A^r + \epsilon)a_{n-1}, \quad n \geq N,\ 0 \leq r \leq m-1.$$

Case 2. $b \leq 0 \leq B$**:** Then

$$(b - \epsilon)(A^r + \epsilon)a_{n-1} < b_{n-r-1} < (B + \epsilon)(A^r + \epsilon)a_{n-1}, \quad n \geq N,\ 0 \leq r \leq m-1.$$

Case 3. $b \leq B < 0$**:** Choose ϵ with $0 < \epsilon < -B$. Then

$$(b - \epsilon)(A^r + \epsilon)a_{n-1} < b_{n-r-1} < (B + \epsilon)(A^r - \epsilon)a_{n-1}, \quad n \geq N,\ 0 \leq r \leq m-1.$$

In each case, inserting $J_0 = J_1 = \cdots = J_{n-m-1} = 0$ in (2.3) we have for $n \geq N$,

$$\begin{aligned} &x_{1n}(d\alpha) \\ &\geq \max_J \frac{\sum_{k=n-m}^{n-1} b_k J_k^2 + 2\sum_{k=n-m}^{n-2} a_{k+1} J_k J_{k+1}}{\sum_{k=n-m}^{n-1} J_k^2} \\ &\geq \min_{0 \leq k \leq m-1} (b - \epsilon)(A^k \pm \epsilon)a_{n-1} + 2 \min_{0 \leq k \leq m-2} (A^k - \epsilon)a_{n-1} \max_J \frac{\sum_{k=0}^{m-2} J_k J_{k+1}}{\sum_{k=0}^{m-1} J_k^2} \\ &= a_{n-1}\left[\min_{0 \leq k \leq m-1} (b - \epsilon)(A^k \pm \epsilon) + 2 \min_{0 \leq k \leq m-2} (A^k - \epsilon) \cos\frac{\pi}{m+1}\right]. \end{aligned}$$

Letting ϵ tend to 0 we obtain (2.9). Similarly, we have

$$
\begin{aligned}
&x_{nn}(d\alpha) \\
&\leq \min_J \frac{\sum_{k=n-m}^{n-1} b_k J_k^2 + 2\sum_{k=n-m}^{n-2} a_{k+1} J_k J_{k+1}}{\sum_{k=n-m}^{n-1} J_k^2} \\
&\leq \max_{0\leq k\leq m-1} (B+\epsilon)(A^k \pm \epsilon)a_{n-1} + 2 \min_{0\leq k\leq m-1} (A^k - \epsilon)a_{n-1} \min_J \frac{\sum_{k=0}^{m-2} J_k J_{k+1}}{\sum_{k=0}^{m-1} J_k^2} \\
&= a_{n-1}\left[\max_{0\leq k\leq m-1}(B+\epsilon)(A^k \pm \epsilon) - 2\cos\frac{\pi}{m+1}\min_{0\leq k\leq m-1}(A^k - \epsilon)\right],
\end{aligned}
$$

from which (2.10) follows. □

COROLLARY 2.4. *Under the same hypotheses as in* Theorem 2.3 *we have:*

(1) *if* $b \geq 0$ *and* $A \geq 1$, *then* $\liminf_{n\to\infty} \frac{x_{1n}(d\alpha)}{a_{n-1}} \geq b+2$;

(2) *if* $B \leq 0$ *and* $A \geq 1$, *then* $\limsup_{n\to\infty} \frac{x_{nn}(d\alpha)}{a_{n-1}} \leq B-2$.

Corollary 2.4 generalizes a result by Freud [2] in which he proved: If $d\alpha(x)$ is symmetric (so that $b_n = 0$, $n \geq 0$ and $b = B = 0$) and $A = 1$, then $\liminf_{n\to\infty} \frac{x_{1n}}{a_{n-1}} \geq 2$.

THEOREM 2.5. *Under the same hypotheses as in* Theorem 2.3, *if* $b = B$, $A = 1$ *and* $\lim_{n\to\infty} a_n = \infty$, *then*

$$
(2.11)\qquad \begin{cases} \lim_{n\to\infty} \dfrac{x_{1n}(d\alpha)}{\max_{0\leq k\leq n-1} a_k(d\alpha)} = b+2 & \text{if} \quad b \geq 0 \\[2ex] \lim_{n\to\infty} \dfrac{x_{nn}(d\alpha)}{\max_{0\leq k\leq n-1} a_k(d\alpha)} = b-2 & \text{if} \quad b \leq 0. \end{cases}
$$

Proof. First assume that $b \geq 0$. Choose an increasing sequence $\{n_s\}_{s=1}^{\infty}$ of integers inductively as: $n_1 = 1$ and n_{s+1} is the smallest integer greater than n_s such that $a_{n_{s+1}} > a_{n_s}$. Then

$$\max_{0\leq k\leq n_s} a_k = \max_{0\leq k< n_{s+1}} a_k = a_{n_s}.$$

For any $n \geq 2$, we can choose an integers $s \geq 1$ such that $n_s \leq n-1 < n_{s+1}$. Since $x_{1n} \geq x_{1,n_s+1}$, we have by Corollary 2.4,

$$\liminf_{n\to\infty} \frac{x_{1n}}{\max\limits_{0\leq k\leq n-1} a_k} \geq \liminf_{n\to\infty} \frac{x_{1,n_s+1}}{a_{n_s}} \geq \liminf_{n\to\infty} \frac{x_{1n}}{a_n} \geq b+2. \tag{2.12}$$

On the other hand, by (2.5)

$$\frac{x_{1n}}{\max\limits_{0\leq k\leq n-1} a_k} \leq \frac{\max\limits_{0\leq k\leq n-1} b_k}{\max\limits_{0\leq k\leq n-1} a_k} + 2\cos\frac{\pi}{n+1} \leq \frac{\max\limits_{0\leq k\leq n-1} b_k}{\max\limits_{0\leq k\leq n-1} a_k} + 2, \quad n \geq 2$$

so that

$$\limsup_{n\to\infty} \frac{x_{1n}}{\max\limits_{0\leq k\leq n-1} a_k} \leq \limsup_{n\to\infty} \frac{\max\limits_{0\leq k\leq n-1} b_k}{\max\limits_{0\leq k\leq n-1} a_k} + 2. \tag{2.13}$$

If $\{|b_n|\}_{n=0}^{\infty}$ is bounded, say $|b_n| < M$, $n \geq 0$, then $b = 0$ and

$$\limsup_{n\to\infty} \frac{\max\limits_{0\leq k\leq n-1} b_k}{\max\limits_{0\leq k\leq n-1} a_k} \leq \limsup_{n\to\infty} \frac{M}{\max\limits_{0\leq k\leq n-1} a_k} \leq \limsup_{n\to\infty} \frac{M}{a_{n-1}} = 0. \tag{2.14}$$

If $\{|b_n|\}_{n=0}^{\infty}$ is not bounded then we can choose an increasing sequence $\{n_s\}_{s=1}^{\infty}$ of integers as before for $\{|b_n|\}_{n=0}^{\infty}$ instead of $\{a_n\}_{n=0}^{\infty}$. For any $n \geq 2$, choose n_s such that $n_s \leq n-1 < n_{s+1}$. Then

(2.15)

$$\limsup_{n\to\infty} \frac{\max\limits_{0\leq k\leq n-1} b_k}{\max\limits_{0\leq k\leq n-1} a_k} \leq \limsup_{n\to\infty} \frac{\max\limits_{0\leq k\leq n-1} |b_k|}{\max\limits_{0\leq k\leq n-1} a_k} \leq \limsup_{n\to\infty} \frac{|b_{n_s}|}{a_{n_s}} = |b| = b.$$

Combining (2.12)~(2.15) we obtain (2.11) if $b \geq 0$. The case for $b \leq 0$ can also be proved by a similar process. □

Theorem 2.5 is proved by Freud [2] in a special case when $d\alpha(x)$ is symmetric so that $b = B = 0$. In the following examples, we write $a_n(w)$ and $x_{kn}(w)$ for $a_n(d\alpha)$ and $x_{kn}(d\alpha)$ in case $d\alpha(x) = w(x)dx$.

Example 2.1. Consider Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ $(\alpha > -1)$ satisfying (see [1])

$$\begin{aligned} xL_n^{(\alpha)}(x) &= \sqrt{(n+1)(n+\alpha+1)}L_{n+1}^{(\alpha)}(x) \\ &\quad +(2n+\alpha+1)L_n^{(\alpha)}(x) + \sqrt{n(n+\alpha)}L_{n-1}^{(\alpha)}(x), \; n \geq 0, \end{aligned}$$

which are orthogonal with respective to the weight $x^\alpha e^{-x}$, $x > 0$. Clearly, the conditions in Theorem 2.5 are satisfied with $A = 1$ and $b = 2$. Since $\{a_n\}_{n=0}^\infty$ is monotone increasing, we have by (2.11)

$$\lim_{n\to\infty} \frac{x_{1n}(x^\alpha e^{-x})}{n} = 4,$$

which is more precise than (see [12, p. 128]): $x_{1n} < 2n + \alpha + 1 + \{(2n + \alpha + 1)^2 + 1/4 - \alpha^2\}^{\frac{1}{2}} \simeq 4n$.

EXAMPLE 2.2. Consider Meixner polynomials $\{m_n^{(\gamma,\mu)}(x)\}_{n=0}^\infty$ ($\gamma > 0$, $0 < \mu < 1$) satisfying (see [10])

$$\begin{aligned} xm_n^{(\gamma,\mu)}(x) =& \frac{\sqrt{\mu(n+1)(n+\gamma)}}{1-\mu} m_{n+1}^{(\gamma,\mu)}(x) + \frac{n+\mu(n+\gamma)}{1-\mu} m_n^{(\gamma,\mu)}(x) \\ &+ \frac{\sqrt{\mu n(n-1+\gamma)}}{1-\mu} m_{n-1}^{(\gamma,\mu)}(x), \quad n \geq 0, \end{aligned}$$

which are orthogonal with respective to the weight

$$W^{\gamma,\mu}(x) = \frac{\mu^x \Gamma(\gamma + x)}{\Gamma(x+1)\Gamma(\gamma)}, \quad \gamma > 0, \quad 0 < \mu < 1, \quad x = 0, 1, \cdots.$$

Clearly, the conditions in Theorem 2.5 are satisfied with $A = 1$ and $b = \frac{1+\mu}{\sqrt{\mu}}$. Since $\{a_n\}_{n=0}^\infty$ is monotone increasing we have by (2.11)

$$\lim_{n\to\infty} \frac{x_{1n}(W^{\gamma,\mu})}{n} = \frac{(1+\sqrt{\mu})^2}{1-\mu}.$$

EXAMPLE 2.3. Consider Meixner-Pollaczek polynomials $\{P_n^{(\lambda,\phi)}(x)\}_{n=0}^\infty$ satisfying (see [3, p. 32])

$$\begin{aligned} xP_n^{(\lambda,\phi)}(x) =& \frac{\sqrt{(n+1)(n+2\lambda)}}{2\sin\phi} P_{n+1}^{(\lambda,\phi)}(x) - \frac{(n+\lambda)\cos\phi}{\sin\phi} P_n^{(\lambda,\phi)}(x) \\ &+ \frac{\sqrt{n(n+2\lambda-1)}}{2\sin\phi} P_{n-1}^{(\lambda,\phi)}(x), \quad n \geq 0, \end{aligned}$$

which are orthogonal with respective to the weight

$$W^{\lambda,\phi}(x) = e^{(2\phi-\pi)x}|\Gamma(\lambda + ix)|^2, \quad 0 < \phi < \pi, \ \lambda > 0, \quad -\infty < x < \infty.$$

Clearly, the conditions in Theorem 2.5 are satisfied with $A = 1$ and $b = -2\cos\phi$. Since $\{a_n\}_{n=0}^{\infty}$ is monotone increasing we have (2.11)

$$\lim_{n\to\infty} \frac{x_{nn}(W^{\lambda,\phi})}{n} = -\frac{1+\cos\phi}{\sin\phi} \quad \text{if} \quad 0 < \phi \le \frac{\pi}{2} \tag{2.16}$$

$$\lim_{n\to\infty} \frac{x_{1n}(W^{\lambda,\phi})}{n} = \frac{1-\cos\phi}{\sin\phi} \quad \text{if} \quad \pi/2 \le \phi < \pi. \tag{2.17}$$

Freud [2] have shown (2.17) for $\phi = \pi/2$.

3. Zeros of orthogonal polynomials for the weight $g^2(x)d\alpha(x)$

In this section we consider another distribution function $d\beta(x) = g^2(x)d\alpha(x)$, where $g(x)$ is a polynomial with only real zeros and then compare the asymptotic behavior of extreme zeros of $\{P_n(d\alpha; x)\}_{n=0}^{\infty}$ and $\{P_n(d\beta; x)\}_{n=0}^{\infty}$.

We first need to find the connections between $a_n(d\alpha)$, $a_n(d\beta)$ and $b_n(d\alpha)$, $b_n(d\beta)$ when $d\beta(x) = (x-\xi)^2 d\alpha(x)$, $\xi \in \mathbb{R}$.

Proposition 3.1. *Let $d\beta(x) = (x-\xi)^2 d\alpha(x)$, $\xi \in \mathbb{R}$. Then for $n \ge 0$*

$$\gamma_n(d\beta) = \gamma_{n+1}(d\alpha)\left[\frac{K_{n+1}(d\alpha;\xi,\xi)}{K_{n+2}(d\alpha;\xi,\xi)}\right]^{1/2}, \tag{3.1}$$

$$a_n(d\beta) = a_{n+1}(d\alpha)\left[\frac{K_n(d\alpha;\xi,\xi)K_{n+2}(d\alpha;\xi,\xi)}{K_{n+1}^2(d\alpha;\xi,\xi)}\right]^{1/2}, \tag{3.2}$$

and

$$b_n(d\beta) = b_n(d\alpha) + \frac{\gamma_n(d\beta)}{\gamma_{n+1}(d\alpha)} d_{n+1} - \frac{\gamma_{n-1}(d\beta)}{\gamma_n(d\alpha)} d_n \quad (\gamma_{-1}(d\beta) = 0), \tag{3.3}$$

where

(3.4)
$$d_{n+1} := \frac{\gamma_n(d\beta)}{\gamma_{n+1}(d\alpha)}\left[(b_{n+1}(d\alpha) - \xi) - a_{n+1}(d\alpha)\frac{P_n(d\alpha;\xi)P_{n+1}(d\alpha;\xi)}{K_{n+1}(d\alpha;\xi,\xi)}\right].$$

Proof. Expanding $(x-\xi)^2 P_n(d\beta;x)$ in terms of $\{P_k(d\alpha;x)\}_{n=0}^{n+2}$, we can obtain easily

$$(3.5)\quad \begin{aligned}&(x-\xi)^2 P_n(d\beta;x)\\ &= \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)}P_n(d\alpha;x) + d_{n+1}P_{n+1}(d\alpha;x) + \frac{\gamma_n(d\beta)}{\gamma_{n+2}(d\alpha)}P_{n+2}(d\alpha;x),\end{aligned}$$

where d_{n+1} is a constant. Since the left hand side of the equation (3.5) has $x=\xi$ as a double root,

$$(3.6)\quad \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)}P_n(d\alpha;\xi) + d_{n+1}P_{n+1}(d\alpha;\xi) + \frac{\gamma_n(d\beta)}{\gamma_{n+2}(d\alpha)}P_{n+2}(d\alpha;\xi) = 0;$$

$$(3.7)\quad \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)}P'_n(d\alpha;\xi) + d_{n+1}P'_{n+1}(d\alpha;\xi) + \frac{\gamma_n(d\beta)}{\gamma_{n+2}(d\alpha)}P'_{n+2}(d\alpha;\xi) = 0.$$

Multiplying (3.6) by $P'_{n+1}(d\alpha;\xi)$, (3.7) by $P_{n+1}(d\alpha;\xi)$ and then substracting these two equations give

$$\frac{\gamma_{n+1}(d\alpha)}{\gamma_n(d\beta)}K_{n+1}(d\alpha;\xi,\xi) = \frac{\gamma_n(d\beta)}{\gamma_{n+1}(d\alpha)}K_{n+2}(d\alpha;\xi,\xi),$$

which gives (3.1) and (3.2). Now, multiplying (3.6) by $P'_n(d\alpha;\xi)$, (3.7) by $P_n(d\alpha;\xi)$ and then substracting these two equations give

$$(3.8)\quad d_{n+1} = \frac{\gamma_n(d\beta)}{\gamma_{n+2}(d\alpha)}\frac{P'_{n+2}(d\alpha;\xi)P_n(d\alpha;\xi) - P_{n+2}(d\alpha;\xi)P'_n(d\alpha;\xi)}{P_{n+1}(d\alpha;\xi)P'_n(d\alpha;\xi) - P_n(d\alpha;\xi)P'_{n+1}(d\alpha;\xi)}.$$

Using the three term recurrence relation (1.1) ,and the confluent kernel formula (1.2) we can obtain the equation (3.4) from (3.8). Since $\{P_n(d\beta;x)\}_{n=0}^{\infty}$ is also an orthonormal polynomial system we have

$$(3.9)\quad \begin{aligned}&(x-\xi)^2[a_{n+1}(d\beta)P_{n+1}(d\beta;x) - xP_n(d\beta;x)\\ &\qquad + b_n(d\beta)P_n(d\beta;x) + a_n(d\beta)P_{n-1}(d\beta;x)] = 0.\end{aligned}$$

Expanding the left hand side of (3.9) in terms of $\{P_k(d\alpha;x)\}_{k=0}^{n+3}$ by using (1.1) and (3.5), we can obtain (3.3) from the coefficient of $P_n(d\alpha;x)$. □

PROPOSITION 3.2. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be orthonormal polynomials satisfying*

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x), \quad n=0,1,\cdots,$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$. Let $0 < s < 1$ and $0 < p < \infty$. If $\lim_{n\to\infty} a_{n+1}/a_n = 1$ and $\lim_{n\to\infty} b_n/a_n = b$, then

$$(3.10) \qquad \lim_{n\to\infty} \max_{x\in[b_n-2sa_n, b_n+2sa_n]} \frac{|P_n(x)|^p}{\sum_{k=0}^{n-1} |P_k(x)|^p} = 0.$$

If moreover $0 < \lim_{n\to\infty} a_n = a < \infty$, then

$$(3.11) \qquad \lim_{n\to\infty} \max_{x\in[a(b-2), a(b+2)]} \frac{|P_n(x)|^p}{\sum_{k=0}^{n-1} |P_k(x)|^p} = 0.$$

Proof. See Theorem 8 in [7] for (3.10) and Theorem 2.1 in [9] for (3.11). □

THEOREM 3.3. *Assume that $\lim_{n\to\infty} \frac{a_{n+1}(d\beta)}{a_n(d\alpha)} = 1$, $\lim_{n\to\infty} \frac{b_n(d\beta)}{a_n(d\alpha)} = b$ and $\lim_{n\to\infty} a_n(d\alpha) = a$, $0 < a \leq \infty$. If $a = \infty$, then we also assume $|b| < 2$. Then for any $\xi \in [a(b-2), a(b+2)]$ and $d\beta(x) = (x-\xi)^2 d\alpha(x)$, we have*

$$(3.12) \qquad \lim_{n\to\infty} \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)} = a, \qquad \lim_{n\to\infty} \frac{a_n(d\beta)}{a_n(d\alpha)} = 1$$

and

$$(3.13) \quad \lim_{n\to\infty} a_n(d\beta) = a, \qquad \lim_{n\to\infty} \frac{a_{n+1}(d\beta)}{a_n(d\beta)} = 1, \qquad \lim_{n\to\infty} \frac{b_n(d\beta)}{a_n(d\beta)} = b.$$

Moreover, if $a = \infty$ (and so $|b| < 2$), then

$$(3.14) \qquad \begin{cases} \lim_{n\to\infty} \frac{x_{1n}(d\alpha)}{x_{1n}(d\beta)} = 1 & \text{if} \quad 0 \leq b < 2 \\ \\ \lim_{n\to\infty} \frac{x_{nn}(d\alpha)}{x_{nn}(d\beta)} = 1 & \text{if} \quad -2 < b \leq 0. \end{cases}$$

Proof. If $0 < a < \infty$, then by (3.11) we have for $\xi \in [a(b-2), a(b+2)]$

$$0 \leq \lim_{n\to\infty} \frac{P_n^2(d\alpha;\xi)}{K_n(d\alpha;\xi,\xi)} \leq \lim_{n\to\infty} \max_{[a(b-2),a(b+2)]} \frac{P_n^2(d\alpha;x)}{K_n(d\alpha;x,x)} = 0.$$

If $a = \infty$ and $|b| < 2$, then $a(b-2) = -\infty$ and $a(b+2) = \infty$. In this case choose s with $0 \leq |b|/2 < s < 1$. Then,

$$\lim_{n\to\infty} (b_n - 2sa_n) = \lim_{n\to\infty} a_n(b_n/a_n - 2s) = -\infty$$

and

$$\lim_{n\to\infty} (b_n + 2sa_n) = \lim_{n\to\infty} a_n(b_n/a_n + 2s) = \infty.$$

Hence, for any $\xi \in \mathbb{R}$ there exists an integer $N = N(\xi) \geq 1$ such that $\xi \in [b_n - 2sa_n, b_n + 2sa_n]$, $n \geq N$. By (3.10) we have

$$0 \leq \lim_{n\to\infty} \frac{P_n^2(d\alpha;\xi)}{K_n(d\alpha;\xi,\xi)} \leq \lim_{n\to\infty} \max_{[b_n-2sa_n,b_n+2sa_n]} \frac{P_n^2(d\alpha;x)}{K_n(d\alpha;x,x)} = 0.$$

Hence, for $\xi \in [a(b-2), a(b+2)]$ $(0 < a \leq \infty)$ we have

$$\lim_{n\to\infty} \frac{K_{n+1}(d\alpha;\xi,\xi)}{K_n(d\alpha;\xi,\xi)} = 1 + \lim_{n\to\infty} \frac{P_n^2(d\alpha;\xi)}{K_n(d\alpha;\xi,\xi)} = 1$$

so that all the limits in (3.12) and (3.13) except the last one of (3.13) follow from (3.2) and the following identities:

$$\frac{a_n(d\beta)}{a_n(d\alpha)} = \frac{a_{n+1}(d\alpha)}{a_n(d\alpha)} \frac{\sqrt{K_n(d\alpha;\xi,\xi)K_{n+2}(d\alpha;\xi,\xi)}}{K_{n+1}(d\alpha;\xi,\xi)}$$

$$\frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)} = a_{n+1}(d\alpha)\left[\frac{K_{n+2}(d\alpha;\xi,\xi)}{K_{n+1}(d\alpha;\xi,\xi)}\right]^{1/2}$$

$$\frac{a_{n+1}(d\beta)}{a_n(d\beta)} = \frac{a_{n+2}(d\alpha)}{a_{n+1}(d\alpha)}\left(\frac{K_{n+1}^3(d\alpha;\xi,\xi)K_{n+3}(d\alpha;\xi,\xi)}{K_n(d\alpha;\xi,\xi)K_{n+2}^3(d\alpha;\xi,\xi)}\right)^{\frac{1}{2}}.$$

Since

$$0 \leq \lim_{n\to\infty} \frac{|P_n(d\alpha;\xi)P_{n+1}(d\alpha;\xi)|}{K_{n+1}(d\alpha;\xi,\xi)} \leq \lim_{n\to\infty} \frac{P_n^2(d\alpha;\xi) + P_{n+1}^2(d\alpha;\xi)}{2K_{n+1}(d\alpha;\xi,\xi)} = 0$$

and

$$\lim_{n\to\infty} \frac{\gamma_n(d\beta)}{\gamma_{n+1}(d\alpha)} = 1,$$

we have

$$\lim_{n\to\infty} \frac{d_{n+1}}{a_n(d\beta)} = b - \frac{\xi}{a}, \qquad \lim_{n\to\infty} \frac{d_n}{a_n(d\beta)} = b - \frac{\xi}{a}.$$

Hence, the last limit in (3.13) follows from the relations (3.3) and (3.4).

Now assume that $a = \infty$ and $0 \leq b < 2$ (the case for $-2 < b \leq 0$ can be handled similarly). First note that both $d\alpha(x)$ and $d\beta(x)$ satisfy the conditions in Theorem 2.5 so that (2.11) holds for both $d\alpha(x)$ and $d\beta(x)$. The second limit in (3.12) implies that for any $\epsilon > 0$, there is an integer $N = N(\epsilon)$ such that

$$a_n(d\alpha) \leq (1+\epsilon)a_n(d\beta), \quad n \geq N.$$

Let $\{n_s\}_{s=1}^{\infty}$ be the increasing sequence of integers as in the proof of Theorem 2.5. Since $\{n_s\}_{s=1}^{\infty}$ is increasing, for sufficiently large n, there exists n_s such that $N \leq n_s \leq n-1 < n_{s+1}$. Hence, we have for sufficiently large n,

$$\max_{0\leq k\leq n-1} a_k(d\alpha) = a_{n_s}(d\alpha) \leq (1+\epsilon)a_{n_s}(d\beta) \leq (1+\epsilon) \max_{0\leq k\leq n-1} a_k(d\beta)$$

so that

$$\limsup_{n\to\infty} \frac{x_{1n}(d\alpha)}{x_{1n}(d\beta)}$$
$$= \limsup_{n\to\infty} \frac{x_{1n}(d\alpha)}{\max_{0\leq k\leq n-1} a_k(d\alpha)} \cdot \frac{\max_{0\leq k\leq n-1} a_k(d\beta)}{x_{1n}(d\beta)} \cdot \frac{\max_{0\leq k\leq n-1} a_k(d\alpha)}{\max_{0\leq k\leq n-1} a_k(d\beta)} \leq 1+\epsilon.$$

Hence,

$$\limsup_{n\to\infty} \frac{x_{1n}(d\alpha)}{x_{1n}(d\beta)} \leq 1.$$

Interchanging the roles of $d\alpha(x)$ and $d\beta(x)$ we can also show

$$\liminf_{n\to\infty} \frac{x_{1n}(d\alpha)}{x_{1n}(d\beta)} \geq 1.$$

Therefore, (3.14) is proved for $0 \leq b < 2$. □

REMARK 3.1. Under the same hypotheses on $d\alpha(x)$ as in Theorem 3.3, the conclusions of Theorem 3.3 remains true by induction for $d\beta(x) = g^2(x)d\alpha(x)$, where $g(x)$ is any polynomial, of which all zeros are in $[a(b-2), a(b+2)]$.

EXAMPLE 3.1. Let $W_g^{\lambda,\phi}(x) = g^2(x)W^{\lambda,\phi}(x)$, where $g(x)$ is a polynomial of which zeros are all real and $W^{\lambda,\phi}(x)$ is the orthogonalizing weight for Meixner-Pollaczek polynomials as in Example 2.3. Then, by Remark 3.1, we have:

$$\lim_{n\to\infty} \frac{x_{nn}(W_g^{\lambda,\phi})}{n} = -\frac{1+\cos\phi}{\sin\phi}, \quad \text{if} \quad 0 < \phi \leq \frac{\pi}{2}$$

and

$$\lim_{n\to\infty} \frac{x_{1n}(W_g^{\lambda,\phi})}{n} = \frac{1-\cos\phi}{\sin\phi}, \quad \text{if} \quad \frac{\pi}{2} \leq \phi < \pi.$$

EXAMPLE 3.2. Let $d\alpha(x) = e^{-x^2}dx$, $-\infty < x < \infty$. Then the corresponding orthonormal polynomials are Hermite polynomials $\{H_n\}_{n=0}^{\infty}$ satisfying

$$xH_n(x) = \sqrt{\frac{n+1}{2}}H_{n+1}(x) + \sqrt{\frac{n}{2}}H_{n-1}(x), \quad n \geq 0.$$

Then for any polynomial $g(x)$ whose zeros are all real we have by Remark 3.1

$$\lim_{n\to\infty} \frac{x_{1n}(e^{-x^2})}{\sqrt{2n}} = \lim_{n\to\infty} \frac{x_{1n}(g^2(x)e^{-x^2})}{\sqrt{2n}} = 1.$$

It is also shown in [5, 6] for any $\rho > -1$

$$\lim_{n\to\infty} \frac{x_{1n}(|x|^\rho e^{-x^2})}{\sqrt{2n}} = 1.$$

When $d\alpha(x)$ is symmetric and $\xi = 0$ we can obtain a result similar to Theorem 3.3 under a more relaxed condition.

LEMMA 3.4. *Let $d\alpha(x)$ and $d\beta(x)$ be two distributional functions and $n \geq 1$ a fixed integer.*

(1) *If there exists a positive constant A_n such that*

$$a_k(d\alpha) \leq A_n a_k(d\beta) \quad \text{and} \quad b_k(d\alpha) \leq A_n b_k(d\beta), \quad k = 0, 1, \cdots, n-1, \tag{3.15}$$

then $x_{1n}(d\alpha) \leq A_n x_{1n}(d\beta)$.

(2) *If there exists a positive constant A_n such that*

$$a_k(d\alpha) \leq A_n a_k(d\beta) \quad \text{and} \quad b_k(d\alpha) \geq A_n b_k(d\beta), \quad k = 0, 1, \cdots, n-1, \tag{3.16}$$

then $x_{nn}(d\alpha) \geq A_n x_{1n}(d\beta)$.

Proof. Recall that the minimum in (2.4) is obtained when J_k's are alternating, i.e., $J_k J_{k+1} \leq 0$, $k = 0, 1, \cdots, n-2$. Hence,

$$\begin{aligned} x_{nn}(d\alpha) &= \min_J \frac{\sum_{k=0}^{n-1} b_k(d\alpha)J_k^2 + 2\sum_{k=0}^{n-2} a_{k+1}(d\alpha)J_k J_{k+1}}{\sum_{k=0}^{n-1} J_k^2} \\ &\geq \min_J \frac{\sum_{k=0}^{n-1} A_n b_k(d\beta)J_k^2 + 2\sum_{k=0}^{n-2} A_n a_{k+1}(d\beta)J_k J_{k+1}}{\sum_{k=0}^{n-1} J_k^2} \\ &= A_n x_{nn}(d\beta). \end{aligned}$$

Similarly, we can prove (1) since the maximum in (2.3) is obtained when $J_k \geq 0$, $k = 0, 1, \cdots, n-1$. □

THEOREM 3.5. *Assume that $d\alpha(x)$ is symmetric and let $d\beta(x) = x^{2\rho} d\alpha(x)$, where ρ is a positive integer. If $a_n(d\alpha) \sim a_{n+1}(d\alpha)$ then*

$$a_n(d\alpha) \sim a_n(d\beta) \sim a_{n+1}(d\beta) \tag{3.17}$$

so that

$$x_{1n}(d\beta) \sim x_{1n}(d\alpha) \quad \text{and} \quad x_{nn}(d\beta) \sim x_{nn}(d\alpha). \tag{3.18}$$

Proof. First we assume that $\rho = 1$ so that $d\beta(x) = x^2 d\alpha(x)$. Since $d\alpha(x)$ is symmetric, $P_{2k+1}(d\alpha; 0) = 0$ and so $K_{2k+1}(d\alpha; 0, 0) = K_{2k+2}(d\alpha; 0, 0)$, $k \geq 0$. Hence we obtain from (3.2) the following identities:

$$\frac{a_{2k}(d\alpha)}{a_{2k}(d\beta)} = \frac{a_{2k}(d\alpha)}{a_{2k+1}(d\alpha)} \left(1 + \frac{P_{2k}^2(d\alpha; 0)}{K_{2k+1}(d\alpha; 0, 0)}\right)^{-\frac{1}{2}}; \tag{3.19}$$

$$\frac{a_{2k+1}(d\alpha)}{a_{2k+1}(d\beta)} = \frac{a_{2k+1}(d\alpha)}{a_{2k+2}(d\alpha)} \left(1 - \frac{P_{2k+2}^2(d\alpha; 0)}{K_{2k+3}(d\alpha; 0, 0)}\right)^{\frac{1}{2}} \quad k \geq 0. \tag{3.20}$$

On the other hand, since $a_n(d\alpha) \sim a_{n+1}(d\alpha)$ there exist positive constants C_1 and C_2 such that

$$C_1 \leq \frac{P_{2k-2}^2(d\alpha; 0)}{P_{2k}^2(d\alpha; 0)} = \frac{a_{2k}^2(d\alpha)}{a_{2k-1}^2(d\alpha)} \leq C_2.$$

Hence, we have

$$1 - \frac{1}{C_1 + 1} \leq 1 - \frac{P_{2k}^2(d\alpha; 0)}{P_{2k-2}^2(d\alpha; 0) + P_{2k}^2(d\alpha; 0)} \leq 1 - \frac{P_{2k}^2(d\alpha; 0)}{K_{2k+1}(d\alpha; 0, 0)} \leq 1, \quad k \geq 1,$$

so that

$$1 - \frac{P_{2k}^2(d\alpha; 0)}{K_{2k+1}(d\alpha; 0, 0)} \sim 1. \tag{3.21}$$

Thus, (3.17) is proved by (3.19), (3.20), and (3.21) and so is (3.18) by Lemma 3.4. Finally, the general case for $\rho \geq 1$ follows by induction. □

EXAMPLE 3.3. Let $W_Q(x) = \exp(-Q(x))$ be a Freud weight such that

(i) $Q : \mathbb{R} \to \mathbb{R}$ is even and continuous
(ii) Q'' is continuous in $(0, \infty)$

(iii) $Q' > 0$ in $(0, \infty)$

(iv) there are positive constants $C_1 \geq 1$ and C_2 such that

$$C_1 \leq \frac{d/dx(xQ'(x))}{Q'(x)} \leq C_2, \quad x > 0.$$

For any $u \geq 0$, let a_u^* be the Mhaskar-Rahmanov-Saff number for $W_Q(x)$, which is the unique positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u^* t Q'(a_u^* t)}{\sqrt{1-t^2}} \, dt, \quad u > 0.$$

Then, it is shown in [4, Theorem 12.3 (b) and Lemma 5.2 (c)] that

$$x_{1n}(W_Q^2) \sim a_n(W_Q^2) \sim a_{n+1}(W_Q^2) \sim a_n^*, \quad \lim_{n\to\infty} a_n(W_Q^2) = \infty,$$

and

$$\lim_{n\to\infty} \frac{x_{1n}(W_Q^2)}{a_n^*} = 1. \tag{3.22}$$

Hence, $W_{\rho Q}(x) := x^\rho W_Q(x)$ with ρ a nonnegative integer, we have by Theorem 3.5

$$a_n(W_{\rho Q}^2) \sim a_n(W_Q^2) \sim a_n^*$$

and

$$x_{1n}(W_{\rho Q}^2) \sim x_{1n}(W_Q^2) \sim a_n^*.$$

Moreover, we have by Theorem 3.3: If

$$\lim_{n\to\infty} \frac{a_{n+1}(W_Q^2)}{a_n(W_Q^2)} = 1, \tag{3.23}$$

then for any polynomial $g(x)$ with only real zeros

$$\lim_{n\to\infty} \frac{a_{n+1}(g^2 W_Q^2)}{a_n(W_Q)^2} = 1$$

and

$$\lim_{n\to\infty} \frac{x_{1n}(g^2 W_Q^2)}{x_{1n}(W_Q^2)} = \lim_{n\to\infty} \frac{x_{nn}(g^2 W_Q^2)}{x_{nn}(W_Q^2)} = 1. \tag{3.24}$$

Let $W_\beta(x) := \exp(-\frac{1}{2}|x|^\beta)$, $\beta > 1$. Then the corresponding Mhaskar-Rahmanov-Saff number is

$$a_u^* = 2\left[\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta}{2}+1)}{\Gamma(\beta+1)}\right]^{\frac{1}{\beta}} u^{\frac{1}{\beta}}, \quad u > 0 \tag{3.25}$$

and Lubinsky et. al. [5] (see also [6]) proved

$$\lim_{n\to\infty} n^{-\frac{1}{\beta}} a_n(W_\beta^2) = \left[\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta}{2}+1)}{\Gamma(\beta+1)}\right]^{\frac{1}{\beta}} \tag{3.26}$$

so that the condition (3.23) is satisfied for W_β. Therefore, we have by (3.12), (3.24), (3.25), and (3.26)

(3.27)

$$\lim_{n\to\infty} n^{-\frac{1}{\beta}} x_{1n}(g^2W_\beta^2) = -\lim_{n\to\infty} n^{-\frac{1}{\beta}} x_{nn}(g^2W_\beta^2) = 2\left[\frac{\Gamma(\frac{\beta}{2})\Gamma(\frac{\beta}{2}+1)}{\Gamma(\beta+1)}\right]^{\frac{1}{\beta}}.$$

Freud [2] proved (3.27) for $g(x) = |x|^{\rho/2}$, $\rho > -1$, and $\beta = 2, 4, 6$ and conjectured (3.27) for $g(x) = |x|^{\rho/2}$, $\rho > -1$ and any $\beta > 0$. Hence, (3.27) partially extends Freud's conjecture, which is now settled down for $|x|^\rho W_\beta^2(x)$ $(\rho > -1, \beta > 1)$ by the following result of Rahmanov.

THEOREM 3.6.([11]) Let $w(x)$ (not necessarily even) be a weight on $\mathbb{R}$ and $X_n := \max\{|x_{1n}(w)|, |x_{nn}(w)|\}$. If there is a constant $\beta > 1$ such that $\lim_{x\to\infty} x^{-\beta}\log w(x) = -1$, then

$$\lim_{n\to\infty} n^{-1/\beta} X_n = 2[\Gamma(\beta/2)\Gamma(\beta/2+1)/\Gamma(\beta+1)]^{1/\beta}. \tag{3.28}$$

In particular, when $g(x)$ is a polynomial with only real zeros, which is neither even nor odd, (3.27) is a little sharper than (3.28).

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