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CONVOLUTION OPERATORS WITH THE AFFINE ARCLENGTH MEASURE ON PLANE CURVES

YOUNGWOO CHOI

ABSTRACT. Let $\gamma : I \to \mathbb{R}^2$ be a sufficiently smooth curve and σ_γ be the affine arclength measure supported on γ. In this paper, we study the L^p–improving properties of the convolution operators T_{σ_γ} associated with σ_γ for various curves γ. Optimal results are obtained for all finite type plane curves and homogeneous curves (possibly blowing up at the origin). As an attempt to extend this result to infinitely flat curves we give an example of a family of flat curves whose affine arclength measure has the same L^p-improvement property. All of these results will be based on uniform estimates of damping oscillatory integrals.

1. Introduction

Let $\gamma : I \to \mathbb{R}^2$ be a sufficiently smooth curve and let

$$\lambda(t) = \left|\det \begin{bmatrix} \gamma'(t) \\ \gamma''(t) \end{bmatrix}\right|^{\frac{1}{3}}.$$

The affine arclength measure σ_γ on γ is given by

$$\iint_{\mathbb{R}^2} f \, d\sigma_\gamma = \int_I f\big(\gamma(t)\big)\, \lambda(t)\, dt$$

for $f \in C_0^\infty(\mathbb{R}^2)$. Consider the convolution operator T_{σ_γ} defined by

$$(1.1) \qquad T_{\sigma_\gamma} f(x) = f * \sigma_\gamma(x) = \int_I f\big(x - \gamma(t)\big)\, \lambda(t)\, dt.$$

Under certain circumstances T_{σ_γ} is L^p-improving in the sense that for some p and q with $1 \le p < q \le \infty$, T_{σ_γ} is a bounded operator

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from $L^p(\mathbb{R}^2)$ into $L^q(\mathbb{R}^2)$. Littman [6] proved that if γ is a compact nondegenerate plane curve (in which case the affine arclength measure is essentially the same with the euclidean arclength measure) T_{σ_γ} maps $L^{\frac{3}{2}}(\mathbb{R}^2)$ boundedly into $L^3(\mathbb{R}^2)$. Global version of this [1, 3] is also available:

(1.2)

$$\left(\iint_{\mathbb{R}^2} \left|\int_{\mathbb{R}} f\left(x_1 - t, x_2 - t^2\right)\, dt\right|^3 dx_1 dx_2\right)^{\frac{1}{3}} \leq C\, \|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)}.$$

In this paper, we generalize Littman's result to degenerate plane curves. To mitigate the effect of the degeneracies, we will consider the affine arclength measure rather than the euclidean arclength measure.

The use of the affine arclength measure in convolution operators was suggested by Drury [2]. In many cases it turned out to be effective in extending results for nondegenerate curves to those for degenerate curves.

Mainly we will prove

$$\left\|T_{\sigma_\gamma} f\right\|_{L^3(\mathbb{R}^2)} \leq C\, \|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \tag{1.3}$$

when γ is a compact plane curve of finite type. Global estimates like (1.2) will be also obtained and certain examples of flat curves with the same mapping properties will be given. As expected the results are degeneracy-independent in all successful cases. The author would like to thank Prof. Stephen Wainger and Prof. Andreas Seeger for their guidance on this project during his stay at Department of Mathematics, University of Wisconsin–Madison as a graduate student.

2. Estimates of damping oscillatory integrals

The main purpose of this section is to provide a couple of estimates for certain oscillatory integrals. The first one can be thought of as a perturbed version of van der Corput's lemma.

PROPOSITION 2.1. *Let $g(t)$ and $h(t)$ be twice continuously differentiable functions on (α, β), $1 \leq \alpha < \beta < \infty$, satisfying the following conditions for some K and L with $0 < K < L < \infty$:*

1. $g'(t) \geq 0$;
2. $\frac{g'(t)}{t}$ *is monotone* ;

3. $g''(t) = \frac{g'(t)}{t}\{1-\mu(t)\}$ with $K \leq |\mu(t)| \leq L$;
4. $\left|\frac{h'(t)}{t}\right| \leq \min\left(\frac{K}{2}, \frac{1}{2}\right)$;
5. $|h''(t)| \leq \min\left(\frac{K}{2}, \frac{1}{2}\right)$.

Then, for $s(t) = t^2 \pm g(t) + h(t)$ *and* $w \in C^1(\alpha, \beta)$, *we have*

$$\left|\int_\alpha^\beta e^{is(t)}\, w(t)\, dt\right| \leq C\left(\|w\|_{L^\infty(\alpha,\beta)} + \|w'\|_{L^1(\alpha,\beta)}\right)$$

where $C = C(K, L)$ *is a constant depending only on* K *and* L.

Proof of Proposition 2.1. We may assume that $K \leq 1 \leq L$ without loss of generality. Let's first consider the case $s(t) = t^2 + g(t) + h(t)$. Conditions 1, 3, 4, and 5 imply

$$s'(t) \geq t + g'(t)$$

and

$$|s''(t)| \leq \frac{(L+2)\big(t+g'(t)\big)}{t}.$$

An integration by parts gives:

$$\begin{aligned}\left|\int_\alpha^\beta e^{is(t)}\, w(t)\, dt\right| &\leq \left(2 + \int_\alpha^\beta \frac{|s''(t)|}{s'(t)^2}\, dt\right)\left(\|w\|_{L^\infty} + \|w'\|_{L^1}\right)\\ &\leq \left(2 + (L+2)\int_\alpha^\beta \frac{1}{t^2}\, dt\right)\left(\|w\|_{L^\infty} + \|w'\|_{L^1}\right)\\ &\leq (L+4)\left(\|w\|_{L^\infty} + \|w'\|_{L^1}\right).\end{aligned}$$

Now, we turn to the other case $s(t) = t^2 - g(t) + h(t)$.
Let $\beta' \in (\alpha, \beta]$ and write

$$\int_\alpha^{\beta'} e^{is(t)}\, dt = I_1 + I_2\,,$$

with

$$I_l = \int_{J_l} e^{is(t)}\, dt\,, \quad l = 1,\ 2,$$

$$J_1 = \left\{ t \in (\alpha, \beta') : \left| 1 - \frac{g'(t)}{2t} \right| \le \frac{K}{3} \right\}$$

and

$$J_2 = (\alpha, \beta') \setminus J_1.$$

Each of J_1 and J_2 is, by condition 2.1, either empty, a subinterval, or a union of two disjoint subintervals. To estimate I_1, we observe that for $t \in J_1$,

$$\left| g''(t) - 2 \right| \ge \frac{2K}{3}.$$

So, we have

$$\left| s''(t) \right| \ge \frac{K}{6}.$$

An application of van der Corput's lemma gives us

$$\left| I_1 \right| \le \frac{10\sqrt{6}}{\sqrt{K}}.$$

It remains to estimate I_2. By conditions 3, 4 and 5, we have

$$\begin{aligned} \left| s'(t) \right| &\ge 2t \left(\left| 1 - \frac{g'(t)}{2t} \right| - \left| \frac{h'(t)}{2t} \right| \right) \\ &\ge 2t \left(\frac{K}{3} - \frac{K}{4} \right) = \frac{tK}{6} ; \\ \left| s'(t) \right| &\ge \frac{t}{2} \left| 1 - \frac{g'(t)}{2t} \right| ; \end{aligned}$$

and

$$\begin{aligned} \left| s''(t) \right| &\le 2 + \frac{g'(t)}{t} (1 + L) + 1 \\ &\le 2 (L + 2) \left(1 + \frac{g'(t)}{2t} \right) \\ &\le 2 (L + 2) \left(2 + \left| \frac{g'(t)}{2t} - 1 \right| \right) \end{aligned}$$

for $t \in J_2$. Integrating by parts, we obtain

$$\begin{aligned}
|I_2| &\leq \frac{24}{K} + \int_{J_2} \frac{|s''(t)|}{s'(t)^2}\, dt \\
&\leq \frac{24}{K} + \frac{168\,(L+2)}{K^2} \int_{J_2} \frac{1}{t^2}\, dt \\
&\leq \frac{24}{K} + \frac{168\,(L+2)}{K^2}.
\end{aligned}$$

Thus, we get

$$\left| \int_\alpha^{\beta'} e^{is(t)}\, dt \right| \leq C$$

for any $\beta' \in (\alpha, \beta]$. Additional integration by parts provides the desired estimate. □

As an application of Proposition 2.1, we get the following estimate which will be useful in treating plane curves and space curves.

PROPOSITION 2.2. *Let $0 \leq a < b < \infty$ and let $\phi : (a,b) \to \mathbb{R}$ be a real valued C^3 function such that the following conditions hold for some constants C_1 and C_2 with $0 < C_1 < C_2 < \infty$:*

1. *$\phi''(x)$ never vanishes ;*
2. *$C_1 \phi'(x)^2 \leq |\phi(x)\phi''(x)| \leq C_2 \phi'(x)^2$;*
3. *$|\phi(x)\phi'''(x)| \leq C_2 |\phi'(x)\phi''(x)|$;*
4. *$\int_a^b \left| \frac{d}{dx} \frac{\sqrt{\phi(x)\phi''(x)}}{\phi'(x)} \right| dx \leq C_2$.*

For $y \in \mathbb{R}$, we let

$$I_y(\xi, \eta) = |\eta|^{\frac{1}{2}} \int_a^b e^{i(\xi x + \eta\phi(x))}\, |\phi''(x)|^{\frac{1}{2} + yi}\, dx.$$

Then,

$$|I_y(\xi, \eta)| \leq C\, (1 + |y|)^{\frac{3}{2}},$$

where $C = C(C_1, C_2)$ is a constant depending only on C_1 and C_2.

Proof of Proposition 2.2. We may assume without loss of generality that $\eta > 0$, $\xi \neq 0$, and $\phi(x)$ and $\phi'(x)$ never vanish. A change of variable $t^2 = \eta\,|\phi(x)|$, allows us to write

$$I_y(\xi,\eta) \;=\; \pm\int_\alpha^\beta e^{is(t)}\; w(t)\; dt,$$

where

$$\begin{aligned} s(t) &= t^2 \pm g(t) + h(t),\\ g(t) &= \operatorname{sgn}(\phi(x)\phi'(x))\,|\xi|\,x,\\ h(t) &= 2y\;\log\big|\phi'(x)\big| - y\;\log\big|\phi(x)\big|, \end{aligned}$$

and

$$w(t) \;=\; 2\left(\frac{\sqrt{|\phi(x)\phi''(x)|}}{|\phi'(x)|}\right)^{\frac{1}{2}+iy}$$

It is straightforward to verify the following identities:

$$\begin{aligned} g'(t) &= \frac{2\,|\xi|}{\eta}\cdot\frac{t}{|\phi'(x)|}\;;\\ g''(t) &= \frac{2\,|\xi|}{\eta}\cdot\frac{1}{|\phi'(x)|}\cdot\left\{1-2\cdot\frac{\phi(x)\phi''(x)}{\phi'(x)^2}\right\}\\ &= \frac{g'(t)}{t}\cdot\left\{1-2\cdot\frac{\phi(x)\phi''(x)}{\phi'(x)^2}\right\}\\ &= \frac{g'(t)}{t}\cdot\big\{1-\mu(t)\big\}, \end{aligned}$$

where

$$\mu(t) \;\equiv\; 2\cdot\frac{\phi(x)\phi''(x)}{\phi'(x)^2}.$$

We have

$$2C_1 \;\le\; \big|\mu(t)\big| \le 2C_2.$$

Also,

$$\begin{aligned} h'(t) &= 2y\cdot\frac{2t}{\eta\phi'(x)}\cdot\frac{\phi''(x)}{\phi'(x)} - y\cdot\frac{2t}{\eta\phi'(x)}\cdot\frac{\phi'(x)}{\phi(x)}\\ &= \frac{4y}{t}\cdot\frac{\phi(x)\phi''(x)}{\phi'(x)^2} - \frac{2y}{t}\;; \end{aligned}$$

$$\begin{aligned} h''(t) &= \frac{2y}{t^2} + 4y\left\{-\frac{1}{t^2}\frac{\phi(x)\phi''(x)}{\phi'(x)^2}\right. \\ &\qquad \left. + \frac{2}{\eta\phi'(x)}\left(\frac{\phi''(x)}{\phi'(x)} + \frac{\phi(x)\phi'''(x)}{\phi'(x)^2} - \frac{2\phi(x)\phi''(x)^2}{\phi'(x)^3}\right)\right\} \\ &= \frac{2y}{t^2}\left\{1 - 2\cdot\frac{\phi(x)\phi''(x)}{\phi'(x)^2}\right. \\ &\qquad \left. + 4\cdot\frac{\phi(x)}{\phi'(x)}\left(\frac{\phi''(x)}{\phi'(x)} + \frac{\phi(x)\phi''(x)}{\phi'(x)^2} - 2\cdot\frac{\phi(x)\phi''(x)^2}{\phi'(x)^3}\right)\right\} \\ &= \frac{2y}{t^2}\left\{1 + 2\cdot\frac{\phi(x)\phi''(x)}{\phi'(x)^2}\right. \\ &\qquad \left. + 4\cdot\frac{\phi(x)\phi''(x)}{\phi'(x)^2}\cdot\frac{\phi(x)\phi'''(x)}{\phi'(x)\phi''(x)} - 8\left(\frac{\phi(x)\phi''(x)}{\phi'(x)^2}\right)^2\right\}. \end{aligned}$$

Conditions 2 and 3 imply

$$\left|h'(t)\right| \le C_3\,|y|\,t^{-1},$$

and

$$\begin{aligned} \left|h''(t)\right| &\le C_3\,|y|\,t^{-2}, \\ C_3 &\equiv 2\left(1 + 2\,C_2 + 12C_2^2\right). \end{aligned}$$

Thus, for $t \in (\alpha,\beta)\cap(\delta,\infty)$

$$\left|h'(t)\right| \le \min\left(C_1, \frac{1}{2}\right)t,$$

and

$$\left|h''(t)\right| \le \min\left(C_1, \frac{1}{2}\right),$$

where

$$\delta \equiv \max\left\{\sqrt{\frac{C_3}{C_1}}\,|y|^{\frac{1}{2}}, \sqrt{2\,C_3}\,|y|^{\frac{1}{2}}\right\}.$$

Hence, by Lemma 2.1,

$$\left|\int_{(\alpha,\beta)\cap(\delta,\infty)\cap(1,\infty)} e^{is(t)}\,dt\right| \le \tilde{C}.$$

Here, $\tilde{C}$ depends only on C_1 and C_2. A trivial estimate on the possibly remaining interval provides us

$$\left| \int_\alpha^\beta e^{is(t)} \, dt \right| \le \tilde{C} + (\delta + 1) \le C'\big(1 + |y|\big)^{\frac{1}{2}},$$

where C' is a constant depending only on C_1 and C_2. The same argument goes through for

$$\left| \int_\alpha^{\beta'} e^{is(t)} \, dt \right| \le C'\big(1 + |y|\big)^{\frac{1}{2}}$$

whenever $\beta' \in (\alpha, \beta]$. Hence, by an integration by parts, we obtain:

$$\begin{aligned} \left| \int_\alpha^\beta e^{is(t)} \, w(t) \, dt \right| &\le C'\big(1 + |y|\big)^{\frac{1}{2}} \left(\sup_{\alpha < t < \beta} w(t) + \|w'\|_{L^1(\alpha,\beta)} \right) \\ &\le 2C'C_2\big(1 + |y|\big)^{\frac{3}{2}} \\ &\le C\big(1 + |y|\big)^{\frac{3}{2}}. \end{aligned}$$

The proof is now finished. □

3. Preliminary estimates

Consider $\gamma : [a, b] \to \mathbb{R}^2$ of the form $\gamma(t) = \big(t, \phi(t)\big)$ with ϕ being a sufficiently smooth function. Let T_{μ_γ} and T_{σ_γ} be defined by

$$T_{\mu_\gamma} f(x) = \int_a^b f\big(x - \gamma(t)\big) \, dt$$

and

$$T_{\sigma_\gamma} f(x) = \int_a^b f\big(x - \gamma(t)\big) \, \big|\phi''(t)\big|^{\frac{1}{3}} \, dt.$$

Then, we have the following:

Lemma 3.1 (Littman). *Suppose there exists a positive constant C such that $\big|\phi''(t)\big| \ge C$ for $a \le t \le b$. Then, T_{μ_γ} maps $L^{\frac{3}{2}}(\mathbb{R}^2)$ boundedly into $L^3(\mathbb{R}^2)$.*

Lemma 3.2. *Suppose there exist constants C_1 and C_2 with $0 < C_1 < C_2 < \infty$ such that*

1. *$\phi''(x)$ never vanishes ;*

2. $C_1\phi'(x)^2 \;\le\; \left|\phi(x)\phi''(x)\right| \;\le\; C_2\phi'(x)^2$;
3. $\left|\phi(x)\phi'''(x)\right| \;\le\; C_2\left|\phi'(x)\phi''(x)\right|$;
4. $\int_a^b \left| \frac{d}{dx} \frac{\sqrt{\phi(x)\phi''(x)}}{\phi'(x)} \right| \, dx \;\le\; C_2.$

Then, T_{σ_γ} maps $L^{\frac{3}{2}}(\mathbb{R}^2)$ boundedly into $L^3(\mathbb{R}^2)$.

Proof of Lemma 3.1 *and Lemma* 3.2. Let $\left\{D_z : -\frac{3}{2} \le \Re\, z \le 0\right\}$ be the analytic family of distributions given by

$$\left\langle D_z, h \right\rangle \;=\; \frac{1}{\Gamma\left(\frac{z+1}{2}\right)} \int_{\mathbb{R}} h(s)\;|s|^z \; ds$$

as in [4]. Consider $\left\{T^z_{\mu_\gamma} : -\frac{3}{2} \le \Re\, z \le 0\right\}$ and $\left\{T^z_{\sigma_\gamma} : -\frac{3}{2} \le \Re\, z \le 0\right\}$ defined by

$$T^z_{\mu_\gamma}\, f\,(x_1, x_2) \;=\; \int_a^b \left\langle D_z, f\,(x_1 - t, x_2 - \phi(t) - s)\,\right\rangle_s \, dt$$

and

$$T^z_{\sigma_\gamma}\, f\,(x_1, x_2) \;=\; \int_a^b \left\langle D_z, f\,(x_1 - t, x_2 - \phi(t) - s)\,\right\rangle_s \left|\phi''(t)\right|^{-\frac{z}{3}} \, dt.$$

Note the inequalities

$$\left\| T^z_{\mu_\gamma} f \right\|_{L^\infty} \;\le\; \frac{1}{\left|\Gamma\left(\frac{1+z}{2}\right)\right|}\; \|f\|_{L^1}$$

and

$$\left\| T^z_{\mu_\gamma} f \right\|_{L^\infty} \;\le\; \frac{1}{\left|\Gamma\left(\frac{1+z}{2}\right)\right|}\; \|f\|_{L^1}\,,$$

when $\Re\, z = 0$. Let $I^y_{\mu_\gamma}$ and $I^y_{\sigma_\gamma}$ be the Fourier multipliers of $T^{-\frac{3}{2}+iy}_{\mu_\gamma}$ and $T^{-\frac{3}{2}+iy}_{\sigma_\gamma}$, respectively. Then,

$$I^y_{\mu_\gamma}\,(\xi, \eta) \;=\; \frac{|\eta|^{\frac{1}{2}-iy}(2\pi)^{\frac{1}{2}-iy}}{\Gamma\left(\frac{3}{4} - \frac{y}{2}i\right)} \int_a^b e^{-2\pi i(\xi t + \eta\phi(t))}\; dt$$

and

$$I^y_{\sigma_\gamma}\,(\xi, \eta) \;=\; \frac{|\eta|^{\frac{1}{2}-i\gamma}(2\pi)^{\frac{1}{2}-iy}}{\Gamma\left(\frac{3}{4} - \frac{y}{2}i\right)} \int_a^b e^{-2\pi i(\xi t + \eta\phi(t))}\; \left|\phi''(t)\right|^{\frac{1}{2}-\frac{\gamma}{3}i}\; dt.$$

By van der Corput's lemma and Proposition 2.2, we obtain

$$\left|I^y_{\mu_\gamma}(\xi,\eta)\right| \le \frac{C}{\left|\Gamma\left(\frac{3}{4}-\frac{y}{2}i\right)\right|}$$

and

$$\left|I^y_{\sigma_\gamma}(\xi,\eta)\right| \le \frac{C\,(1+|y|)^{\frac{3}{2}}}{\left|\Gamma\left(\frac{3}{4}-\frac{y}{2}i\right)\right|},$$

which mean by Plancherel's theorem

$$\left\|T^{-\frac{3}{2}+iy}_{\mu_\gamma} f\right\|_{L^2} \le \frac{C}{\left|\Gamma\left(\frac{3}{4}-\frac{y}{2}i\right)\right|}\ \|f\|_{L^2}$$

and

$$\left\|T^{-\frac{3}{2}+iy}_{\sigma_\gamma} f\right\|_{L^2} \le \frac{C\,(1+|y|)^{\frac{3}{2}}}{\left|\Gamma\left(\frac{3}{4}-\frac{y}{2}i\right)\right|}\ \|f\|_{L^2}.$$

Therefore, Stein's analytic interpolation theorem [9] applies and we obtain the desired $L^{\frac{3}{2}} - L^3$ boundedness of T_{μ_γ} and T_{σ_γ}. □

4. Convolution estimates for finite type plane curves

Based on Lemmas 3.1 and 3.2, we will prove inequality (1.3) for various plane curves. Finite type curves will be discussed in this section and the flat ones will be in the following section. First we give a global example extending inequality (1.2).

PROPOSITION 4.1. *Let $k \in \mathbb{R}$ and $\gamma(t) = \left(t, t^k\right)$, $t > 0$. Then, there exists a constant C such that*

$$\left\|T_{\sigma_\gamma} f\right\|_{L^3(\mathbb{R}^2)} \le C\,\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \tag{4.1}$$

for any $f \in L^{\frac{3}{2}}\left(\mathbb{R}^2\right)$.

Proof of Proposition 4.1. This is an immediate consequence of Lemma 3.2. □

REMARK 4.2. 1. The estimate (4.1) is global.
2. When $k \notin \{-1, 0, 1\}$, (4.1) is the only L^p-L^q estimate.

In the rest of this paper, we focus our interest on local results.

PROPOSITION 4.3. *Let $k > 1$, and suppose that ϕ_1 is a sufficiently smooth function defined near $t = 0$ satisfying $\phi_1^{(j)}(t) = O\left(t^{k-j+\epsilon}\right)$ as $t \to 0+$, $j = 0, 1, 2, 3$, for some $\epsilon > 0$. Let $\gamma(t) = \left(t, t^k + \phi_1(t)\right)$. Then, for some $t_0 > 0$, T_{σ_γ} defined by*

$$T_{\sigma_\gamma} f(x) = \int_0^{t_0} f\left(x - \gamma(t)\right) \lambda(t)\, dt$$

maps $L^{\frac{3}{2}}\left(\mathbb{R}^2\right)$ boundedly into $L^3\left(\mathbb{R}^2\right)$.

Proof of Proposition 4.3. It suffices to show that

$$\int_0^{t_0} \left| \frac{d}{dt} \frac{\phi(t)\phi''(t)}{\phi'(t)^2} \right| dt < \infty$$

for some $t_0 > 0$, where $\phi(t) = t^k + \phi_1(t)$. But,

$$\begin{aligned} \frac{d}{dt} \frac{\phi(t)\phi''(t)}{\phi'(t)^2} &= \frac{\phi'(t)^2\phi''(t) + \phi(t)\phi'(t)\phi'''(t) - 2\phi(t)\phi''(t)^2}{\phi'(t)^2} \\ &= \frac{O\left(t^{3k-4+\epsilon}\right)}{\phi'(t)^3} \\ &= O\left(t^{-1+\epsilon}\right). \end{aligned}$$

Clearly for some $t_0 > 0$, $\frac{d}{dt}\frac{\phi(t)\phi''(t)}{\phi'(t)^2}$ is integrable over the interval $(0, t_0)$ which completes the proof of Proposition 4.3. □

Now, we are in a position to state the following result on finite-type curves.

THEOREM 4.4. *Let γ be a sufficiently smooth compact plane curve of finite-type. Then, the corresponding convolution operator with the affine arclength measure maps $L^{\frac{3}{2}}\left(\mathbb{R}^2\right)$ boundedly into $L^3\left(\mathbb{R}^2\right)$.*

Proof of Theorem 4.4. Compactness implies that there are only finitely many degenerate points on the curve. According to Lemma 3.1, we can assume, without loss of any generality, that γ is degenerate only at $t = 0$ and the interval $(0, t_0)$ is as small as we want. Moreover, by using a linear motion in $\mathbb{R}^2$, we can further assume that γ is given by

$$\gamma(t) = \left(t^k + \gamma_1(t), t^l + \gamma_2(t)\right)$$

for $0 < t < t_0$ with some integers $0 < k < l$. Here, $\gamma_1(t)$ and $\gamma_2(t)$ satisfy

$$\gamma_1^{(j)}(t) = O\left(t^{k-j+1}\right)$$

and

$$\gamma_2^{(j)}(t) = O\left(t^{l-j+1}\right)$$

as $t \to 0+$, for $j = 0, 1, 2, 3$. Making a change of variable

$$s = t^k + \gamma_1(t)$$

in the integral defining $T_{\sigma_\gamma} f$ and writing

$$\phi(s) = t^l + \gamma_2(t)$$

we can bring the operator into the form

$$T' f(x_1, x_2) = \int_0^{s_0} f\big(x_1 - s, x_2 - \phi(s)\big) \left|\phi''(s)\right|^{\frac{1}{3}} ds.$$

Notice that $\phi(s)$ satisfies

$$\phi(s) = s^{\frac{l}{k}} + \phi_1(s)$$

where

$$\phi_1^{(j)}(s) = O\left(s^{\frac{l}{k} - j + \epsilon}\right)$$

as $t \to 0+$, for $j = 0, 1, 2, 3$ and for some positive ϵ. Proposition 4.4 finishes the proof. □

Real analytic plane curves not contained in any straight line is of finite type. So, we have the following:

COROLLARY 4.5. *Let γ be a compact real-analytic plane curve and T_{σ_γ} be the convolution operator associated with the affine arclength measure supported on γ. Then, T_{σ_γ} maps $L^{\frac{3}{2}}(\mathbb{R}^2)$ into $L^3(\mathbb{R}^2)$.*

REMARK 4.6. Notice that the type set of T_{σ_γ} in Proposition 4.3, Theorem 4.4 and Corollary 4.5 is the triangle with vertices at $(0,0)$, $(1,1)$, and $\left(\frac{2}{3}, \frac{1}{3}\right)$.

5. Convolution estimates for flat plane curves

Let $l > 0$ and consider the family of functions $\{\gamma_{kl}\}_{k \in \mathbb{N} \cup \{0\}}$ defined by

$$\gamma_{kl}(t) = \exp\left(-h_k\left(\frac{1}{t^l}\right)\right),$$

where $h_k(t)$ are defined inductively with

$$\begin{aligned} h_0(t) &= t \\ h_{k+1}(t) &= \exp\left(h_k(t)\right), \qquad \text{for } k = 0, 1, \cdots . \end{aligned}$$

Then, we have:

THEOREM 5.1. *For any $t_0 > 0$, $T_{kl}^{t_0}$ given by*

$$T_{kl}^{t_0} f(x_1, x_2) = \int_0^{t_0} f\left(x_1 - t, x_2 - \gamma_{kl}(t)\right) \left|\gamma_{kl}''(t)\right|^{\frac{1}{3}} dt$$

maps $L^{\frac{3}{2}}(\mathbb{R}^2)$ boundedly into $L^3(\mathbb{R}^2)$.

Proof of Theorem 5.1. According to Corollary 4.5, we have only to prove the existence of $t_0 > 0$ such that $T_{kl}^{t_0}$ is bounded from $L^{\frac{3}{2}}(\mathbb{R}^2)$ into $L^3(\mathbb{R}^2)$. We begin with some calculations:

$$\begin{aligned} \gamma_{kl}'(t) &= \gamma_{kl}(t) \cdot h_k'\left(\frac{1}{t^l}\right) \cdot \frac{l}{t^{l+1}} \\ &= \gamma_{kl}(t) \prod_{j=1}^{k} h_j\left(\frac{1}{t^l}\right) \cdot \frac{l}{t^{l+1}} ; \\ \gamma_{kl}''(t) &= \gamma_{kl}'(t)\, h_k'\left(\frac{1}{t^l}\right) \cdot \frac{l}{t^{l+1}} \\ &\quad - \gamma_{kl}(t) \left\{ \sum_{j=1}^{k} h_j'\left(\frac{1}{t^l}\right) \prod_{1 \le j_1 \le k, j_1 \ne j} h_j\left(\frac{1}{t^l}\right) \right\} \cdot \left(\frac{l}{t^{l+1}}\right)^2 \\ &\quad - \gamma_{kl}(t) \cdot \frac{l(l+1)}{t^{l+2}} \prod_{j=1}^{k} h_j\left(\frac{1}{t^l}\right) \\ &= \gamma_{kl}(t) \left\{ \left(\prod_{j=1}^{k} h_j\left(\frac{1}{t^l}\right) \cdot \frac{l}{t^{l+1}} \right)^2 \right. \\ &\qquad - \sum_{j=1}^{k} \prod_{j_1=1}^{j} h_{j_1}\left(\frac{1}{t^l}\right) \prod_{1 \le j_1 \le k, j_1 \ne j} h_{j_1}\left(\frac{1}{t^l}\right) \left(\frac{l}{t^{l+1}}\right)^2 \\ &\qquad \left. - \frac{l(l+1)}{t^{l+2}} \prod_{j=1}^{k} h_j\left(\frac{1}{t^l}\right) \right\} \end{aligned}$$

$$\sim \frac{\gamma'_{kl}(t)^2}{\gamma_{kl}(t)} ;$$

$$\gamma'''_{kl}(t) \sim \frac{\gamma'_{kl}(t)^3}{\gamma_{kl}(t)^2}$$

as $t \to 0+$.

Thus, there exist $t_0 > 0$, C_1 and C_2 with $0 < C_1$, $C_2 < \infty$ such that for $0 < t < t_0$ we have

1. $\gamma_{kl}(t) > 0$, $\gamma'_{kl}(t) > 0$, $\gamma''_{kl}(t) > 0$;
2. $C_1 \leq \frac{\gamma_{kl}(t)\ \gamma''_{kl}(t)}{\gamma'_{kl}(t)^2} \leq C_2$; and
3. $\left|\frac{\gamma_{kl}(t)\ \gamma'''_{kl}(t)}{\gamma'_{kl}(t)\ \gamma''_{kl}(t)}\right| \leq C_2$.

According to Proposition 3.2, it therefore suffices to show that $\frac{\gamma_{kl}(t)\gamma''_{kl}(t)}{\gamma'_{kl}(t)^2}$ is monotonic. But,

$$\frac{\gamma_{kl}(t)\ \gamma''_{kl}(t)}{\gamma'_{kl}(t)^2} = 1 - \sum_{j=1}^{k} \frac{1}{\prod_{j_1=j+1}^{k} h_{j_1}\left(\frac{1}{t^l}\right)} \cdot \frac{1}{h_j\left(\frac{1}{t^l}\right)} - \frac{l+1}{l} \cdot \frac{t^l}{\prod_{j=1}^{k} h_j\left(\frac{1}{t^l}\right)}$$

is decreasing. This finishes the proof of Theorem 5.1. □

REMARK 5.2. The type set for $T_{kl}^{t_0}$ is now determined to be the triangle with vertices at $(0,0)$, $(1,1)$ and $\left(\frac{2}{3}, \frac{1}{3}\right)$.

References

[1] M. Christ, *Endpoint bounds for singular fractional integral operators,* Preprint, 1988.

[2] S. W. Drury, *Degenerate curves and harmonic analysis,* Math. Proc. Camb. Phil. Soc. **108** (1990), 89–96.

[3] S. W. Drury and K. Guo, *Convolution estimates related to surfaces of half the ambient dimension,* Math. Proc. Camb. Phil. Soc. **110** (1991), 151–159.

[4] I. M. Gelfand and G. E. Shilov, *Generalized functions, Vol.* I, *Properties and operations,* Academic Press, New York-London, 1964.

[5] H. W. Guggenheimer, *Differential geometry,* Dover Publications Inc., New York, 1977.

[6] W. Littman, *L^p-L^q-estimates for singular integral operators arising from hyperbolic equations,* In Proc. Sympos. Pure Math. XXIII, 479–481, Amer. Math. Soc., Providence, RI, 1973.

[7] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals,* III, *Fractional integration along manifolds,* J. Funct. Anal. **86** (1989), 360–389.

[8] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, 1993.
[9] E. M. Stein and G. Weiss, *Introduction to Fourier analysis in Euclidean spaces*, Princeton Univ. Press, Princeton, NJ, 1971.

Department of Mathematics
Ajou University
Suwon 442-749, Korea
E-mail: youngwoo@madang.ajou.ac.kr