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SIGN PATTERNS OF IDEMPOTENT MATRICES

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ABSTRACT. Sign patterns of idempotent matrices, especially symmetric idempotent matrices, are investigated. A number of fundamental results are given and various constructions are presented. The sign patterns of symmetric idempotent matrices through order 5 are determined. Some open questions are also given.

0. Introduction

A matrix whose entries come from the set $\{+, -, 0\}$ is called a *sign pattern matrix* (or sign pattern, or pattern). We denote the set of all $n \times n$ sign pattern matrices by Q_n. For a real matrix B, by sgn B we mean the sign pattern matrix in which each positive (respectively, negative, zero) entry of B is replaced by $+$ (respectively, $-, 0$). If $A \in Q_n$, then the *sign pattern class of A* is defined by

$$Q(A) = \{B \in M_n(R) \mid \text{sgn } B = A\}.$$

Suppose P is a property referring to a real matrix. Then A is said to *require* P if every real matrix in $Q(A)$ has property P, or to *allow* P if some real matrix in $Q(A)$ has property P.

A permutation sign pattern matrix P is obtained by replacing the 1's in a real permutation matrix by $+$ signs. Then P^TAP gives a "permutation similarity" of the pattern A. A signature pattern S is an $n \times n$ diagonal pattern with nonzero diagonal entries. Hence, the product S^2 is an $n \times n$ diagonal pattern with $+$ diagonal entries (indicated subsequently by I_n or I). Then SAS gives a "signature similarity" of the pattern A.

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If A is an $n \times n$ sign pattern, then A is sign nonsingular if every $B \in Q(A)$ is nonsingular. Sign nonsingular matrices have been heavily studied (see [9] for example), and it is well known that they have unambiguously signed determinants; that is, there is at least one nonzero term in the determinant, and all nonzero terms have the same sign.

By the minimum rank of an $n \times n$ sign pattern matrix A we mean $\min_{B \in Q(A)}\{\text{rank } B\}$, and denote this by mrA.

If A is an $n \times n$ matrix, then A is permutation similar to a Frobenius normal form matrix

$$\begin{pmatrix} A_{11} & & * \\ & \ddots & \\ 0 & & A_{mm} \end{pmatrix},$$

where the (square) diagonal blocks A_{ii} are the irreducible components of A. The one-by-one irreducible components can be zero blocks.

We introduce the symbol # to represent a qualitatively ambiguous sum, that is, $\# = (+) + (-)$. We next recall [4] that a *generalized sign pattern matrix* $\hat{A} = (\hat{a}_{ij})$ is a matrix whose entries are in the set $\{+, -, 0, \#\}$, and $Q(\hat{A}) = \{B = (b_{ij}) \in M_n(R) | b_{ij}$ is arbitrary if $\hat{a}_{ij} = \#$; sgn $b_{ij} = \hat{a}_{ij}$ if $\hat{a}_{ij} \in \{+, -, 0\}\}$. Two generalized sign pattern matrices $\hat{A}_1$ and $\hat{A}_2$ are said to be *compatible*, denoted by $\hat{A}_1 \overset{c}{\leftrightarrow} \hat{A}_2$, if there exists a matrix $B \in Q(\hat{A}_1) \cap Q(\hat{A}_2)$. Hence

$$\hat{A}_1 = \begin{pmatrix} \# & + \\ - & \# \end{pmatrix} \overset{c}{\leftrightarrow} \begin{pmatrix} + & + \\ \# & 0 \end{pmatrix} = \hat{A}_2.$$

As in [3], $\mathcal{ID}$ is the class of all square patterns A for which there exists $B \in Q(A)$ where $B^2 = B$ (A allows a real idempotent). We further let $\mathcal{SID}$ denote the class of all symmetric patterns A for which there exists symmetric $B \in Q(A)$ where $B^2 = B$ (A allows a real symmetric idempotent). Clearly, $\mathcal{SID} \subseteq \mathcal{ID}$. Further, if $A \in \mathcal{ID}$, then by necessity, $A^2 \overset{c}{\leftrightarrow} A$ must hold.

The nonnegative patterns in $\mathcal{ID}$ were characterized in [3]. A square sign pattern matrix A is said to be sign idempotent when $A^2 = A$; these patterns were originally discussed in [1]. The sign idempotent patterns in $\mathcal{ID}$ were recently characterized in [7].

In this paper we more generally investigate the classes $\mathcal{ID}$ and $\mathcal{SID}$. In particular, a number of basic general results are given, various constructions are presented, and all patterns in $\mathcal{SID}$ through order 5 are determined. A number of open questions are given.

1. General results

The proof of the following lemma should be clear.

LEMMA 1.1. *Each of the classes $\mathcal{ID}$ and $\mathcal{SID}$ is closed under the following operations:*

(i) *permutation similarity,*
(ii) *signature similarity,*
(iii) *transposition,*
(iv) *direct sum,*
(v) *Kronecker product.*

THEOREM 1.2. *Suppose that $A \in Q_n$ and $mrA = 1$. Then $A \in \mathcal{ID}$ if and only if $A^2 \overset{c}{\leftrightarrow} A$.*

Proof. We have already observed that $A \in \mathcal{ID}$ implies $A^2 \overset{c}{\leftrightarrow} A$. Now let $A = uv^T$. Then $A^2 \overset{c}{\leftrightarrow} A \iff uv^Tuv^T \overset{c}{\leftrightarrow} uv^T \iff v^Tu \overset{c}{\leftrightarrow} +$. With $B = xy^T$, where sgn $x = u$, sgn $y = v$, and $y^Tx = 1$, we have $B^2 = B \in Q(A)$. □

It is clear that a symmetric pattern A is in $\mathcal{SID}$ if and only if each irreducible component of A is in $\mathcal{SID}$. We let J_n denote the all $+$ pattern of order n.

COROLLARY 1.3. *Suppose that A is an $n \times n$ irreducible symmetric sign pattern matrix, and $mrA = 1$. Then $A \in \mathcal{SID}$ if and only if A is signature similar to J_n.*

Proof. Assume $A \in \mathcal{SID}$. Then $A^2 \overset{c}{\leftrightarrow} A$. Since A is irreducible and symmetric, it is then easily seen that all the diagonal entries of A are $+$. With $mrA = 1$, we have $A = uu^T$, where each entry in u is nonzero. Hence A is signature similar to J_n.

Conversely, suppose A is signature similar to J_n. If B is the $n \times n$ matrix each of whose entries is $1/n$, then B is a symmetric idempotent in $Q(J_n)$. Thus, $A \in \mathcal{SID}$. □

Since the only nonsingular idempotent matrix is the identity matrix, the following proposition is clear.

PROPOSITION 1.4. *The only sign nonsingular sign pattern matrix in $\mathcal{ID}$ is I_n.*

COROLLARY 1.5. *The only 2×2 sign patterns in $\mathcal{ID}$ are $0, I_2$, or the 2×2 sign patterns A where $mr A = 1$ and $A^2 \overset{c}{\leftrightarrow} A$.*

Under equivalence (permutation similarity, signature similarity, and transposition), we find six representatives of the 2×2 sign patterns in $\mathcal{ID}$:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}, \begin{pmatrix} + & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} + & + \\ + & + \end{pmatrix}, \begin{pmatrix} 0 & + \\ 0 & + \end{pmatrix}, \begin{pmatrix} + & + \\ - & - \end{pmatrix}.$$

Clearly, the first four patterns are in $\mathcal{SID}$.

Every $n \times n$ matrix B is a principal submatrix of a $2n \times 2n$ idempotent matrix of rank n, as it is easily checked that

$$\begin{pmatrix} B & B \\ I - B & I - B \end{pmatrix}$$

is idempotent and has trace n. We then have the following result.

PROPOSITION 1.6. *Every $n \times n$ sign pattern matrix is a principal submatrix of a $2n \times 2n$ pattern in $\mathcal{ID}$.*

The following is an interesting related question. Is every $n \times n$ symmetric sign pattern matrix with positive diagonal a principal submatrix of a $2n \times 2n$ pattern in $\mathcal{SID}$? At the end of the next section we will show that this is indeed the case.

We now give a general necessary condition for bordering a square real matrix into an idempotent matrix.

LEMMA 1.7. *If $\begin{pmatrix} B & C \\ D & E \end{pmatrix}$ is idempotent, where B is square, then C has at least min $\{\text{rank}(X^2 - X) | X \in Q(sgn\ B)\}$ columns.*

Proof. The block matrix is idempotent implies $B^2 + CD = B$, so that rank $(B^2 - B) =$ rank $(CD) \leq$ rank $C \leq$ the number of columns of C. The result then follows. $\square$

EXAMPLE 1.8. Consider the $n \times n$ sign pattern matrix

$$A_1 = \begin{pmatrix} 0 & + & & & \\ & 0 & + & & \\ & & 0 & \ddots & \\ & & & \ddots & + \\ - & & & & 0 \end{pmatrix}$$

Then, if $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathcal{ID}$, A_2 must have at least n columns. To see this, observe that for each $B \in Q(A_1)$, the eigenvalues of B are the n-th roots of a negative number, so that B has no nonnegative eigenvalue. Hence, $B(I - B)$ is invertible, that is, $\operatorname{rank}(B^2 - B) = n$. That A_2 must have at least n columns follows from lemma 1.7.

Proposition 1.6 and example 1.8 show that $2n$ is the smallest positive integer k such that every $n \times n$ sign pattern matrix is a principal submatrix of a $k \times k$ pattern in $\mathcal{ID}$.

We now present an important bordering result, where we "stretch" the last block row and column. A similar result was proved in [8] for normal matrices. The following lemma can be proved by direct block multiplication.

LEMMA 1.9. *Let B_1 and X be square matrices, and let k be any positive integer. If the real matrix*

$$B = \begin{pmatrix} B_1 & U \\ V & X \end{pmatrix}$$

is idempotent, then the matrix

$$\tilde{B} = \begin{pmatrix} B_1 & \frac{1}{\sqrt{k}}U & \frac{1}{\sqrt{k}}U & \cdots & \frac{1}{\sqrt{k}}U \\ \frac{1}{\sqrt{k}}V & \frac{1}{k}X & \frac{1}{k}X & \cdots & \frac{1}{k}X \\ \frac{1}{\sqrt{k}}V & \frac{1}{k}X & \frac{1}{k}X & \cdots & \frac{1}{k}X \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{k}}V & \frac{1}{k}X & \frac{1}{k}X & \cdots & \frac{1}{k}X \end{pmatrix}$$

of block size $(k+1)\times(k+1)$ is idempotent.

We remark that rank $\tilde{B} = tr(\tilde{B}) = tr(B) =$ rank B. As a consequence of lemma 1.9 we have the following.

THEOREM 1.10. *If*

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathcal{ID},$$

where A_1 is square, then the square pattern

$$\begin{pmatrix} A_1 & A_2 & \dots & A_2 \\ A_3 & A_4 & \dots & A_4 \\ \vdots & \vdots & \ddots & \vdots \\ A_3 & A_4 & \dots & A_4 \end{pmatrix} \in \mathcal{ID}.$$

COROLLARY 1.11. *If*

$$\begin{pmatrix} A_1 & A_2 \\ A_2^T & A_4 \end{pmatrix} \in \mathcal{SID},$$

where A_1 is square, then the square pattern

$$\begin{pmatrix} A_1 & A_2 & \dots & A_2 \\ A_2^T & A_4 & \dots & A_4 \\ \vdots & \vdots & \ddots & \vdots \\ A_2^T & A_4 & \dots & A_4 \end{pmatrix} \in \mathcal{SID}.$$

We will particularly apply this corollary in the next sections.

2. Special results on symmetric patterns

Idempotent matrices, particularly symmetric idempotents, play an important role in a number of applications. In [5], a number of fundamental results on symmetric idempotents, as well as applications to statistics, are given. In the following lemma, we also provide some basic information on symmetric idempotents.

LEMMA 2.1. *Let $B = (b_{ij})$ be a real symmetric idempotent matrix. If $b_{ij} \neq 0$ for some i and j, $i \neq j$, then $0 < b_{ii}, b_{jj} < 1$ and $|b_{ij}| \leq min\{\frac{1}{2}, \sqrt{b_{ii}b_{jj}}\}$.*

Proof. Comparing the (i,i) entries of $B = B^2$, we see that

$$b_{ii} - {b_{ii}}^2 = {b_{ij}}^2 + \sum_{\substack{k \neq i \\ k \neq j}} {b_{ik}}^2 > 0.$$

It follows that $0 < b_{ii} < 1$. Similarly, we get $0 < b_{jj} < 1$. The above equation also shows that ${b_{ij}}^2 \leq b_{ii} - {b_{ii}}^2$. Since the maximum of the quadratic function $x - x^2$ is $\frac{1}{4}$, we see that $|b_{ij}| \leq \frac{1}{2}$. Finally, note that the matrix $\begin{pmatrix} b_{ii} & b_{ij} \\ b_{ij} & b_{jj} \end{pmatrix}$ is positive semidefinite, since it is a principal submatrix of the positive semidefinite matrix B. Hence, det $\begin{pmatrix} b_{ii} & b_{ij} \\ b_{ij} & b_{jj} \end{pmatrix} = b_{ii}b_{jj} - {b_{ij}}^2 \geq 0$. Therefore, $|b_{ij}| \leq \sqrt{b_{ii}b_{jj}}$. □

THEOREM 2.2. *If*

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & + \end{pmatrix} \in \mathcal{SID},$$

where A_2 is a nonzero column, then

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 & A_2 \\ A_2^T & + & - \\ A_2^T & - & + \end{pmatrix} \in \mathcal{SID}.$$

Proof. If

$$B = \begin{pmatrix} B_1 & Y \\ Y^T & \alpha \end{pmatrix} \in Q(A)$$

is a real symmetric idempotent matrix, where $0 < \alpha < 1$ (by lemma 2.1), then it is easily verified that

$$\tilde{B} = \begin{pmatrix} B_1 & \frac{1}{\sqrt{2}}Y & \frac{1}{\sqrt{2}}Y \\ \frac{1}{\sqrt{2}}Y^T & \frac{1}{2}(1+\alpha) & \frac{1}{2}(\alpha - 1) \\ \frac{1}{\sqrt{2}}Y^T & \frac{1}{2}(\alpha - 1) & \frac{1}{2}(1+\alpha) \end{pmatrix} \in Q(\tilde{A})$$

is symmetric idempotent. □

Note that rank $\tilde{B} = tr(\tilde{B}) = tr(B_1) + \alpha + 1 = \text{rank } B + 1$. Applying theorem 2.2 to $\tilde{A}$, and so forth, we obtain the following more general result.

COROLLARY 2.3. *If*

$$\begin{pmatrix} A_1 & A_2 \\ A_2^T & + \end{pmatrix} \in \mathcal{SID},$$

where A_2 is a nonzero column, then the square pattern

$$\begin{pmatrix} A_1 & A_2 & A_2 & \cdots & A_2 \\ A_2^T & + & - & \cdots & - \\ A_2^T & - & + & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & - \\ A_2^T & - & \cdots & - & + \end{pmatrix} \in \mathcal{SID}.$$

COROLLARY 2.4. *For each $1 < k < n$, where $n \geq 3$, the $n \times n$ pattern*

$$\begin{pmatrix} J_{n-k} & & J_{n-k,k} & & \\ & + & - & \cdots & - \\ J_{k,n-k} & - & + & \ddots & \vdots \\ & \vdots & \ddots & \ddots & - \\ & - & \cdots & - & + \end{pmatrix} \in \mathcal{SID}.$$

Proof. Apply corollary 2.3 to

$$\begin{pmatrix} & & & + \\ & J_{n-k} & & \vdots \\ & & & + \\ + & \cdots & + & + \end{pmatrix} \in \mathcal{SID}.$$

□

Some comments concerning corollaries 2.3 and 2.4 are now in order.

(1) Although somewhat laborious, taking similarities of the matrices of the form diag $\{0,\cdots,0,1,\cdots,1\}$ via Householder transformations, one can obtain the same patterns as in corollary 2.4. Different symmetric idempotent matrices are obtained.

(2) We can also see the result in corollary 2.3 as follows. If

$$B=\begin{pmatrix} B_1 & Y \\ Y^T & \alpha \end{pmatrix},\quad 0<\alpha<1$$

is real symmetric idempotent, then so is the matrix

$$\hat{B}=\begin{pmatrix} B_1 & \frac{1}{\sqrt{k}}Y & \frac{1}{\sqrt{k}}Y & \cdots & \frac{1}{\sqrt{k}}Y \\ \frac{1}{\sqrt{k}}Y^T & \frac{k-1+\alpha}{k} & \frac{\alpha-1}{k} & \cdots & \frac{\alpha-1}{k} \\ \frac{1}{\sqrt{k}}Y^T & \frac{\alpha-1}{k} & \frac{k-1+\alpha}{k} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{\alpha-1}{k} \\ \frac{1}{\sqrt{k}}Y^T & \frac{\alpha-1}{k} & \cdots & \frac{\alpha-1}{k} & \frac{k-1+\alpha}{k} \end{pmatrix},$$

where $\hat{B}$ has block size $(k+1)\times(k+1)$. This symmetric idempotent matrix is different from that obtained through the proof of corollary 2.3.

EXAMPLE 2.5. From corollary 2.4 and the use of a permutation similarity, we see that

$$\begin{pmatrix} & & & & & & + \\ & & & & & & \vdots \\ & & & & & & + \\ & & & J_n & & & - \\ & & & & & & + \\ & & & & & & \vdots \\ + & \cdots & + & - & + & \cdots & + \end{pmatrix} \in \mathcal{SID}$$

where the $-$ is in say the k-th position $(1 < k < n+1)$ of the last row/column. Then, by the stretching corollary 1.11, we have that

$$\begin{pmatrix} & J_n & & \begin{matrix} J_{k-1,q} \\ -J_{1,q} \\ J_{n-k,q} \end{matrix} \\ J_{q,k-1} & -J_{q,1} & J_{q,n-k} & J_{q,q} \end{pmatrix} \in \mathcal{SID}$$

for any positive integer q.

In the next section we will give, up to equivalence, all the irreducible patterns of orders 3, 4, and 5 in the $\mathcal{SID}$ class. The compatibility condition $A^2 \overset{c}{\leftrightarrow} A$ turns out to be sufficient for order ≤ 5, except for one particular 5×5 irreducible symmetric pattern (up to equivalence). It is shown in section 3 that this particular 5×5 pattern does not even allow idempotence. However, we now give another example in which a different proof technique is used.

PROPOSITION 2.6. *Let*

$$A = \begin{pmatrix} + & 0 & 0 & + & + & + & + \\ 0 & + & + & 0 & + & + & - \\ 0 & + & + & 0 & + & - & + \\ + & 0 & 0 & + & + & - & - \\ + & + & + & + & + & + & + \\ + & + & - & - & + & + & + \\ + & - & + & - & + & + & + \end{pmatrix}.$$

Then A is irreducible and symmetric, $A^2 \overset{c}{\leftrightarrow} A$, but $A \notin \mathcal{ID}$.

Proof. It is easy to check that A is irreducible and symmetric, and $A^2 \overset{c}{\leftrightarrow} A$. Assume that A allows an idempotent matrix

$$B = \begin{pmatrix} & & & & & a & b \\ & & & & & * & * \\ & & B_1 & & & * & * \\ & & & & & -g & -h \\ & & & & & * & * \\ * & c & -e & * & * & * & * \\ * & -d & f & * & * & * & * \end{pmatrix}$$

where B_1 is 5×5 and the $*$ entries are immaterial. Equating the (1,2), (1,3), (4,2) and (4,3) entries in $B^2 = B$, we get

$$\begin{aligned} s_1 + ac - bd &= 0 \\ s_2 - ae + bf &= 0 \\ s_3 - cg + dh &= 0 \\ s_4 + eg - fh &= 0 \end{aligned}$$

for some $s_i > 0, \quad 1 \le i \le 4$. Hence,

$$\begin{aligned} bd &> ac && (1) \\ ae &> bf && (2) \\ cg &> dh && (3) \\ fh &> eg && (4) \end{aligned}$$

Multiplying (1) and (2), and cancelling, we have $de > cf$. Similarly, multiplying (3) and (4), and cancelling, we get $cf > de$, contradicting $de > cf$. $\square$

The following proposition is useful in finding patterns in $\mathcal{SID}$.

PROPOSITION 2.7. *Suppose*

$$B = \begin{pmatrix} B_1 & x \\ x^T & y \end{pmatrix}$$

is a real symmetric idempotent matrix, where x is a nonzero column. Then B_1 has exactly one eigenvalue α different from 0 and 1, and $\alpha(1-\alpha) = x^T x$. Furthermore, $y = 1 - \alpha$, and rank B_1 = rank B.

Proof. Since B_1 is real symmetric, there exists an orthogonal matrix Q_1 that diagonalizes B_1, that is,

$$Q_1^T B_1 Q_1 = D_1 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Then

$$\begin{pmatrix} Q_1^T & \\ & 1 \end{pmatrix} B \begin{pmatrix} Q_1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} D_1 & Q_1^T x \\ x^T Q_1 & y \end{pmatrix},$$

which is also a real symmetric idempotent matrix. So, $D_1 = D_1^2 + Q_1^T x x^T Q_1$, or

$$\begin{pmatrix} \lambda_1 - \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n - \lambda_n^2 \end{pmatrix} = Q_1^T x x^T Q_1.$$

The right hand side of this last equation has rank 1, and hence so does the left hand side. Hence, there is exactly one λ_j such that $\lambda_j - \lambda_j^2 \neq 0$ (that is, λ_j is different from 0 and 1), and $\lambda_j - \lambda_j^2 = tr(Q_1^T x^T x Q_1) = x^T x$. Let α denote this eigenvalue of B_1. We then have $\alpha - \alpha^2 = x^T x > 0$, so that $0 < \alpha < 1$. Comparison of the (2,2) blocks of B and B^2 gives $y - y^2 = x^T x$. Since both α and y are solutions to the equation $t - t^2 = x^T x$, it follows that $y = \alpha$ or $y = 1 - \alpha$.

We claim that $y = 1 - \alpha$. If $\alpha = 1 - \alpha$, we are done. Otherwise $\alpha \neq \frac{1}{2}$, $0 < \alpha < 1$. If $y = \alpha$, then

$$\text{rank } B = tr(B) = tr(B_1) + y = k + \alpha + \alpha,$$

where k is the algebraic multiplicity of 1 as an eigenvalue of B_1. Since 2α is not an integer, we have a contradiction. Thus $y = 1 - \alpha$. It is now clear that

$$\begin{aligned} \text{rank } B_1 &= 1 + (\text{algebraic multiplicity of 1 as an eigenvalue of } B_1) \\ &= 1 + (tr(B_1) - \alpha) \\ &= tr(B_1) + (1 - \alpha) \\ &= tr(B) \\ &= \text{rank } B. \end{aligned}$$

□

Finally, in this section, we return to the question raised in section 2 on symmetric patterns.

LEMMA 2.8. *Let B_1 be positive definite and $\rho(B_1) < 1$. Let U be the positive definite square root of $B_1 - B_1^2$. Then the block matrix*

$$\begin{pmatrix} B_1 & U \\ U & I - B_1 \end{pmatrix}$$

is a real symmetric idempotent matrix.

Proof. Straightforward. Note that B_1 and U commute. $\square$

PROPOSITION 2.9. *Let A_1 be an $n \times n$ symmetric sign pattern matrix with positive diagonal. Then there exist $n \times n$ symmetric sign pattern matrices A_2 and A_4 such that*

$$\begin{pmatrix} A_1 & A_2 \\ A_2 & A_4 \end{pmatrix} \in \mathcal{SID},$$

and where $A_1 + A_4 \overset{c}{\leftrightarrow} I_n$.

Proof. Fixing all the diagonal entries equal to say $\frac{1}{2}$, and de-emphasizing the off-diagonal entries, we can find a matrix $B_1 \in Q(A_1)$ that satisfies the conditions in the above lemma. The rest is clear. $\square$

REMARK. We may also use [6] to prove the above proposition, by multiplying the second block row in (1.6.13) or (1.6.14) by -1 to get a real symmetric orthogonal matrix Q first, and then $\frac{1}{2}(I + Q)$ gives a real symmetric idempotent matrix.

3. Sign patterns in $\mathcal{ID}$ or $\mathcal{SID}$ of orders ≤ 5

The 2×2 sign patterns in $\mathcal{ID}$ and $\mathcal{SID}$ are given in section 1. We now consider 3×3 sign patterns in $\mathcal{ID}$, and patterns of orders 4 and 5 in $\mathcal{SID}$.

Since the sign patterns of nonnegative idempotent matrices are known (see [3]), it suffices to consider 3×3 sign patterns that are not signature similar to nonnegative patterns.

PROPOSITION 3.1. *Up to permutation similarity, signature similarity, and transposition, there are 33 3×3 sign patterns A such that $A^2 \overset{c}{\leftrightarrow} A$, A has at least one "+" diagonal entry, and A is not signature similar to a nonnegative pattern. Out of these 33 patterns, 13 are in $\mathcal{ID}$, and 1 is in $\mathcal{SID}$.*

We obtain the 33 sign patterns mentioned above by using several Matlab programs. In turn, for each of the 33 patterns, we either produce an idempotent matrix (using some basic observations and Maple) or show that the pattern does not allow idempotence. The following idempotent matrices represent the 13 sign patterns in $\mathcal{ID}$ (the last one represents the one pattern in $\mathcal{SID}$):

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1/2 & -1/2 \\ -1 & -1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 3 \\ -4/3 & -1 & -2 \\ -2/3 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 2 & 2 \\ -3/4 & -1/2 & -3/2 \\ -1/4 & -1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 \\ -1/2 & 0 & -1 \\ -1/2 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 3/2 & 3/2 & 3/2 \\ -1/3 & 0 & -1 \\ -1/6 & -1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix}.$$

In constructing several of the above matrices, the following two facts were used:

(i) $\begin{pmatrix} I & 0 \\ u & B \end{pmatrix}$ is idempotent iff B is idempotent and $Bu = 0$;

(ii) $\begin{pmatrix} B & 0 \\ v & 0 \end{pmatrix}$ is idempotent iff B is idempotent and $(B^T - I)v^T = 0$.

As an illustration of the 20 patterns not in $\mathcal{ID}$ mentioned in proposition 3.1, consider

$$A = \begin{pmatrix} + & + & + \\ - & + & - \\ + & + & + \end{pmatrix}.$$

Suppose $B \in Q(A)$ is idempotent. Then it can be seen that rank B = 2=tr(B). Therefore, rank($B - I$)=rank($I - B$)=1, so that we must have

$$\text{sgn } (B - I) = \begin{pmatrix} + & + & + \\ - & - & - \\ + & + & + \end{pmatrix}.$$

Hence, $b_{11} > 1, b_{33} > 1$. Since $b_{22} > 0$, we then get $\text{tr}(B) > 2$, a contradiction.

Due to the large number of 4×4 and 5×5 sign patterns A such that $A^2 \stackrel{c}{\longleftrightarrow} A$, we restrict our attention to irreducible symmetric patterns.

PROPOSITION 3.2. *Up to permutation similarity and signature similarity, there are* 5 4×4 *irreducible symmetric sign patterns* A *such that* $A^2 \overset{c}{\longleftrightarrow} A$. *All of these* 5 *patterns are in* $\mathcal{SID}$.

As in the 3×3 case, we obtain the 5 patterns mentioned above by using several Matlab programs. Using specific theory from section 2, along with Maple, we find the following idempotent matrices representing the 5 sign patterns in $\mathcal{SID}$:

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} 4-s & 1 & 1 & 0 \\ 1 & s & 0 & -1 \\ 1 & 0 & s & 1 \\ 0 & -1 & 1 & 4-s \end{pmatrix}, \text{ where } s = 2 \pm \sqrt{2},$$

$$\frac{1}{4}\begin{pmatrix} 2+\sqrt{2} & 1 & 1 & 0 \\ 1 & 3-\sqrt{2} & -1 & -t \\ 1 & -1 & 3-\sqrt{2} & t \\ 0 & -t & t & \sqrt{2} \end{pmatrix}, \text{ where } t = \sqrt{2\sqrt{2}-1}.$$

PROPOSITION 3.3. *Up to permutation similarity and signature similarity, there are* 20 5×5 *irreducible symmetric sign patterns* A *such that* $A^2 \overset{c}{\longleftrightarrow} A$. *All of these* 20 *patterns, except one, are in* $\mathcal{SID}$.

The following are the 20 sign patterns mentioned in proposition 3.3:

$$\begin{pmatrix} + & + & + & 0 & 0 \\ + & + & - & - & - \\ + & - & + & + & + \\ 0 & - & + & + & - \\ 0 & - & + & - & + \end{pmatrix}, \begin{pmatrix} + & + & + & 0 & 0 \\ + & + & - & - & - \\ + & - & + & + & + \\ 0 & - & + & + & + \\ 0 & - & + & + & + \end{pmatrix}, \begin{pmatrix} + & + & + & 0 & 0 \\ + & + & 0 & - & - \\ + & 0 & + & + & + \\ 0 & - & + & + & - \\ 0 & - & + & - & + \end{pmatrix},$$

$$\begin{pmatrix} + & + & + & 0 & 0 \\ + & + & 0 & - & - \\ + & 0 & + & + & + \\ 0 & - & + & + & + \\ 0 & - & + & + & + \end{pmatrix}, \begin{pmatrix} + & + & + & 0 & 0 \\ + & + & + & - & - \\ + & + & + & + & + \\ 0 & - & + & + & - \\ 0 & - & + & - & + \end{pmatrix}, \begin{pmatrix} + & + & + & 0 & 0 \\ + & + & + & - & - \\ + & + & + & + & + \\ 0 & - & + & + & + \\ 0 & - & + & + & + \end{pmatrix},$$

$$
\begin{pmatrix} + & + & + & + & 0 \\ + & + & - & - & - \\ + & - & + & - & - \\ + & - & - & + & + \\ 0 & - & - & + & + \end{pmatrix},
\begin{pmatrix} + & + & + & + & 0 \\ + & + & - & - & - \\ + & - & + & 0 & - \\ + & - & 0 & + & + \\ 0 & - & - & + & + \end{pmatrix},
\begin{pmatrix} + & + & + & + & 0 \\ + & + & - & + & + \\ + & - & + & - & + \\ + & + & - & + & - \\ 0 & + & + & - & + \end{pmatrix},
$$

$$
\begin{pmatrix} + & + & + & + & 0 \\ + & + & - & - & - \\ + & - & + & + & + \\ + & - & + & + & + \\ 0 & - & + & + & + \end{pmatrix},
\begin{pmatrix} + & + & + & + & 0 \\ + & + & + & 0 & - \\ + & + & + & + & - \\ + & 0 & + & + & + \\ 0 & - & - & + & + \end{pmatrix},
\begin{pmatrix} + & + & + & + & 0 \\ + & + & - & + & + \\ + & - & + & + & - \\ + & + & + & + & + \\ 0 & + & - & + & + \end{pmatrix},
$$

$$
\begin{pmatrix} + & + & + & + & + \\ + & + & - & - & - \\ + & - & + & - & - \\ + & - & - & + & - \\ + & - & - & - & + \end{pmatrix},
\begin{pmatrix} + & + & + & + & + \\ + & + & - & - & - \\ + & - & + & - & - \\ + & - & - & + & + \\ + & - & - & + & + \end{pmatrix},
\begin{pmatrix} + & + & + & + & + \\ + & + & - & - & - \\ + & - & + & - & + \\ + & - & - & + & + \\ + & - & + & + & + \end{pmatrix},
$$

$$
\begin{pmatrix} + & + & + & + & + \\ + & + & - & - & - \\ + & - & + & + & + \\ + & - & + & + & + \\ + & - & + & + & + \end{pmatrix},
\begin{pmatrix} + & + & + & + & + \\ + & + & - & - & + \\ + & - & + & + & - \\ + & - & + & + & - \\ + & + & - & - & + \end{pmatrix},
\begin{pmatrix} + & + & + & + & + \\ + & + & - & - & + \\ + & - & + & + & - \\ + & - & + & + & + \\ + & + & - & + & + \end{pmatrix},
$$

$$
\begin{pmatrix} + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & + \end{pmatrix},
\begin{pmatrix} + & + & + & + & 0 \\ + & + & - & 0 & - \\ + & - & + & + & + \\ + & 0 & + & + & - \\ 0 & - & + & - & + \end{pmatrix}.
$$

We number these patterns by rows. By corollary 1.11 and proposition 3.2 (possibly together with a permutation similarity), each of the patterns 2, 4, 6, 10, 14, 16 and 17 is in $\mathcal{SID}$. Notice for example that in pattern 10, the third and fourth rows/columns can be permuted to the fourth and fifth rows/columns; hence, it follows from corollary 1.11 that this pattern is in $\mathcal{SID}$. Similarly, by theorem 2.2 and proposition 3.2 (possibly together with a permutation similarity), each of the patterns 1, 3, 5, 7 and 15 is in $\mathcal{SID}$. Pattern 19 is in $\mathcal{SID}$ by corollary 1.3, while pattern 13 is in $\mathcal{SID}$ by corollary 2.4.

It is well known that every real symmetric idempotent matrix B of order n and rank r can be written as $B = C(C^TC)^{-1}C^T$ for some $n \times r$ real matrix C of rank r. In particular, we used this idea to show that patterns 12 and 18 are in $\mathcal{SID}$. With $C = \begin{pmatrix} 4 & 0 & 1 & 3 & 4 \\ -2 & 2 & -4 & -3 & -1 \end{pmatrix}^T$ and $B = C(C^TC)^{-1}C^T$, then sgn B is pattern 18. Clearly, rank $B = 2$. Note that $I - B$ is also a symmetric idempotent matrix, and rank $(I - B) = 3$. It can be seen that sgn $(I - B)$ is equivalent to pattern 18 (up to permutation similarity and signature similarity). Therefore, pattern 18 allows symmetric idempotents of ranks 2 and 3. Hence, in general, if $A \in \mathcal{ID}$, there may exist idempotent matrices B_1 and B_2 in $Q(A)$ such that rank $B_1 \neq$ rank B_2.

We found that the remaining patterns, except pattern 20, are in $\mathcal{SID}$ by somewhat complicated Maple constructions. Indeed, the following real symmetric idempotent matrices have sign patterns 8, 9 and 11, respectively:

$$\frac{1}{10}\begin{pmatrix} 5+t & 2 & t-4 & 1 & 0 \\ 2 & 2 & -2 & -2 & -2 \\ t-4 & -2 & 5+t & 0 & -1 \\ 1 & -2 & 0 & 9-t & t \\ 0 & -2 & -1 & t & 9-t \end{pmatrix}, \quad \text{where } t = 2+\sqrt{6},$$

$$\frac{1}{155}\begin{pmatrix} 145-5s & 30s-2 & 25s+19 & 31 & 0 \\ 30s-2 & 105-25s & -150s-21 & 31 & 155s \\ 25s+19 & -150s-21 & 91-125s & -31 & 31 \\ 31 & 31 & -31 & 31 & -31 \\ 0 & 155s & 31 & -31 & 93+155s \end{pmatrix},$$

$$\text{where } s = \frac{\sqrt{33}-1}{20},$$

$$\frac{1}{10}\begin{pmatrix} 3 & 1 & 2 & 4 & 0 \\ 1 & 3 & 2 & 0 & -4 \\ 2 & 2 & 2 & 2 & -2 \\ 4 & 0 & 2 & 6 & 2 \\ 0 & -4 & -2 & 2 & 6 \end{pmatrix}.$$

Finally, we show that pattern 20, denoted by A, is not in $\mathcal{SID}$. We have

$$A = \begin{pmatrix} + & + & + & + & 0 \\ + & + & - & 0 & - \\ + & - & + & + & + \\ + & 0 & + & + & - \\ 0 & - & + & - & + \end{pmatrix}.$$

We in fact show that $A \notin \mathcal{ID}$. Assume, to the contrary, that there is an idempotent matrix $B \in Q(A)$. Notice that the submatrix $B(\{1,2,5\}, \{1,4,5\})$ has sign pattern

$$\begin{pmatrix} + & + & 0 \\ + & 0 & - \\ 0 & - & + \end{pmatrix},$$

which is sign nonsingular. Therefore, rank $B \geq 3$. It follows that $I - B$ is idempotent and rank $(I - B) \leq 2$. However, since A is symmetric, each diagonal entry b_{ii} of B satisfies $0 < b_{ii} < 1$, just as in the proof of lemma 2.1. Hence, the submatrix $(I - B)(\{1,2,5\}, \{1,4,5\})$ has sign pattern

$$\begin{pmatrix} + & - & 0 \\ - & 0 & + \\ 0 & + & + \end{pmatrix},$$

which is also sign nonsingular, contradicting the fact that rank $(I - B) \leq 2$.

4. Concluding remarks

Complete characterizations of the classes $\mathcal{ID}$ and $\mathcal{SID}$ still remain open, as well as a number of other open questions involving these classes. For example, if $A \in \mathcal{ID}$, does there always exist an idempotent $B \in Q(A)$ such that rank $B = mrA$? For an $n \times n$ nonnegative pattern $A \in \mathcal{ID}$, rank $B = mrA$ for all idempotents $B \in Q(A)$, and furthermore, for each $1 \le r \le n$, there exists an $r \times r$ principal submatrix of A which is in $\mathcal{ID}$. These facts follow from Theorem 3.1 in [3].

Let B be a real square matrix. Then, B is idempotent if and only if $(2B-I)^2 = I$, and, B is symmetric idempotent if and only if $2B-I$ is a symmetric orthogonal matrix. Hence, B is idempotent $\Longrightarrow$ sgn $(2B-I)$ allows an inverse pair (see [4]), and, B is symmetric idempotent $\Longrightarrow$ sgn $(2B-I)$ allows a symmetric orthogonal matrix. Note that sgn B and sgn $(2B-I)$ can differ only on the diagonal. The patterns that allow an orthogonal matrix have recently been investigated (see [2] where a number of references are given).

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