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MEASURE DERIVATIVE AND ITS APPLICATIONS TO σ-MULTIFRACTALS

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Abstract. The fractal space is often associated with natural phenomena with many length scales and the functions defined on this space are usually not differentiable. First we define a σ-multifractal from σ-iterated function systems with probability. We introduce the measure derivative through the invariant measure of the σ-multifractal. We show that the non-differentiable function on the σ-multifractal can be differentiable with respect to this measure derivative. We apply this result to some examples of ordinary differential equations and diffusion processes on σ-multifractal spaces.

1. Introduction

Recently there have been many attempts to investigate the fractal sets and apply them to the multiple length-scale phenomena of the nature [5]. The archetype property of the fractal sets is identified as the *self similarity.* The simplest sets with these self-similar properties can be obtained by using the iterated function systems [2]. The iterated function systems have been widely studied since they can be applied to the various branches of science, for example, the image and signal processings [3] and the crystal lattice dynamics [4]. Now the mathematical research on the anomalies of fractals includes the studies of quantum mechanics with fractal support [9] and the Schrödinger equation in complex fractal graphs [10]. There have been continuous efforts to apply the fractal geometry to understand diffusion in fractal media by means of fractal operations [6].

In the studies of fractal phenomena, it became increasingly evident that the mathematical calculus, which is not appropriate on fractals,

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should be extended to the fractal space. For example, Kigami attempted to develop calculus on the self-similar fractals based on the harmonic measure [8] and Giona generalized the ordinary differential equation to the fractal support with the help of corresponding integral equations [7].

In this paper we first define the *σ-multifractals* with the help of σ-iterated function systems with probability and introduce a measure derivative on these sets. We show that the non-differentiable function on σ-multifractal becomes differentiable with respect to the measure derivative. As an illustration, we apply the measure derivative method to some examples of ordinary differential equations and diffusion processes on σ-multifractal processes.

2. σ-iterated function systems and σ-multifractals

Let $X = \cup_{i=1}^{\infty} X_i$ be the union of non-overlapping metric space X_i's with metric d. And suppose that for each X_i, there exists a finite number n_i of contraction maps $\mathbf{w}_i = (w_i^1, w_i^2, \cdots, w_i^{n_i})$ on X_i, $w_i^j : X_i \to X_i$. Then there exists the invariant set $\mathcal{F}_i$, called the *self-similar set*, such that

$$\mathcal{F}_i = \cup_{j=1}^{n_i} w_i^j(\mathcal{F}_i).$$

This self-similar set is a typical type of fractal sets and in some cases is simply called the fractal. In order to treat a more general form of these similar sets, we define the σ-similar set, $\mathcal{F} = \cup_{i=1}^{\infty} \mathcal{F}_i$, and consider the measure structure on this set. For each i, let the probability vector $\mathbf{p}_i = (p_1^i, p_2^i, \cdots, p_{n_i}^i)$, in which $0 < p_j^i < 1$ and $\sum_{j=1}^{n_i} p_j^i = 1$, be associated with the map $\mathbf{w}_i$. Then $(\mathbf{w}_i, \mathbf{p}_i)$ is said to be the *iterated function system with probability* (IFSP) [2]. We also call $\{\mathbf{w} = (\mathbf{w}_i), \mathbf{p} = (\mathbf{p}_i)\}$ the *σ-iterated function system with probability* (σ-IFSP).

Let $\mathcal{M}(X_i)$ be the space of all probability measures defined on the σ-algebra of the Borel sets of X_i. Then the Markov operator $T_i : \mathcal{M}(X_i) \to \mathcal{M}(X_i)$ defined by

$$T_i[m(A)] = \sum_{j=1}^{n_i} p_j^i m \circ (w_j^i)^{-1}(A), \quad m \in \mathcal{M}(X),$$

has a unique invariant meaure μ_i with support on $\mathcal{F}_i$ such that

$$T_i[\mu_i] = \mu_i.$$

And define the measure $\mu = \sum_{i=1}^{\infty} \mu_i$ on the attractor of σ-IFSP $(\mathbf{w}, P)$ by

$$\mu(E) = \sum_{i=1}^{\infty} \mu_i(X_i \cap E).$$

Then this measure has support on the σ-similar set, $\mathcal{F} = \cup_{i=1}^{\infty} \mathcal{F}_i$. We can see that $\mathcal{F}$ has a multi-fractal structure. So from now on, we call this set, $\mathcal{F}$, the *σ-multifractal with invariant measure μ* or simply the *μ-multifractal.*

3. Measure derivative on σ-multifractals

Let $\mathcal{F}$ be a μ-multifractal, that is, σ-multifractal with invariant measure μ as in Section 2. And let λ be a μ-absolutely continuous measure on $\mathcal{F}$, that is, $\lambda << \mu$. Then by the Radon-Nikodym theorem [10], there exists a μ-measurable function f on $\mathcal{F}$, denoted formally by $\frac{d\lambda}{d\mu} = f$, such that

$$\lambda(E) = \int_E f(\omega) d\mu(\omega).$$

From now on, we focus on the μ-multifractal $\mathcal{F}$ in $\Re$.

DEFINITION 1. Let $\mathcal{F}$ be a σ-multifractal, λ be the measure defined above and let F be a function defined on $\Re$. Then for $x \in \mathcal{F}$, the *measure derivative* of $F(x)$, $\frac{d}{d\lambda[x]} F(x) \equiv D_\lambda F(x)$, is defined by

$$\frac{d}{d\lambda[x]} F(x) = \lim_{h \to 0} [F(x+h) - F(x-h)] / \lambda[(x-h,\ x+h)],$$

when the limit exists.

REMARK 1. When λ is Borel measure and $Z = \{\cdots, -1, 0, 1, \cdots\}$, the measure derivative of the continuous function F at a.e. x can be defined by using the dyadic intervals, that is,

$$\frac{d}{d\lambda[x]} F(x) = \lim_{n \to \infty} [F(k + 2^{-n}(i+1)) \\ - F(k + 2^{-n} i) / \lambda[(k + 2^{-n} i,\ k + 2^{-n}(i+1))],$$

where $(k + 2^{-n} i,\ k + 2^{-n}(i+1))$ is a dyadic interval of length 2^{-n} containing x, $k \in Z$, $i = 0, 1, \cdots, 2^n - 1$.

Let $X_i = [a_i,\ b_i]$ be the closed interval and $\{w_i^k\}_{k=1}^{n_i}$ be given such that $w_i^1(a_i) = a_i, w_i^{n_i}(b_i) = b_i$ and

$$\mathrm{dist}\{w_i^j(X_i), w_i^l(X_i)\} \geq \max\{\mathrm{diam}[w_i^j(X_i)], \mathrm{diam}[w_i^l(X_i)]\}$$

for each neighboring interval $w_i^j(X_i)$ and $w_i^l(X_i)$. Then $\mathcal{F}$ is a generalized Cantor set as a μ-multifractal.

THEOREM 1. *Let $\mathcal{F}$ be a generalized Cantor set defined as above and F be a continuous function defined on $\Re$ such that*

$$F(x) = F(w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(b_i))$$

for each $x \in (w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(b_i),\ w_i^{l_1} \circ w_i^{l_2'} \circ \cdots \circ w_i^{l_k'}(a_i))$ where $w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(X_i)$ and $w_i^{l_1'} \circ w_i^{l_2'} \circ \cdots \circ w_i^{l_k'}(X_i)$ are the neighboring basic intervals. Then

$$\frac{d}{d\mu[x]}F(x) = \lim_{k\to\infty} \frac{F(w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(b_i)) - F(w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(a_i))}{p_i^{j_1} p_i^{j_2} \cdots p_i^{j_k}},$$

for almost all $x \in \mathcal{F}$ such that $x \in \cap_{k=1}^{\infty}[w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(a_i),\ w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(b_i)]$.

Proof. For each $x \in \mathcal{F}_i$, there exists a sequence $(j_1, j_2, \cdots),\ j_k = 1, 2, \cdots, n_i$, such that $x \in \cap_{k=1}^{\infty} w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(X_i)$. Then for each basic interval $w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(X_i) = [w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(a_i),\ w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(b_i)]$, let $h_k = w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(b_i) - w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(a_i)$. Then $h_k \searrow 0$ as $k \to \infty$ and by the invariance of μ,

$$\begin{aligned}
&\mu(x-h,\ x+h) \\
&= \mu([w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(a_i),\ w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(b_i)]) \\
&= p_i^{j_1} p_i^{j_2} \cdots p_i^{j_k}.
\end{aligned}$$

Then as in Remark 1,

$$\frac{d}{d\mu[x]}F(x) = \lim_{k\to\infty} \frac{F(w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(b_i)) - F(w_i^{j_1} \circ w_i^{j_2} \circ \cdots \circ w_i^{j_k}(a_i))}{p_i^{j_1} p_i^{j_2} \cdots p_i^{j_k}}. \square$$

PROPOSITION 1. *For D_λ-differentiable functions, F and G on $\mathcal{F}$, and two real numbers a and b, we have*

(a) $\frac{d}{d\lambda[x]}[aF(x)+bG(x)] = a\frac{d}{d\lambda[x]}F(x) + b\frac{d}{d\lambda[x]}G(x)$,

(b) $\frac{d}{d\lambda[x]}[F(x)G(x)] = [\frac{d}{d\lambda[x]}F(x)]G(x) + F(x)[\frac{d}{d\lambda[x]}G(x)]$.

In [6], Giona defined the derivative $\mathcal{D}_\mu$ on IFSP with invariant measure μ as the inverse operator of integration, that is, when

$$\mathcal{I}_\mu F(x) = \int_0^x F(w)d\mu = G(x)$$

he defined $\mathcal{I}_\mu^{-1}G(x) = \mathcal{D}_\mu G(x) = F(x)$. And he used this derivative to transform a differential equation to the integral equation. In followings, we show that the measure derivative coincides with the Giona's derivative in this sense. Therefore, we can efficiently transform the integral equation into the differential equation on fractals and investigate the diffusion process on fractals through the differential equation under the measure derivative.

PROPOSITION 2. *Let μ and λ be defined as above and define*

$$\nu(x) = \int_0^x f(\omega)d\mu(\omega),$$

and $[\nu(x)]^n = \nu^n(x)$. Then

(a) $D_\lambda[\nu^n(x)] = \mathcal{D}_\lambda[\nu^n(x)]$.

(b) *Let $F(x)$ be an infinitely D_λ-differentiable at $x = a$. Then $F(x)$ can be extended to the following form of Taylor series on a neighborhood of x in which the series converges:*

$$F(x) = \sum_{n=0}^{\infty} a_n[\nu(x) - \nu(a)]^n \tag{1}$$

where $a_n = D_\lambda^{(n)}F[\nu(a)]/n!$.

Proof. (a) $\frac{d}{d\lambda[x]}\nu(x) = \lim_{h\to 0}\int_{x-h}^{x+h} d\lambda(\omega)/\lambda[(x-h,\ x+h)] = 1$. Then

$$\frac{d}{d\lambda[x]}\nu^n(x) = n[\frac{d}{d\lambda[x]}\nu(x)]\nu^{n-1}(x) = n\nu^{n-1}(x),$$

by Proposition 1. Now put $\nu_0(x) = 1$, $\nu_1(x) = \nu(x)$ and define $\nu_n(x)$ by

$$\begin{aligned}\nu_n(x) &= \int_0^x d\lambda(x_n) \int_0^{x_n} d\lambda(x_{n-1}) \cdots \int_0^{x_2} f(x_1) d\mu(x_1) \\ &= \int_0^x d\lambda(x_n) \int_0^{x_n} d\lambda(x_{n-1}) \cdots \int_0^{x_2} d\lambda(x_1).\end{aligned}$$

Since

$$\frac{\nu^n(x)}{n!} = \nu_n(x) = \int_0^x \nu_{n-1}(\omega) d\lambda(\omega) = \int_0^x \frac{\nu^{n-1}(\omega)}{(n-1)!} d\lambda(\omega),$$

we have

$$\nu^n(x) = \int_0^x n\nu^{n-1}(\omega) d\lambda(\omega).$$

Therefore $\mathcal{D}_\lambda[\nu^n(x)] \equiv \mathcal{I}_\lambda^{-1}[\nu^n(x)] = n\nu^{n-1}(x)$ and

(2) $$\frac{d}{d\lambda[x]}[\nu^n(x)] = D_\lambda[\nu^n(x)] = n\nu^{n-1}(x).$$

(b) A series expansion can be done similarly to the usual calculus. □

4. Applications

It can be shown that when $F(x)$ defined on $\Re$ is differentiable in usual sense, it is also D_λ-differentiable since $\Re$ can be regarded as the σ-multifractal with Lebesgue measure. But the converse does not hold in general. For example, the Cantor ternary function $F(x)$ is not differentiable on the Cantor set in usual sense. However, for the s-dimensional Hausdorff measure, λ, with $s = \log 2/\log 3$, we have for every $x \in C$,

$$\frac{d}{d\lambda[x]} F(x) = \lim_{n\to\infty} 2^{-n}/3^{-ns} = 1.$$

Thus the measure derivative is well-defined and can be seen as a generalized form of the derivative on fractals, which can be applicable to the motion on the σ-multifractal $\mathcal{F}$.

4.1 Differential equation on σ-multifractals with respect to the measure derivative

Let $x \in \Re$, $\mathbf{y}$ and $\mathbf{H}$ are n-dimensional vector functions defined on $\Re$ and $\Re^n$, respectively. Then the solution of the Cauchy problem

$$\frac{d\mathbf{y}}{dx} = \mathbf{H}[\mathbf{y}(x)], \qquad \mathbf{y}(0) = \mathbf{y}_0, \tag{3}$$

has a unique solution in a neighborhood of $x = 0$ when $\mathbf{H}$ is Lipschitz continuous. Note that the above differential equation in (1) can be recast in the form of a nonlinear Volterra-type integral equation

$$\mathbf{y}(x) = \mathbf{y}_0 + \int_0^x \mathbf{H}[\mathbf{y}(\xi)]d\xi,$$

which is particularly useful in functional analysis.

From this point, we can extend the initial value problem involving ordinary differential equations on $\Re$ to one in σ-multifractal. That is, supposing that y and H are real valued functions defined on $\Re$, the differential equation in (3) is extended to the σ-multifractal $\mathcal{F}$ as follows:

$$\frac{dy}{d\lambda[x]} = H[y(x)], \qquad y(0) = y_0, \tag{4}$$

or

$$y = y_0 + \int_0^x H[y(\omega)]d\lambda(\omega),$$

where the integral on the right hand side also exists under the condition of Lipschitz continuity of H.

EXAMPLE 1. Let $X_i = [i,\ i+1]$, $w_i^1(x) = \frac{1}{2}x + \frac{i}{2}$, $w_i^2(x) = \frac{1}{2}x + \frac{i}{2} + \frac{1}{2}$ and $p_i^1 = p_i^2 = 1/2$ for all $i \in Z$. Then $\mathcal{F} = \Re$ and μ is the Lebesgue measure. Thus for $\lambda = \mu$, $\nu(x) = \lambda([0,\ x]) = x$ and $d\lambda(x) = dx$. Thus in this case, the equations (3) and (4) coincide.

Since the solution of the usual Cauchy problem in $\Re$ has the form of

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

we can see that the solution of a similar Cauchy problem in the σ-multifractal $\mathcal{F}$ also has the form of

$$y(x) = \sum_{n=0}^{\infty} a_n [\nu(x)]^n,$$

from Proposition 3.

EXAMPLE 2. As in $\Re$, we define the exponential type function on the σ-multifractal $\mathcal{F}$ by

$$\exp[\nu(x)] = \sum_{n=0}^{\infty} [\nu(x)]^n / n!. \tag{5}$$

Then it can be easily shown that $y = y_0 \exp[k\nu(x)]$ is the solution of the differential equation

$$\frac{dy}{d\lambda[x]} = ky(x), \quad y(0) = y_0,$$

on the σ-multifractal $\mathcal{F}$ with respect to the measure.

In the following figures, we show the graphs of two functions on the subset $[0,\ 2]$ of σ-multifractal generated by the IFSP in Example 1 where probabilities, for example, are chosen to be $p_0^1 = 1/5$ and $p_1^1 = 3/4$.

$\nu(x)$

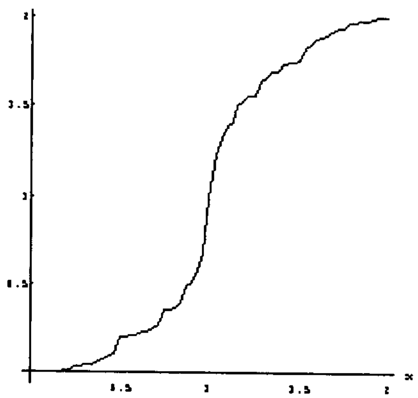

Fig 1. $y = \nu(x)$

$\exp[\nu(x)]$

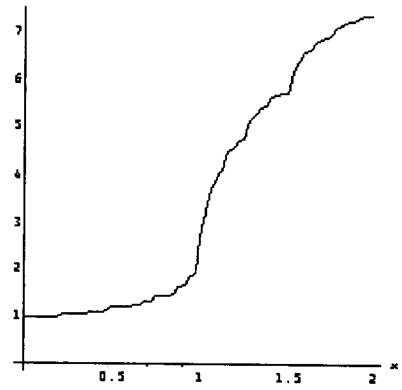

Fig 2. $y = \exp[\nu(x)]$

4.2 Multifractal diffusion process on the σ-multifractals

Let $\mathcal{F}$ be the σ-multifractal with invariant measure μ and let λ and ν be defined as in Section 2. For the random variable X_t of $\mathcal{F}$ defined on the probability space Ω with probability measure P, define $X_\nu(t)$ by $[X_\nu(t)](\omega) = \nu[X_t(\omega)]$ for each $\omega \in \Omega$.

DEFINITION 2. The process $\{X_\nu(t) : t \geq 0\}$ is the *multifractal diffusion process on $\mathcal{F}$ with drift δ* if

(i) it has a stationary independent increment,
(ii) it has a distribution of

$$\langle X_\nu(t)\rangle = X_\nu(0) + \delta t, \langle [X_\nu(t) - X_\nu(0)]^2\rangle = t \text{ and}$$
$$\langle [X_\nu(t) - X_\nu(0)]^k\rangle \sim o(t) \text{ for } k > 2.$$

NOTE. When $X_\nu(t)$ is normally distributed with $X_\nu(0) = 0$, it can be seen that $X_\nu(t) = B(t) + \delta t$ for the standards Brownian motion process $\{B(t)\}$ on $\Re$. However, in this paper $X_\nu(t)$ does not need to be the Brownian motion process.

Let $X_\nu(t)$ be distributed with the density function $\tilde{p}[\nu(x), t]$ on $\nu(\mathcal{F})$. Since $P[X_\nu(t) \in B] = P[X(t) \in \nu^{-1}(B)]$ a.e.$[\lambda]$, we can define the probability density function $p(x,t)$ on $\mathcal{F}$ such that

$$\int_{\nu^{-1}(B)} p(x,t)d\lambda(x) = \int_B \tilde{p}[\nu(x), t]d\nu(x).$$

From the independent and stationary increment, the conditional probability density of X_ν satisfies

$$\begin{aligned}\tilde{p}[\nu(x), t+s; \nu(y), s] &\equiv \tilde{p}[X_\nu(t+s) = \nu(x); X_\nu(s) = \nu(y)] \\ &= \tilde{p}[\nu(x), t; \nu(y), 0] \equiv \tilde{p}[\nu(x), t; \nu(y)]\end{aligned}$$

which corresponds to the conditional density function $p(x,t;y)$.

Now we will show that the above definition is reasonable in that the conditional density function satisfies two types of the diffusion equation, Kormogrov's backward and forward diffusion equations (also called Fokker-Planck equations).

THEOREM 2. *The conditional density function* $p(x,t;y)$ *defined above satisfies the Kormogrov's backward diffusion equation, that is,*

$$\frac{1}{2}\frac{\partial^2}{\partial\lambda[x]^2}p(x,t;y)+\delta\frac{\partial}{\partial\lambda[x]}p(x,t;y)=\frac{\partial}{\partial t}p(x,t;y) \text{ a.e. } [\lambda].$$

Proof. From $P(X(t)\in A;y)=\int_A p(x,t;y)d\lambda(x)=\int_A\int_{\mathcal{F}}p(x_h,h;y)\cdot p(x,t-h;x_h)d\lambda(x_h)d\lambda(x)=\int_A E[p(x,t-h;x_h)]d\lambda(x)$, we have

$$p(x,t;y)=E[p(x,t-h;X(h))] \text{ a.e.}[\lambda].$$

Recall that $\langle X_\nu(h)-\nu(y)\rangle=\delta h$, $\langle[X_\nu(h)-v(y)]^2\rangle=h$ and $\langle[X_\nu(h)-v(y)]^k\rangle\sim o(h)$ for $k>2$ from the definition. We expand the right hand side of the above equation in Taylor series about $(x,t;y)$ as in Proposition 3(1) and use the change of variables so that

$$\begin{aligned}
&p(x,t;y)\\
&=E[p(x,t-h;X(h))]\\
&=E[p(x,t;y)+(-h)\frac{\partial}{\partial t}p(x,t;y)+(X(h)-y)\frac{\partial}{\partial\lambda[x]}p(x,t;y)\\
&\quad+\frac{h^2}{2}\frac{\partial^2}{\partial t^2}p(x,t;y)+\frac{(X(h)-y)^2}{2}\frac{\partial^2}{\partial\lambda[x]^2}p(x,t;y)+\cdots]\\
&=p(x,t;y)-h\frac{\partial}{\partial t}p(x,t;y)+E[\{X_\nu(h)-\nu(y)\}]\frac{\partial}{\partial\lambda[x]}p(x,t;y)\\
&\quad+\frac{h^2}{2}\frac{\partial^2}{\partial t^2}p(x,t;y)+\frac{E[\{(X_\nu(h)-\nu(y)\}^2]}{2}\frac{\partial^2}{\partial\lambda[x]^2}p(x,t;y)+\cdots\\
&=p(x,t;y)-h\frac{\partial}{\partial t}p(x,t;y)+\delta h\frac{\partial}{\partial\lambda[x]}p(x,t;y)\\
&\quad+\frac{h}{2}\frac{\partial^2}{\partial\lambda[x]^2}p(x,t;y)+o(h).
\end{aligned}$$

Rearranging terms and then taking the limit $h\to 0$, we have

$$\frac{1}{2}\frac{\partial^2}{\partial\lambda[x]^2}p(x,t;y)+\delta\frac{\partial}{\partial\lambda[x]}p(x,t;y)=\frac{\partial}{\partial t}p(x,t;y).$$

□

THEOREM 3. *The conditional density function $p(x,t;y)$ defined above satisfies the Kormogrov's forward diffusion equation, that is,*

$$\frac{1}{2}\frac{\partial^2}{\partial\lambda[x]^2}p(x,t;y) - \delta\frac{\partial}{\partial\lambda[x]}p(x,t;y) = \frac{\partial}{\partial t}p(x,t;y).$$

Proof. The proof is similar to that of Theorem 1 except that we work with conditioning on $X(t-h)$. □

As an example of the diffusion process on multifractals, consider a random walk on the σ-multifractal.

EXAMPLE 3. Let Z be the set of integers. And let $X_i = [i,\ i+1]$ for $i \in Z$, $w_i^1(x) = r_i^1 x + (1-r_i^1)i$ and $w_i^2(x) = r_i^2 x + (1-r_i^2)(1+i)$ with $0 < p_i^1,\ p_i^2 < 1$, $0 < r_i^1,\ r_i^2$ and $r_i^1 + r_i^2 \le 1$ for all $i \in Z$. For this σ-IFS, σ-multifractal set $\mathcal{F} \subset \Re$ is a countable union of Cantor type sets with an independent geometric structure. Now we define some Brownian type motion on this fractal with respect to its invariant measure. Suppose that at every time step of Δt, a particle standing at x_i goes to either x_{i+1} with probability p, $0 < p < 1$, if $\nu(x_{i+1}) - \nu(x_i) = \Delta\nu$, or x_{i+1} with probability $1-p$ if $\nu(x_i) - \nu(x_{i+1}) = \Delta\nu$. Suppose that $\nu(x_0) = 0$ for the initial state x_0 of the process. Let

$$X_i = \begin{cases} 1 & \text{if } \nu(x_{i-1}) - \nu(x_i) = \Delta\nu, \\ -1 & \text{if } \nu(x_i) - \nu(x_{i-1}) = \Delta\nu. \end{cases}$$

When $X(t)$ denotes the position during time t, we have

$$X_\nu(t) = \Delta\nu(X_1 + X_2 + \cdots + X_{[t/\Delta t]}),$$

in which $[t/\Delta t]$ denotes the largest integer not greater than $t/\Delta t$. Since for each i,

$$\langle X_i\rangle = 2p - 1 \text{ and } \langle X_i^2\rangle = 1 - (2p-1)^2,$$

we have

$$\langle X_\nu(t)\rangle = \Delta\nu[t/\Delta t](2p-1)$$

and

$$\langle X_\nu(t)^2\rangle = \Delta\nu^2[t/\Delta t](1-(2p-1)^2).$$

Put $\Delta\nu = \sqrt{\Delta t}$, $p = 1/2(1 + \delta\sqrt{\Delta t})$ and let $\Delta t \to 0$. From the central limit theorem the distribution of $X_\nu(t)$ converges to a normal distribution such that

$$\langle X_\nu(t)\rangle \to \delta t \text{ and } \langle X_\nu(t)^2\rangle \to t.$$

Thus we define λ-Brownian-type motion $\{X_\nu(t) : t \geq 0\}$ on the σ-multifractal $\mathcal{F}$ so that $X_\nu(t)$ is normally distributed with a mean $X_\nu(0) + \delta t$ and a variance t with respect to the measure λ, that is,

$$p(x, y; t) = 1/\sqrt{2\pi t}\exp[-\{\nu(x) - \nu(y) - \delta t\}^2/2t],$$

where $\exp[\nu(x)]$ is defined as in (5). Clearly the above process $\{X_\nu(t) : t \geq 0\}$ is the multifractal diffusion process from Definition 2 and so $p(x, t; y)$ satisfies the diffusion equations, which can also be shown by direct differentiation using (2).

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