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PARALLEL BLOCK ILU PRECONDITIONERS FOR A BLOCK-TRIDIAGONAL M-MATRIX

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ABSTRACT. We propose new parallel block ILU (Incomplete LU) factorization preconditioners for a nonsymmetric block-tridiagonal M-matrix. Theoretical properties of these block preconditioners are studied to see the convergence rate of the preconditioned iterative methods. Lastly, numerical results of the right preconditioned GMRES and BiCGSTAB methods using the block ILU preconditioners are compared with those of these two iterative methods using a standard ILU preconditioner to see the effectiveness of the block ILU preconditioners.

1. Introduction

The discretization of partial differential equations in 2D or 3D, by finite difference or finite element approximation, leads often to large sparse block-tridiagonal linear systems. In this paper, we consider the linear system of equations

$$(1) \qquad Ax = b, \quad x, b \in \mathbb{R}^n$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse nonsymmetric block-tridiagonal M-matrix blocked in the form

$$(2) \qquad A = \begin{pmatrix} B_1 & -C_1 & 0 & \cdots & 0 \\ -E_1 & B_2 & -C_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -E_{m-2} & B_{m-1} & -C_{m-1} \\ 0 & \cdots & 0 & -E_{m-1} & B_m \end{pmatrix}.$$

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It is assumed that the diagonal blocks B_i of A are square matrices, and C_i's and E_i's are nonnegative matrices. Since A is a large sparse matrix, direct solvers become prohibitively expensive because of the large amount of work and storage required. As an alternative, we usually consider nonstationary iterative methods such as the GMRES [15], BCG [9], CGS [16], and BiCGSTAB [17]. Given an initial guess x_0, these algorithms compute iteratively new approximations x_k to the true solution $x^* = A^{-1}b$. The iterate x_k is accepted as a solution if the residual $r_k = b - Ax_k$ satisfies $\|r_k\|/\|b\| \leq$ (Tolerance). In general, the convergence is not guaranteed or may be extremely slow. Hence, the original problem (1) must be transformed into a more tractable form. To do so, we consider an easily invertible matrix K called the *preconditioning matrix* or *preconditioner* and apply the iterative solvers either to the left preconditioned linear system $K^{-1}Ax = K^{-1}b$ or to the right preconditioned linear system $AK^{-1}y = b$, where $y = Kx$. The preconditioner K should be chosen so that $K^{-1}A$ or AK^{-1} is a good approximation to the identity matrix.

Since the ultimate goal of the preconditioned iterative methods is to reduce the total execution time, the computation of preconditioner K should be done in parallel. One of the powerful preconditioning methods in terms of reducing the number of iterations is the incomplete LU (called ILU) factorization method studied by Meijerink and van der Vorst [12]. A detailed review for the ILU factorization method can be found in [3, 5, 10, 14]. However, it is very difficult to parallelize the ILU factorization algorithm because of the recursive nature of the computation. On the other hand, polynomial preconditioners defined by $K^{-1} = p(A)$, where p is a polynomial, are easy to parallelize since they only involve the computation of matrix-vector operations, but they are not as powerful as the ILU factorization preconditioners. In order to make the ILU factorization method more suitable for vector computers and parallel architectures, incomplete block LU factorizations using matrix blocks as basic entities were proposed [1, 2, 6, 7, 10, 13, 14].

The block incomplete Cholesky factorization preconditioners for a symmetric block-tridiagonal M-matrix have been studied recently by Yun [18]. The purpose of this paper is to propose *new parallel block ILU factorization preconditioners* for a *nonsymmetric* block-tridiagonal M-matrix which extend the ideas for symmetric problems introduced in [18]. The block ILU factorization preconditioners to be proposed in this paper are quite different from the existing incomplete block LU factorization preconditioners which require the approximate inverses of pivot blocks (see [1, 2, 4, 7, 8, 13] for more details). In other words, the block ILU factorization preconditioners to be proposed in this paper are obtained by performing the standard ILU factorization on each matrix block independently, so that they have no block recurrence which requires sparse approximate inverses for pivot blocks and thus they can be computed in parallel based on matrix blocks.

In section 2, we review some properties of the ILU factorization on M-matrices. In section 3, we propose new block ILU factorization preconditioners for a nonsymmetric block-tridiagonal M-matrix and their theoretical properties are studied to see the convergence rate of the preconditioned nonstationary iterative methods. In section 4, we describe how to construct the effective block preconditioners for a special type of matrix which arises from five-point discretization of the second-order partial differential equation. In section 5, we present numerical results of the right preconditioned GMRES(ℓ) and BiCGSTAB methods with the block ILU factorization preconditioners developed in this paper, and their results are compared with those of the right preconditioned GMRES(ℓ) and BiCGSTAB with a standard ILU factorization preconditioner. Lastly, some conclusions are drawn.

2. ILU (Incomplete LU) factorizations

A general algorithm for building ILU factorizations for M-matrices can be derived by performing Gaussian elimination and dropping some elements in predetermined nondiagonal positions. To better understand the ILU factorization process for an M-matrix, we review some important results in this section. Let P_n denote the set of all pairs of indices of off-diagonal matrix entries, that is,

$$P_n = \{(i,j) \mid i \neq j,\ 1 \leq i \leq n,\ 1 \leq j \leq n\}.$$

A representation $A = K - N$ is called a *splitting* of A when K is nonsingular. The *spectral radius* $\rho(A)$ of a matrix A is $\rho(A) = max\{\,|\lambda|\ \mid \lambda \in \sigma(A)\,\}$, where $\sigma(A)$ denotes the *spectrum* of A, that is, the set of eigenvalues of A. For two matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \leq B$ denotes $a_{ij} \leq b_{ij}$ for all i and j, and $A \geq B$ denotes $a_{ij} \geq b_{ij}$ for all i and j. It can be easily shown that $A \geq B$ and $C \geq 0$ implies $AC \geq BC$ and

$CA \geq CB$. A matrix $A = (a_{ij})$ is called an *M-matrix* if $a_{ij} \leq 0$ for $i \neq j$, A is nonsingular, and $A^{-1} \geq 0$. A splitting $A = K - N$ is called a *regular splitting* of A if K is nonsingular, $K^{-1} \geq 0$, and $N \geq 0$. A splitting $A = K - N$ is a *convergent splitting* of A if K is nonsingular and $\rho(K^{-1}N) < 1$. It is well-known that if $A = K - N$ is a convergent splitting, then any stationary iterative method of the form

$$Kx_{k+1} = Nx_k + b, \qquad k \geq 0$$

converges to the exact solution of $Ax = b$ for every choice of x_0. The following theorem shows the existence of the (standard) ILU factorization for an M-matrix.

THEOREM 2.1 ([12]). *Let A be an M-matrix. Then, for every zero pattern set $P \subset P_n$, there exist a unit lower triangular matrix $L = (l_{ij})$, an upper triangular matrix $U = (u_{ij})$, and a matrix $R = (r_{ij})$, with $l_{ij} = u_{ij} = 0$ if $(i, j) \in P$ and $r_{ij} = 0$ if $(i, j) \notin P$, such that $A = LU - R$ is a regular splitting of A. The factor L and U are unique. Moreover, L and U are also M-matrices.*

In Theorem 2.1, $A = LU - R$ is called an *ILU factorization* of A corresponding to a zero pattern set $P \subset P_n$. In particular, if P is an empty set, then $R = 0$ and thus a complete LU factorization of A such that A = LU is obtained. Lemma 2.2 in the following will be used to prove Lemma 3.1.

LEMMA 2.2. *Let A and B be M-matrices. If $A \leq B$, then $B^{-1} \leq A^{-1}$.*

Proof. The matrix A can be splitted into

$$A = A - 0 = B - (B - A),$$

where 0 denotes the zero matrix. Since A and B are M-matrices and $B - A \geq 0$, from Theorem 6.22 in [3] $B^{-1} \leq A^{-1}$. □

The following theorem describes a relation between ILU factorizations of two different M-matrices.

THEOREM 2.3 ([18]). *Let A and B be $n \times n$ M-matrices, and let $A = L_1U_1 - R_1$ and $B = L_2U_2 - R_2$ be ILU factorizations corresponding to the same zero pattern set $P \subset P_n$. If $A \leq B$, then $L_2^{-1} \leq L_1^{-1}$ and $U_2^{-1} \leq U_1^{-1}$.*

A comparison theorem for regular splittings which will be used for the proof of main results in section 3 is presented below.

THEOREM 2.4 ([3]). *Let $A = K_1 - N_1 = K_2 - N_2$ be regular splittings of A. If $K_2^{-1} \leq K_1^{-1}$, then*

$$\rho(K_1^{-1}N_1) \leq \rho(K_2^{-1}N_2).$$

3. Block ILU factorization preconditioners

We first consider block ILU factorization preconditioners for a nonsymmetric block-tridiagonal M-matrix of the simplest form

$$A = \begin{pmatrix} B_1 & -C_1 \\ -E_1 & B_2 \end{pmatrix}. \tag{3}$$

Since A is an M-matrix, B_1 and B_2 are M-matrices. From the ILU factorization process, we can find a unit lower triangular matrix L_i, an upper triangular matrix U_i, and a matrix R_i such that $B_i = L_iU_i - R_i$ is a regular splitting of B_i for each $i = 1, 2$, see Theorem 2.1. If $A = K - N$ is a splitting of A and K is a matrix which is easily invertible, then K can be used as a preconditioner for nonstationary iterative methods. The effectiveness of the preconditioner K depends on how well K approximates A.

LEMMA 3.1. *Let $B_i = L_iU_i - R_i$ be a regular splitting of B_i which can be obtained by the ILU factorization process for each $i = 1, 2$, and let $D_i = diag(U_i)$ be the diagonal matrix which is obtained by taking the diagonal part of U_i for each $i = 1, 2$. Then each of the following holds:*

(a) $L_i^{-1} \geq 0$, $U_i^{-1} \geq 0$ *for* $i = 1, 2$.
(b) $U_i^{-1} \geq D_i^{-1} \geq 0$ *for* $i = 1, 2$.
(c) $I - L_i \geq 0$, $I - D_i^{-1}U_i \geq 0$ *for* $i = 1, 2$.
(d) $L_i^{-1} - I \geq 0$, $(D_i^{-1}U_i)^{-1} - I \geq 0$ *for* $i = 1, 2$.

Proof. For the proof of part (a), from Theorem 2.1, L_i and U_i are M-matrices. Hence, part (a) holds. For the proof of part (b), since U_i is an M-matrix, $D_i = diag(U_i)$ is a diagonal matrix which has positive diagonal elements. Hence, $D_i^{-1} \geq 0$ for $i = 1, 2$. Also, since U_i and D_i are M-matrices and $U_i \leq D_i$, $U_i^{-1} \geq D_i^{-1}$ from Lemma 2.2. Hence, part (b) follows. For the proof of part (c), since L_i is a unit lower triangular M-matrix, $I - L_i \geq 0$. Also, since $D_i^{-1}U_i$ is a unit upper triangular matrix whose off-diagonal elements are nonpositive, $I - D_i^{-1}U_i \geq 0$ for

$i = 1, 2$. For the proof of part (d), since L_i is a unit lower triangular M-matrix, L_i^{-1} is a unit lower triangular matrix whose off-diagonal elements are nonnegative. Thus, $L_i^{-1} - I \geq 0$. Since $U_i^{-1} \geq 0$, $(D_i^{-1}U_i)^{-1}$ is a unit upper triangular matrix whose off-diagonal elements are nonnegative. Hence, $(D_i^{-1}U_i)^{-1} - I \geq 0$ for $i = 1, 2$. □

THEOREM 3.2. *Let A be a nonsymmetric M-matrix of the form (3), and let $B_i = L_iU_i - R_i$ be a regular splitting of B_i which can be obtained by the ILU factorization process for each $i = 1, 2$. Let $D_i = diag(U_i)$ be the diagonal matrix which is obtained by taking the diagonal part of U_i for each $i = 1, 2$. Let*

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix},$$

$$L_\alpha = \begin{pmatrix} L_1 & 0 \\ -E_1D_1^{-1} & L_2 \end{pmatrix}, \quad U_\alpha = \begin{pmatrix} U_1 & -C_1 \\ 0 & U_2 \end{pmatrix},$$

$$L_\beta = \begin{pmatrix} L_1 & 0 \\ -E_1U_1^{-1} & L_2 \end{pmatrix}, \quad U_\beta = \begin{pmatrix} U_1 & -L_1^{-1}C_1 \\ 0 & U_2 \end{pmatrix}.$$

If we let $M = LU$, $M_\alpha^\alpha = L_\alpha U_\alpha$, and $M_\beta^\beta = L_\beta U_\beta$, then each of the following holds:

(a) $R = M - A \geq 0$, $R_\alpha^\alpha = M_\alpha^\alpha - A \geq 0$, *and* $R_\beta^\beta = M_\beta^\beta - A \geq 0$.
(b) $0 \leq L^{-1} \leq {L_\alpha}^{-1} \leq {L_\beta}^{-1}$.
(c) $0 \leq U^{-1} \leq {U_\alpha}^{-1} \leq {U_\beta}^{-1}$.
(d) $0 \leq M^{-1} \leq {M_\alpha^\alpha}^{-1} \leq {M_\beta^\beta}^{-1}$.
(e) $A = M - R = M_\alpha^\alpha - R_\alpha^\alpha = M_\beta^\beta - R_\beta^\beta$ *are regular splittings of A.*
(f) $\rho({M_\beta^\beta}^{-1} R_\beta^\beta) \leq \rho({M_\alpha^\alpha}^{-1} R_\alpha^\alpha) \leq \rho(M^{-1}R) < 1$.

Proof. Since $R = M - A$, $R_\alpha^\alpha = M_\alpha^\alpha - A$, and $R_\beta^\beta = M_\beta^\beta - A$, one obtains

$$R = \begin{pmatrix} R_1 & C_1 \\ E_1 & R_2 \end{pmatrix},$$

$$R_\alpha^\alpha = \begin{pmatrix} R_1 & C_1(I - L_1) \\ E_1(I - D_1^{-1}U_1) & E_1D_1^{-1}C_1 + R_2 \end{pmatrix},$$

$$R_\beta^\beta = \begin{pmatrix} R_1 & 0 \\ 0 & E_1U_1^{-1}L_1^{-1}C_1 + R_2 \end{pmatrix}.$$

Since A is an M-matrix and $R_i \geq 0$ for $i = 1, 2$, $R \geq 0$. From Lemma 3.1, it follows that $R_\alpha^\alpha \geq 0$ and $R_\beta^\beta \geq 0$. Hence, part (a) is proved. For the proof of part (b), if we compute inverse matrices of L, L_α, and L_β, then

$$L^{-1} = \begin{pmatrix} L_1^{-1} & 0 \\ 0 & L_2^{-1} \end{pmatrix},$$

$$L_\alpha{}^{-1} = \begin{pmatrix} L_1^{-1} & 0 \\ L_2^{-1}E_1D_1^{-1}L_1^{-1} & L_2^{-1} \end{pmatrix},$$

$$L_\beta{}^{-1} = \begin{pmatrix} L_1^{-1} & 0 \\ L_2^{-1}E_1U_1^{-1}L_1^{-1} & L_2^{-1} \end{pmatrix}.$$

Using parts (a) and (b) of Lemma 3.1, part (b) is obtained. For the proof of part (c), if we compute inverse matrices of U, U_α, and U_β, then

$$U^{-1} = \begin{pmatrix} U_1^{-1} & 0 \\ 0 & U_2^{-1} \end{pmatrix},$$

$$U_\alpha{}^{-1} = \begin{pmatrix} U_1^{-1} & U_1^{-1}C_1U_2^{-1} \\ 0 & U_2^{-1} \end{pmatrix},$$

$$U_\beta{}^{-1} = \begin{pmatrix} U_1^{-1} & U_1^{-1}L_1^{-1}C_1U_2^{-1} \\ 0 & U_2^{-1} \end{pmatrix}.$$

Using parts (a) and (d) of Lemma 3.1, part (c) is proved. Since $M^{-1} = U^{-1}L^{-1}$, $M_\alpha^{\alpha-1} = U_\alpha{}^{-1}L_\alpha{}^{-1}$, and $M_\beta^{\beta^{-1}} = U_\beta{}^{-1}L_\beta{}^{-1}$, parts (b) and (c) imply part (d). From parts (a) and (d), part (e) is immediately obtained. Since A is an M-matrix and $A = M - R$ is a regular splitting of A, it is easy to show that $\rho(M^{-1}R) < 1$. Hence, from Theorem 2.4, part (f) is proved. □

Many other types of the block ILU factorization preconditioners except those introduced in the above theorem have been discussed in [11].

Next, we consider block ILU factorization preconditioners for a nonsymmetric block-tridiagonal M-matrix of the general form (2). Generalization of Theorem 3.2 to an M-matrix of the form (2) is complicated but easy, so that the following theorem is described without proof.

THEOREM 3.3. *Let A be a nonsymmetric block-tridiagonal M-matrix of the form (2) and let $B_i = L_iU_i - R_i$ be a regular splitting of B_i which can be obtained by the ILU factorization process for each $i = 1, 2, \ldots, m$. Let D_i denote a diagonal matrix consisting of diagonal elements of U_i*

for each $i = 1, 2, \ldots, m$. *Let*

$$L = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_m \end{pmatrix}, \quad U = \begin{pmatrix} U_1 & 0 & \cdots & 0 \\ 0 & U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_m \end{pmatrix},$$

$$L_\alpha = \begin{pmatrix} L_1 & 0 & 0 & \cdots & 0 \\ -E_1 D_1^{-1} & L_2 & 0 & \cdots & 0 \\ 0 & -E_2 D_2^{-1} & L_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -E_{m-1} D_{m-1}^{-1} & L_m \end{pmatrix},$$

$$U_\alpha = \begin{pmatrix} U_1 & -C_1 & 0 & \cdots & 0 \\ 0 & U_2 & -C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & U_{m-1} & -C_{m-1} \\ 0 & 0 & 0 & 0 & U_m \end{pmatrix},$$

$$L_\beta = \begin{pmatrix} L_1 & 0 & 0 & \cdots & 0 \\ -E_1 U_1^{-1} & L_2 & 0 & \cdots & 0 \\ 0 & -E_2 U_2^{-1} & L_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -E_{m-1} U_{m-1}^{-1} & L_m \end{pmatrix},$$

$$U_\beta = \begin{pmatrix} U_1 & -L_1^{-1} C_1 & 0 & \cdots & 0 \\ 0 & U_2 & -L_2^{-1} C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & U_{m-1} & -L_{m-1}^{-1} C_{m-1} \\ 0 & 0 & 0 & 0 & U_m \end{pmatrix},$$

$M = LU$, $M_\alpha^\alpha = L_\alpha U_\alpha$, *and* $M_\beta^\beta = L_\beta U_\beta$. *Then, each of the following holds:*

(a) $R = M - A \geq 0$, $R_\alpha^\alpha = M_\alpha^\alpha - A \geq 0$, and $R_\beta^\beta = M_\beta^\beta - A \geq 0$.

(b) $0 \leq L^{-1} \leq L_\alpha{}^{-1} \leq L_\beta{}^{-1}$.

(c) $0 \leq U^{-1} \leq U_\alpha{}^{-1} \leq U_\beta{}^{-1}$.

(d) $0 \leq M^{-1} \leq M_\alpha^{\alpha^{-1}} \leq M_\beta^{\beta^{-1}}$.

(e) $A = M - R = M_\alpha^\alpha - R_\alpha^\alpha = M_\beta^\beta - R_\beta^\beta$ *are regular splittings of* A.

(f) $\rho(M_\beta^{\beta^{-1}} R_\beta^\beta) \leq \rho(M_\alpha^{\alpha^{-1}} R_\alpha^\alpha) \leq \rho(M^{-1} R) < 1$.

If $B_i = L_iU_i - R_i$ is an ILU factorization of B_i, then L_iU_i can be viewed as a complete LU factorization of $B_i + R_i$ and the following equation holds

$$\begin{pmatrix} B_i + R_i & -C_i \\ -E_i & 0 \end{pmatrix} = \begin{pmatrix} L_i & 0 \\ -E_iU_i^{-1} & I \end{pmatrix} \begin{pmatrix} U_i & -L_i^{-1}C_i \\ 0 & -E_i(L_iU_i)^{-1}C_i \end{pmatrix}.$$

This equation shows that $L_i^{-1}C_i$ and $E_iU_i^{-1}$ required for the construction of L_β and U_β in Theorem 3.3 can be computed efficiently at the time when the ILU factorization of B_i is executed. In other words, since L_i^{-1} is a product of elementary lower triangular matrices which are generated during the ILU factorization process of B_i, $L_i^{-1}C_i$ is not computed explicitly using matrix-solve operations, but computed implicitly using elementary lower triangular matrices. Also, $E_iU_i^{-1}$ can be computed implicitly without using matrix-solve operations because $\begin{pmatrix} L_i \\ -E_iU_i^{-1} \end{pmatrix}$ can be computed during the Gaussian elimination process which transforms $\begin{pmatrix} B_i + R_i \\ -E_i \end{pmatrix}$ into $\begin{pmatrix} U_i \\ 0 \end{pmatrix}$.

Since L_i's and U_i's can be computed *independently of one another*, three types of the block ILU factorization preconditioners M, M_α^α, and M_β^β presented in Theorem 3.3 can be computed *in parallel.* This inherent parallelism is a big advantage of the block ILU factorization preconditioners. In this paper, the right preconditioned iterative methods are used to test the effectiveness of the block preconditioners in Theorem 3.3.

LEMMA 3.4. *Let $A = K - N$ be a splitting of A. Then*

(a) $\sigma(K^{-1}N) = \sigma(NK^{-1})$ *and* $\rho(K^{-1}N) = \rho(NK^{-1})$,

(b) $\sigma(K^{-1}A) = \sigma(AK^{-1})$ *and* $\rho(K^{-1}A) = \rho(AK^{-1})$.

Proof. Since there is a nonsingular matrix K such that $NK^{-1} = K(K^{-1}N)K^{-1}$, $K^{-1}N$ is similar to NK^{-1}. Hence, $\sigma(K^{-1}N) = \sigma(NK^{-1})$ and thus $\rho(K^{-1}N) = \rho(NK^{-1})$. Since $K^{-1}A = I - K^{-1}N$ and $AK^{-1} = I - NK^{-1}$, part (b) is obtained from part (a). □

Let $A = K - N$ be a splitting of A. Then, convergence rate of the right preconditioned iterative methods with the preconditioner K for solving $Ax = b$ largely depends upon how clustered the eigenvalues of AK^{-1} are about 1. Note that $\sigma(AK^{-1}) = \{1 - \lambda \mid \lambda \in \sigma(NK^{-1})\}$. It follows that the smaller the value of $\rho(NK^{-1})$ is, the more clustered the eigenvalues of AK^{-1} are about 1. In Lemma 3.4, we have shown that $\rho(NK^{-1}) = \rho(K^{-1}N)$. Hence, we want to make $\rho(K^{-1}N)$ as small as possible to ensure fast convergence of the right preconditioned iterative methods. From this point of view, the right preconditioned iterative methods with the block ILU factorization preconditioner of type M_β^β will converge to the exact solution faster than those with any other type of the block ILU factorization preconditioner presented in Theorem 3.3.

4. Applications of block ILU factorization preconditioners

The construction of three types of the block ILU factorization preconditioners presented in Theorem 3.3 will be considered in this section for a special type of matrix A described below. The matrix A arises from five-point discretization of the following second-order partial differential equation:

$$-(au_x)_x - (bu_y)_y + (cu)_x + (du)_y + fu = g \tag{4}$$

with $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, and $f(x,y)$ defined on a square region Ω, and with suitable boundary conditions on $\partial\Omega$ which denotes the boundary of Ω. The structure of the resulting matrix A is of the form (2) with B_i's tridiagonal matrices and C_i's and E_i's diagonal matrices. Suppose that $a(x,y)$, $b(x,y)$, $c(x,y)$, $d(x,y)$, and $f(x,y)$ are chosen so that the resulting matrix A becomes a nonsymmetric M-matrix.

Since B_i is a tridiagonal matrix, the complete LU factorization of B_i has no fill-in elements. More specifically, if $B_i = L_iU_i$ is the complete LU factorization of B_i, then L_i is a unit lower bidiagonal matrix and U_i is an upper bidiagonal matrix. Hence, three types of block ILU factorization preconditioners are constructed using the complete LU factorizations of B_i's rather than using the ILU factorizations of B_i's. The block preconditioners defined in Theorems 3.3 which are constructed based on the complete LU factorizations of 1×1 block matrices B_i are from now on called *1-block ILU factorization preconditioners.* Suppose that $B_i = L_iU_i$ is the complete LU factorization of B_i and D_i is a diagonal matrix consisting of diagonal elements of U_i.

For the purpose of getting more effective block preconditioners than 1-block ILU factorization preconditioners mentioned above, we now consider *2-block ILU factorization preconditioners* which are constructed based on the ILU factorizations of 2×2 block matrices rather than 1×1

block matrices B_i. For simplicity of exposition, suppose that A is a 4×4 nonsymmetric block-tridiagonal M-matrix of the form (2), i.e., $m = 4$ is assumed in the form (2). From now on, let d denote the order of matrices B_i, C_i, and E_i. First, A is partitioned into

$$A = \begin{pmatrix} \mathcal{B}_1 & -\mathcal{C}_1 \\ -\mathcal{E}_1 & \mathcal{B}_2 \end{pmatrix}$$

where $\mathcal{B}_1 = \begin{pmatrix} B_1 & -C_1 \\ -E_1 & B_2 \end{pmatrix}$, $\mathcal{B}_2 = \begin{pmatrix} B_3 & -C_3 \\ -E_3 & B_4 \end{pmatrix}$, $\mathcal{C}_1 = \begin{pmatrix} 0 & 0 \\ C_2 & 0 \end{pmatrix}$, and $\mathcal{E}_1 = \begin{pmatrix} 0 & E_2 \\ 0 & 0 \end{pmatrix}$ are 2×2 block submatrices of A. Since A is assumed to be an M-matrix, $\mathcal{B}_i$ $(i = 1, 2)$ is also an M-matrix. It follows from Theorem 2.1 that the ILU factorization of $\mathcal{B}_i$ exists. If $\mathcal{B}_i = \mathcal{L}_i\mathcal{U}_i$ is the complete LU factorization of $\mathcal{B}_i$, then the nonzero structures of $\mathcal{B}_i$, $\mathcal{L}_i$, and $\mathcal{U}_i$ for $d = 7$ are illustrated in Fig. 1.

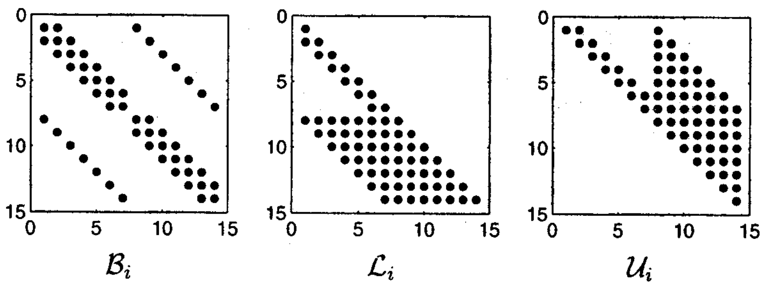


Fig. 1. Nonzero structures of $\mathcal{B}_i$, $\mathcal{L}_i$ and $\mathcal{U}_i$

As can be seen in Fig. 1, the complete LU factorization of $\mathcal{B}_i$ now has a lot of fill-in elements, so that the ILU factorization of $\mathcal{B}_i$ with some fill-ins needs to be considered for the construction of 2-block ILU factorization preconditioners. Note that the complete LU factorization of B_i is used for the construction of 1-block ILU factorization preconditioners since no fill-ins occur during the LU factorization of B_i. For each fixed i, let $\mathcal{B}_i = \mathcal{L}_{ij}\mathcal{U}_{ij} - \mathcal{R}_{ij}$ be the ILU factorization of $\mathcal{B}_i$, where $0 \le j \le d-1$, and the nonzero structures of $\mathcal{L}_{ij}$'s and $\mathcal{U}_{ij}$'s for $d = 7$ are illustrated in Fig. 2 and Fig. 3 respectively.

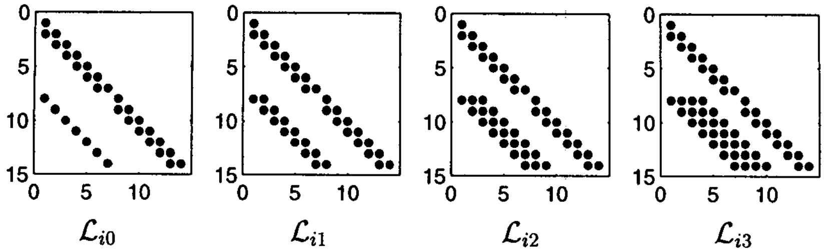


Fig. 2. Nonzero structures of $\mathcal{L}_{ij}$'s

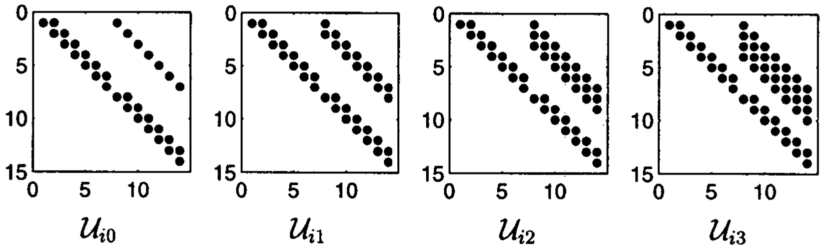


Fig. 3. Nonzero structures of $\mathcal{U}_{ij}$'s

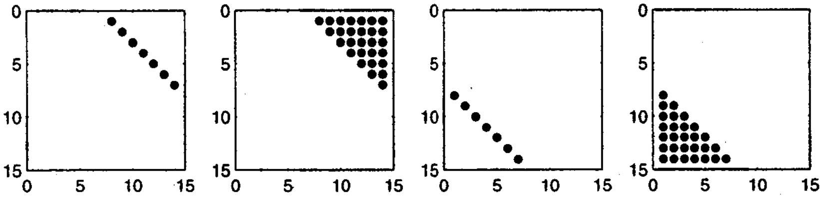
Notice that if $\mathcal{B}_i = \mathcal{L}_i\mathcal{U}_i$ is the complete LU factorization of $\mathcal{B}_i$, then $\mathcal{L}_{i,d-1} = \mathcal{L}_i$, $\mathcal{U}_{i,d-1} = \mathcal{U}_i$, and $\mathcal{R}_{i,d-1} = 0$. Let $\mathcal{D}_{ij}$ denote the diagonal matrix which is obtained by taking diagonal part of $\mathcal{U}_{ij}$. If we let for each $0 \le j \le d-1$

$$\mathcal{L}_j^2 = \begin{pmatrix} \mathcal{L}_{1j} & 0 \\ 0 & \mathcal{L}_{2j} \end{pmatrix}, \quad \mathcal{U}_j^2 = \begin{pmatrix} \mathcal{U}_{1j} & 0 \\ 0 & \mathcal{U}_{2j} \end{pmatrix},$$

$$(\mathcal{L}_\alpha)_j^2 = \begin{pmatrix} \mathcal{L}_{1j} & 0 \\ -\mathcal{E}_1\mathcal{D}_{1j}^{-1} & \mathcal{L}_{2j} \end{pmatrix}, \quad (\mathcal{U}_\alpha)_j^2 = \begin{pmatrix} \mathcal{U}_{1j} & -\mathcal{C}_1 \\ 0 & \mathcal{U}_{2j} \end{pmatrix},$$

$$(\mathcal{L}_\beta)_j^2 = \begin{pmatrix} \mathcal{L}_{1j} & 0 \\ -\mathcal{E}_1\mathcal{U}_{1j}^{-1} & \mathcal{L}_{2j} \end{pmatrix}, \quad (\mathcal{U}_\beta)_j^2 = \begin{pmatrix} \mathcal{U}_{1j} & -\mathcal{L}_{1j}^{-1}\mathcal{C}_1 \\ 0 & \mathcal{U}_{2j} \end{pmatrix},$$

then $M_j^2 = \mathcal{L}_j^2\mathcal{U}_j^2$, $(M_\alpha^\alpha)_j^2 = (\mathcal{L}_\alpha)_j^2(\mathcal{U}_\alpha)_j^2$, and $(M_\beta^\beta)_j^2 = (\mathcal{L}_\beta)_j^2(\mathcal{U}_\beta)_j^2$ are 2-block ILU factorization preconditioners of type M, M_α^α, and M_β^β respectively, where the superscript 2 is used to represent 2-block preconditioners. The nonzero structures of $\mathcal{E}_1\mathcal{D}_{1j}^{-1}$, $\mathcal{E}_1\mathcal{U}_{1j}^{-1}$, $\mathcal{C}_1$, and $\mathcal{L}_{1j}^{-1}\mathcal{C}_1$ for $d = 7$ are illustrated in Fig. 4.



$\mathcal{E}_1\mathcal{D}_{1j}^{-1}$ $\mathcal{E}_1\mathcal{U}_{1j}^{-1}$ $\mathcal{C}_1$ $\mathcal{L}_{1j}^{-1}\mathcal{C}_1$

Fig. 4. Nonzero structures of $\mathcal{E}_1\mathcal{D}_{1j}^{-1}$, $\mathcal{E}_1\mathcal{U}_{1j}^{-1}$, $\mathcal{C}_1$, and $\mathcal{L}_{1j}^{-1}\mathcal{C}_1$

From Fig. 4, it can be seen that the block preconditioner of type M_β^β has much more fill-ins than that of type M_α^α.

In the similar way as was done for 2-block preconditioners, *k-block ILU factorization preconditioners* M_j^k, $(M_\alpha^\alpha)_j^k$, and $(M_\beta^\beta)_j^k$ which are based on the ILU factorizations of $k \times k$ block matrices can be easily constructed. That is, the $m \times m$ block matrix A of the form (2) is first partitioned so that each submatrix of A is a $k \times k$ block matrix (it is assumed that m is divisible by k), and then the ILU factorizations of $k \times k$ block matrices are carried out to construct k-block preconditioners. Since the complete LU factorizations of tridiagonal matrices B_i which have no fill-in elements are used for construction of 1-block preconditioners, for each $0 \le j \le d-1$ $M_j^1 = M$, $(M_\alpha^\alpha)_j^1 = M_\alpha^\alpha$, and $(M_\beta^\beta)_j^1 = M_\beta^\beta$. In other words, the superscript 1 representing 1-block preconditioners can be omitted and 1-block preconditioners of types M, M_α^α, and M_β^β are independent of j. Notice that the construction of $(k+1)$-block preconditioners requires a little more storage and arithmetic than that of k-block preconditioners.

5. Numerical results

In this section, we provide numerical results of the right preconditioned GMRES(20) (called PGMRES(20)) and the right preconditioned BiCGSTAB (called PBSTAB) methods using two different types of the k-block ILU factorization preconditioners M_j^k and $(M_\alpha^\alpha)_j^k$ for solving $Ax = b$ with the special type of matrix A described in section 4. However, numerical experiments for the k-block ILU factorization preconditioner $(M_\beta^\beta)_j^k$ are not provided here since they require a lot of fill-in elements causing too much storage and arithmetic. For each type of block preconditioner, numerical experiments are carried out for $0 \le j \le 3$ and

$1 \leq k \leq 4$. To evaluate the effectiveness of the k-block ILU factorization preconditioners, we also provide numerical results of PGMRES(20) and PBSTAB methods using the standard ILU factorization preconditioner with 0 extra diagonals which is called *ILU(0) preconditioner.* In all cases, the preconditioned iterative methods were started with $x_0 = 0$, and they were stopped when $\|r_i\|_2/\|b\|_2 < 10^{-8}$, where $\|\cdot\|_2$ refers to L_2-norm. All numerical experiments have been carried out in double precision floating point arithmetic. All data presented in Tables 1 to 8 represent *the number of iterations* satisfying the stopping criterion mentioned above, and the data under the column I of Tables 1 to 8 represent the number of iterations of unpreconditioned iterative methods (which can be viewed as preconditioned iterative methods with identity matrix I as a preconditioner). NC in Tables 1 to 8 indicates that iterative methods do not converge within 1000 iterations. For all test problems, the unit square region $\Omega = (0,1) \times (0,1)$ and the Dirichlet boundary condition $u(x,y) = 0$ on $\partial\Omega$ are used. Only the discretized matrix A is of importance, so the right-hand side vector b is created artificially. Therefore, the right-hand side function $g(x,y)$ in equation (4) is not relevant.

EXAMPLE 5.1. We consider equation (4) with $a(x,y) = b(x,y) = 1$, $c(x,y) = 10(x+y)$, $d(x,y) = 10(x-y)$, and $f(x,y) = 0$. We have used two uniform meshes of $\Delta x = \Delta y = 1/49$ and $\Delta x = \Delta y = 1/73$, which lead to two matrices of order $n = 48 \times 48$ and $n = 72 \times 72$, where Δx and Δy refer to the mesh sizes in the x-direction and y-direction, respectively. Once the matrix A is constructed from five-point finite difference discretization of the PDE, the right-hand side vector b is chosen so that $b = A(1,1,\ldots,1)^T$. Numerical results for this problem are listed in Tables 1 and 2.

EXAMPLE 5.2. This example is the same as Example 5.1 except for $a(x,y) = b(x,y)$ which is defined as

$$a(x,y) = \begin{cases} 10^3 & \text{if } 1/4 < x,y < 3/4 \\ 1 & \text{otherwise} \end{cases}.$$

We have used the same uniform meshes as Example 5.1, and the right-hand side vector b is chosen so that $b = A(1,1,\ldots,1)^T$. Numerical results for this problem are listed in Tables 3 and 4.

TABLE 1: Number of iterations of PGMRES(20) for Example 5.1

		PGMRES(20)									
n	j	I	$ILU(0)$	M_j^1	M_j^2	M_j^3	M_j^4	$(M_\alpha^\alpha)_j^1$	$(M_\alpha^\alpha)_j^2$	$(M_\alpha^\alpha)_j^3$	$(M_\alpha^\alpha)_j^4$
48^2	0	224	70	160	113	99	97	72	76	76	73
	1				114	83	81		59	54	45
	2				115	84	74		56	45	40
	3				116	84	75		56	43	38
72^2	0	377	84	253	156	148	140	103	100	97	90
	1				151	134	113		86	73	65
	2				152	133	109		85	63	75
	3				146	128	120		82	75	74

EXAMPLE 5.3. We consider equation (4) with $a(x,y) = 2e^{x+y}$, $b(x,y) = 3e^{x+y}$, $c(x,y) = \sin(x+y)$, $d(x,y) = \cos(x-y)$, and $f(x,y) = 10/(1+x+y)$. We have used the same uniform meshes as Example 5.1, and the right-hand side vector b is chosen so that the exact solution is the discretization of $xe^{xy}\sin(\pi x)\sin(\pi y)$. Numerical results for this problem are listed in Tables 5 and 6.

EXAMPLE 5.4. We consider equation (4) with $c(x,y) = \sin(x+y)$, $d(x,y) = \cos(x-y)$, $f(x,y) = 2/(1+x+y)$, and $a(x,y) = b(x,y)$ defined as

$$a(x,y) = \begin{cases} 3e^{x+y} & \text{if } 1/4 < x, y < 3/4 \\ 6e^{x+y} & \text{otherwise} \end{cases}$$

We have used the same uniform meshes as Example 5.1, and the right-hand side vector b is chosen so that the exact solution is the discretization of $10xy(1-x)(1-y)e^{x-y}$. Numerical results for this problem are listed in Tables 7 and 8.

As can be seen in Tables 1 to 8, the numerical results presented are in good agreement with the theoretical results presented in Theorem 3.3. That is, the block preconditioner of type M_α^α is more effective than the block preconditioner of type M. It can be also seen that PGMRES(20) and PBSTAB with $(k+1)$-block preconditioners converge faster than those with k-block preconditioners. As compared with the standard ILU factorization preconditioner, the block preconditioner $(M_\alpha^\alpha)_j^k$ is relatively effective when $j \geq 1$ and $k \geq 2$.

TABLE 2: Number of iterations of PBSTAB for Example 5.1

		PBSTAB									
n	j	I	$ILU(0)$	M_j^1	M_j^2	M_j^3	M_j^4	$(M_\alpha^\alpha)_j^1$	$(M_\alpha^\alpha)_j^2$	$(M_\alpha^\alpha)_j^3$	$(M_\alpha^\alpha)_j^4$
48^2	0	99	28	74	52	45	40	33	30	28	28
	1				50	42	34		25	23	22
	2				48	40	36		24	22	21
	3				52	38	33		24	21	21
72^2	0	145	42	111	73	66	58	46	45	44	44
	1				71	57	54		39	33	31
	2				71	54	50		34	31	29
	3				73	56	49		34	31	28

TABLE 3: Number of iterations of PGMRES(20) for Example 5.2

		PGMRES(20)									
n	j	I	$ILU(0)$	M_j^1	M_j^2	M_j^3	M_j^4	$(M_\alpha^\alpha)_j^1$	$(M_\alpha^\alpha)_j^2$	$(M_\alpha^\alpha)_j^3$	$(M_\alpha^\alpha)_j^4$
48^2	0	NC	43	110	75	65	61	50	46	45	43
	1				74	60	54		39	37	32
	2				74	59	51		38	33	31
	3				74	58	52		38	32	31
72^2	0	NC	61	155	120	105	96	73	61	58	58
	1				119	89	80		57	50	48
	2				119	90	79		56	50	46
	3				119	91	79		56	49	45

TABLE 4: Number of iterations of PBSTAB for Example 5.2

		PBSTAB									
n	j	I	$ILU(0)$	M_j^1	M_j^2	M_j^3	M_j^4	$(M_\alpha^\alpha)_j^1$	$(M_\alpha^\alpha)_j^2$	$(M_\alpha^\alpha)_j^3$	$(M_\alpha^\alpha)_j^4$
48^2	0	NC	25	53	44	38	35	30	29	27	27
	1				42	36	32		24	21	18
	2				42	33	28		23	22	19
	3				42	33	29		23	20	19
72^2	0	NC	37	89	59	55	51	44	46	38	43
	1				58	50	43		38	32	31
	2				60	48	43		34	31	26
	3				59	50	44		33	30	27

TABLE 5: Number of iterations of PGMRES(20) for Example 5.3

		PGMRES(20)									
n	j	I	$ILU(0)$	M_j^1	M_j^2	M_j^3	M_j^4	$(M_\alpha^\alpha)_j^1$	$(M_\alpha^\alpha)_j^2$	$(M_\alpha^\alpha)_j^3$	$(M_\alpha^\alpha)_j^4$
48^2	0	713	64	245	139	121	109	77	72	69	67
	1				138	113	88		59	54	48
	2				139	113	87		57	51	44
	3				139	113	87		58	49	43
72^2	0	NC	107	421	251	194	176	117	114	113	112
	1				249	175	151		99	84	75
	2				255	181	149		95	77	70
	3				254	181	148		94	76	69

TABLE 6: Number of iterations of PBSTAB for Example 5.3

		PBSTAB									
n	j	I	$ILU(0)$	M_j^1	M_j^2	M_j^3	M_j^4	$(M_\alpha^\alpha)_j^1$	$(M_\alpha^\alpha)_j^2$	$(M_\alpha^\alpha)_j^3$	$(M_\alpha^\alpha)_j^4$
48^2	0	229	37	89	60	54	52	43	39	39	39
	1				59	48	44		33	31	29
	2				60	51	44		31	29	25
	3				63	49	46		31	27	26
72^2	0	331	53	142	92	81	78	61	58	58	58
	1				95	75	66		48	45	42
	2				93	76	63		46	42	39
	3				92	72	66		45	41	38

TABLE 7: Number of iterations of PGMRES(20) for Example 5.4

		PGMRES(20)									
n	j	I	$ILU(0)$	M_j^1	M_j^2	M_j^3	M_j^4	$(M_\alpha^\alpha)_j^1$	$(M_\alpha^\alpha)_j^2$	$(M_\alpha^\alpha)_j^3$	$(M_\alpha^\alpha)_j^4$
48^2	0	NC	64	197	136	106	93	68	68	66	65
	1				136	107	88		60	52	48
	2				137	98	87		58	48	43
	3				138	98	79		58	46	41
72^2	0	NC	103	308	219	187	178	133	128	125	123
	1				219	177	146		94	92	78
	2				218	175	150		92	80	69
	3				218	175	147		92	77	66

TABLE 8: Number of iterations of PBSTAB for Example 5.4

		PBSTAB									
n	j	I	$ILU(0)$	M_j^1	M_j^2	M_j^3	M_j^4	$(M_\alpha^\alpha)_j^1$	$(M_\alpha^\alpha)_j^2$	$(M_\alpha^\alpha)_j^3$	$(M_\alpha^\alpha)_j^4$
48^2	0	218	33	89	61	53	51	43	36	39	34
	1				60	47	40		31	33	30
	2				64	47	42		33	27	30
	3				65	50	41		32	29	27
72^2	0	350	50	140	91	80	69	65	51	55	52
	1				94	74	57		51	50	39
	2				93	66	59		43	42	36
	3				87	76	57		42	40	35

6. Conclusions

We presented in this paper three types of block ILU factorization preconditioners which can be computed *in parallel.* The block ILU factorization preconditioner of type M_β^β may not be used in practical situations since it requires a lot of fill-in elements causing too much storage and arithmetic. Block ILU factorization preconditioner of type M has rich parallelism since both the computation of preconditioner and preconditioner solver step can be done in parallel, but its effectiveness is much worse than the block preconditioner of type M_α^α (see Tables 1 to 8).

The block ILU factorization preconditioner of type M_α^α is relatively effective as compare with the standard ILU factorization preconditioner. Thus, $(M_\alpha^\alpha)_j^k$ with $j \geq 1$ and $k \geq 2$ is recommended as a preconditioner of the nonstationary iterative methods. Notice that the number of arithmetic operations for constructing $(M_\alpha^\alpha)_j^k$ grows as j becomes large. From our experiments, it is not recommended to use large value of j and the *optimal value* of j usually ranges from 1 to 3.

References

[1] O. Axelsson, *Incomplete block matrix factorization preconditioning methods. The ultimate answer?*, J. Comput. Appl. Math. **12 & 13** (1985), 3–18.

[2] ———, *A general incomplete block-matrix factorization method*, Linear Algebra Appl. **74** (1986), 179–190.

[3] ———, *Iterative solution methods*, Cambridge University Press, NY, 1994.

[4] O. Axelsson and B. Polman, *On approximate factorization methods for block-matrices suitable for vector and parallel processors*, Linear Algebra Appl. **77** (1986), 3–26.

[5] R. Barrett et al., *Templates for the solution of linear systems: Building blocks for iterative methods*, SIAM, Philadelphia, PA, 1994.

[6] R. Beauwens and M. Bouzid, *On sparse block factorization, iterative methods*, SIAM J. Numer. Anal. **24** (1987), 1066–1076.

[7] P. Concus, G. H. Golub and G. Meurant, *Block preconditioning for the CG method*, SIAM J. Sci. Stat. Comput. **6** (1985), 220–252.

[8] P. Concus and G. Meurant, *On computing INV block preconditioning for the conjugate gradient method*, BIT **26** (1986), 493–504.

[9] R. Fletcher, *Conjugate gradient methods for indefinite systems*, in: Lecture Notes Math. Springer-Verlag, Berlin **506** (1976), 73–89.

[10] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, MD, 3rd edition, 1996.

[11] Sang W. Kim, *Block ILU factorization preconditioners for a block-tridiagonal M-matrix and a generalized variant of stationary iterative method*, Ph. D. Dissertation, Chungbuk National University, 1998.

[12] J. A. Meijerink and H. A. van der Vorst, *An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix*, Math. Comput. **31** (1977), 148–162.

[13] G. Meurant, *The block preconditioned conjugate gradient method on vector computer*, BIT **24** (1984), 623–633.

[14] Y. Saad, *Iterative methods for sparse linear systems*, PWS Publishing Company, Boston, MA, 1996.

[15] Y. Saad and M. H. Schultz, *GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput. **7** (1986), 856–869.

[16] P. Sonnerveld, *CGS, a fast Lanczos-type solver for nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput. **10** (1989), 36–52.

[17] H. A. Van der Vorst, *Bi-CGSTAB: A fast and smoothly converging variant of Bi-CG for the solution of nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput. **13** (1992), 631–644.

[18] Jae H. Yun, *Block incomplete factorization preconditioners for a symmetric block-tridiagonal M-matrix*, J. Comput. Appl. Math. **94** (1998), 133–152.

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