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FINITE ELEMENT APPROXIMATION AND COMPUTATIONS OF BOUNDARY OPTIMAL CONTROL PROBLEMS FOR THE NAVIER-STOKES FLOWS THROUGH A CHANNEL WITH STEPS

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ABSTRACT. We study a boundary optimal control problem of the fluid flow governed by the Navier-Stokes equations. The control problem is formulated with the flow through a channel with steps. The first-order optimality condition of the optimal control is derived. Finite element approximations of the solutions of the optimality system are defined and optimal error estimates are derived. Finally, we present some numerical results.

1. Introduction

In past years, there has been an increased interest in mathematical analyses and computation of control problems for the Navier-Stokes equations: see [5], [9], [10], [11] and the references therein.

In this paper we consider the minimization of vorticity in viscous, incompressible flow. The control problem, which is formulated in Section 2, involves finite-dimensional control input acting through a part of the boundary as Dirichlet boundary control. Thus we consider the two-dimensional motion of fluid modeled by the (stationary) Navier-Stokes equation,

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

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$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

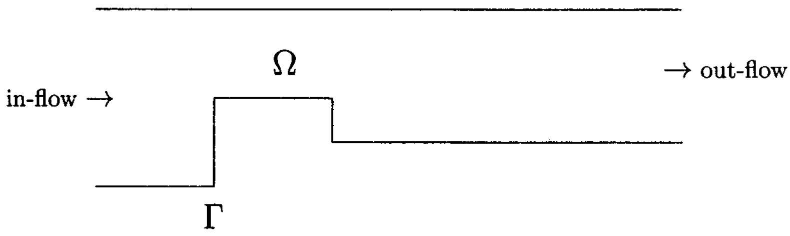

FIGURE 1. channel flow

confined in a channel Ω, depicted in Figure 1. Here $\mathbf{u} = (u_1, u_2)$ is the velocity field, p the pressure, ν the kinematic viscosity of the fluid ($\nu = 1/Re$, where Re is the Reynolds number), and $\mathbf{f}$ the density of external forces (in our example, $\mathbf{f} = \mathbf{0}$). The nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in (1.1) (often called the convective term), is a symbolic notation for the vector

$$\left(u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2}, u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} \right).$$

The divergence-free condition in (1.2) is the equation for law of conservation of mass of incompressible flow.

The paper is organized as follows. In the remainder of this section, we introduce the notation that will be used throughout the paper. Then, in §2 the optimal control problem is described and the existence and first-order optimality condition for the optimal control problem are established. In §3 we consider finite element approximations and derive error estimates. Finally we present numerical results in §4.

Now we collect some notations, definitions and introduce basic theory of the Navier-Stokes equations that we need for our discussion. Let R^n denote the n-dimensional Euclidean space and Ω be a bounded domain of R^n with a Lipschitz continuous boundary Γ. Let $L^2(\Omega)$ be the space of real-valued square integrable functions defined on Ω, and let $\|\cdot\|_{L^2}$ be the norm in this space. We define the Sobolev space $H^m(\Omega)$ for the nonnegative integer m by

$$H^m(\Omega) := \{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega), \text{ for } 0 \le |\alpha| \le m\}$$

where D^α is the weak (or distributional) partial derivative, and α is a multi-index. The norm $\|\cdot\|_m$ associated with $H^m(\Omega)$ given by

$$\|u\|_m^2 = \left\{ \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2}^2 \right\}.$$

Note that $H^0(\Omega) = L^2(\Omega)$ and $\|\cdot\|_0 = \|\cdot\|_{L^2}$. For the vector-valued functions, we define the Sobolev space $\mathbf{H}^m(\Omega)$ (in all cases, boldface indicates vector-valued) by

$$\mathbf{H}^m(\Omega) := \{\mathbf{u} = (u_1, u_2) \mid u_i \in H^m(\Omega), \text{ for } i = 1, 2\},$$

and its associated norm $\|\cdot\|_m$ is given by

$$\|\mathbf{u}\|_m^2 = \sum_{i=1}^{2} \|u_i\|_m^2.$$

We also define particular subspaces:

$$\begin{aligned} L_0^2(\Omega) &= \left\{ f \in L^2(\Omega) : \int_\Omega f dx = 0 \right\}, \\ \mathbf{H}_0^2(\Omega) &= \{\mathbf{u} \in \mathbf{H}^2(\Omega) : \mathbf{u} = \mathbf{0} \text{ on } \Gamma\}. \end{aligned}$$

The matrix $\{\partial u_i / \partial x_j\}$ will be denoted $\nabla \mathbf{u}$. Let $a(\mathbf{u}, \mathbf{v}) : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to R$ be the symmetric bilinear form defined by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} dx, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{1.3}$$

where : means the sum of the product of same components of the matrices. Let $b(\mathbf{u}, p) : \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \to R$ be the bilinear form defined by

$$b(\mathbf{u}, p) = -\int_\Omega (\nabla \cdot \mathbf{u}) p dx, \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega), \forall p \in L_0^2(\Omega). \tag{1.4}$$

The trilinear form c on $[\mathbf{H}^1(\Omega)]^3$ that corresponds to the convective term is defined by

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} dx, \qquad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega). \tag{1.5}$$

Also these forms induce the operators

$$A : \mathbf{H}^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$$

defined by

$$\langle A\mathbf{u},\mathbf{v}\rangle = a(\mathbf{u},\mathbf{v}), \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega), \mathbf{v} \in \mathbf{H}^1_0(\Omega),$$

$$B : \mathbf{H}^1(\Omega) \to L^2_0(\Omega)$$

defined by

$$\langle B\mathbf{u},p\rangle = b(\mathbf{u},p), \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega), p \in L^2_0(\Omega),$$

$$\bar{B} : L^2_0(\Omega) \to \mathbf{H}^{-1}(\Omega)$$

defined by

$$\langle \bar{B}p,\mathbf{u}\rangle = b(\mathbf{u},p), \quad \forall \mathbf{u} \in \mathbf{H}^1_0(\Omega), p \in L^2_0(\Omega),$$

and

$$C : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$$

defined by

$$\langle C(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle = c(\mathbf{u};\mathbf{v},\mathbf{w}), \quad \forall \mathbf{u},\mathbf{v}, \in \mathbf{H}^1(\Omega), \mathbf{w} \in \mathbf{H}^1_0(\Omega).$$

These forms are continuous in the sense that there exist constants C_a, C_b and $C_c > 0$ such that

$$|a(\mathbf{u},\mathbf{v})| \leq C_a\|\mathbf{u}\|_1\|\mathbf{v}\|_1, \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega), \tag{1.6}$$

$$|b(\mathbf{v},q)| \leq C_b\|\mathbf{v}\|_1\|q\|_0, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \forall q \in L^2_0(\Omega), \tag{1.7}$$

and

$$|c(\mathbf{u},\mathbf{v},\mathbf{w})| \leq C_c\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1, \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega). \tag{1.8}$$

Moreover, we have the coercivity properties

$$|a(\mathbf{v},\mathbf{v})| \geq c_a\|\mathbf{v}\|_1^2, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{1.9}$$

and

$$\sup_{\mathbf{0}\neq\mathbf{u}\in\mathbf{H}^1_0(\Omega)} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1} \geq c_b\|q\|_0, \quad \forall q \in L^2_0(\Omega) \tag{1.10}$$

for some constants c_a and $c_b > 0$.

Using our notation, the weak form of the Navier-Stokes equations (1.1)-(1.2) is given as follows:

$$\nu a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + c(\mathbf{u},\mathbf{u},\mathbf{v}) = \langle \mathbf{f},\mathbf{v}\rangle, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega) \tag{1.11}$$

$$b(\mathbf{u},q) = 0, \quad \forall q \in L^2_0(\Omega) \tag{1.12}$$

with Dirichlet boundary condition

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma, \tag{1.13}$$

where $\mathbf{g}$ satisfies $\int_\Gamma \mathbf{g} \cdot \mathbf{n} ds = 0$ since Green's formula

$$\int_\Omega \nabla \cdot \mathbf{u} d\Omega = \int_\Gamma \mathbf{g} \cdot \mathbf{n} ds = 0.$$

Here $\mathbf{n}$ is the outward unit normal vector.

2. Optimal control problem and the optimality system

We assume that the in-flow (at $x = 0$) as boundary condition, is parabolic (Poiseuille flow assumption) with $u_{in} = 4x_2(1 - x_2)$. At the out flow boundary we impose the stress free boundary condition. This channel flow will happens recirculations in both corners, one in front of the forward step and the other right after backward step, and a bubble as soon as the channel narrow. Figure 2 qualitatively illustrate the flow in the channel with $Re = 100$.

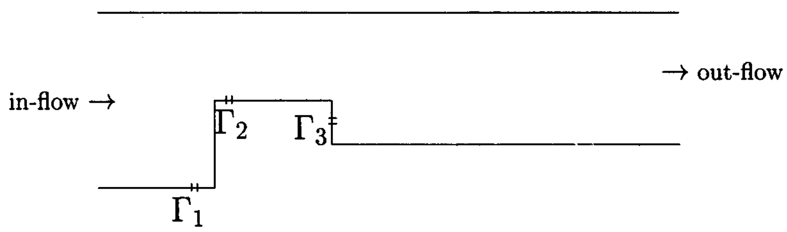

FIGURE 2. channel flow

Our objective is to shape the flow to a regular configuration by means of controlled injection (or suction) along a portion Γ_1 (the horizontal boundary facing the recirculation flow), Γ_2 (facing bubble flow) and Γ_3 (the vertical boundary facing the recirculating flow) (see Figure 2). The regular flow means that the flow has a little recirculation and a little bubble. To do this, we consider the following cost functional corresponding to the total vorticity in the flow given by

$$\mathcal{J}(\mathbf{u}) \doteq \int_\Omega \left| \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right|^2 d\Omega, \tag{2.14}$$

where the vorticity

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

This cost functional is motivated by the fact that potential flows (zero vorticity) are frictionless and incur low energy dissipation.

Now we formulate the optimal control problem as a finite dimensional constrained minimization in a Hilbert space:

$$\begin{aligned}
&\text{Find} && (\mathbf{u}, p, \mathbf{t}) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) \times U \text{ which}\\
&\text{minimizes} && \mathcal{J}(\mathbf{u})\\
(2.15)\ &\text{subject to} && \nu a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega),\\
& && b(\mathbf{u}, q) = 0, \quad \forall q \in L^2_0(\Omega)\\
& && \mathbf{u} = \mathbf{g}_0 + \sum_{i=1}^m t_i \mathbf{g}_i, \quad \text{on } \Gamma,
\end{aligned}$$

where $\mathbf{g}_i \in \mathbf{H}^{1/2}(\Gamma)$ with $\int_\Gamma \mathbf{g}_i \cdot \mathbf{n} ds = 0$ and U is a closed bounded region in R^m. We discuss the Dirichlet boundary control problem and thus the body force is discarded. The function $\mathbf{t} \cdot \mathbf{g} = \sum_{i=1}^m t_i \mathbf{g}_i, \mathbf{t} \in U$ is the control input and influences the equation only through a part of boundary Γ, and the functions $\mathbf{g}_i$ represent distribution functions of control input at Γ. In our example without loss of generality we assume that $U = [-1, 1]^m$.

Let $\mathbf{u} = \mathbf{w} + \bar{\mathbf{u}}^{(0)} + \sum_{i=1}^m t_i \bar{\mathbf{u}}^{(i)}$ with $\mathbf{w} \in \mathbf{H}^1_0(\Omega)$, where $\bar{\mathbf{u}}^{(i)}, 0 \le i \le m$ are the solution of the Stokes equations

$$a(\bar{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, p) = 0 \quad \text{and} \quad b(\bar{\mathbf{u}}, q) = 0, \tag{2.16}$$

for all $(\mathbf{v}, q) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$ with the boundary condition $\bar{\mathbf{u}}^{(i)} = \mathbf{g}_i, 0 \le i \le m$ on Γ, respectively. Note that $\bar{\mathbf{u}}^{(i)}$ is unique [6]. Then the problem (2.15) can be equivalently written as

$$\begin{aligned}
&\text{Find} && (\mathbf{w}, p, \mathbf{t}) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega) \times U \text{ which}\\
&\text{minimizes} && \mathcal{J}(\mathbf{u})\\
(2.17)\ &\text{subject to} && \nu a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega)\\
& && b(\mathbf{w}, q) = 0, \quad \forall q \in L^2_0(\Omega),
\end{aligned}$$

where $\mathbf{u} = \mathbf{w} + \bar{\mathbf{u}}^{(0)} + \sum_{i=1}^{m} t_i \bar{\mathbf{u}}^{(i)}$. In what follows we identify $\mathbf{u}$ with the pair $(\mathbf{w}, \mathbf{t})$ whenever $\mathbf{u} = \mathbf{w} + \bar{\mathbf{u}}^{(0)} + \mathbf{t} \cdot \bar{\mathbf{u}}$, where $\bar{\mathbf{u}} = \mathrm{col}(\bar{\mathbf{u}}^{(1)}, \cdots, \bar{\mathbf{u}}^{(m)})$.

DEFINITION 2.1. *We define the admissibility set*

$$\begin{aligned}\mathcal{U}_{ad} &= \{\mathbf{u} := (\mathbf{w}, \mathbf{t}) \in \mathbf{H}_0^1(\Omega) \times U \mid \mathbf{u} = \mathbf{w} + \bar{\mathbf{u}}^{(0)} + \sum_{i=1}^{m} t_i \bar{\mathbf{u}}^{(i)}, \text{ and} \\ &\qquad \text{there exists } p \in L_0^2(\Omega) \text{ and so that } \mathbf{u} \text{ satisfies the } (2.17)\}.\end{aligned}$$

DEFINITION 2.2. *We define an optimal solution* $(\mathbf{w}^*, \mathbf{t}^*)$ *to be one for which* $\mathcal{J}(\mathbf{u}^*) \leq \mathcal{J}(\mathbf{u})$ *for any* $(\mathbf{w}, \mathbf{t}) \in \mathcal{U}_{ad}$.

Now we state the existence result which can be found in the literature, for example, [5] and [10].

PROPOSITION 2.3. *There exists an optimal solution* $(\mathbf{w}^*, \mathbf{t}^*)$ *for our control problem.*

Assume that $\mathbf{u} := (\mathbf{w}, \mathbf{t}) \in \mathbf{H}_0^1(\Omega) \times U$ is a local solution of (2.17) and that $\mathbf{t} \in \mathrm{int}(U)$. Let $\mathcal{G} : \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U \to \mathbf{H}_0^{-1}(\Omega) \times L_0^2(\Omega)$ be defined as follows: $\mathcal{G}(\mathbf{w}, p, \mathbf{t}) = (\bar{\mathbf{f}}, z)$ if and only if

$$\begin{cases} \nu a(\mathbf{w}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \bar{\mathbf{f}}, \mathbf{v} \rangle, & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ b(\mathbf{w}, q) = \langle z, q \rangle, & \forall q \in L_0^2(\Omega), \end{cases}$$

where $\langle \bar{\mathbf{f}}, \mathbf{v} \rangle = -a(\mathbf{z}, \mathbf{v})$ and $\mathbf{z} = \bar{\mathbf{u}}^{(0)} + \mathbf{t} \cdot \bar{\mathbf{u}}$. Thus the constraints (2.17) can be expressed as $\mathcal{G}(\mathbf{w}, p, \mathbf{t}) = (\bar{\mathbf{f}}, \mathbf{0})$.

Given $\mathbf{u} \in \mathbf{H}^1(\Omega)$, the operator $\mathcal{G}'(\mathbf{u}) \in \mathcal{L}(\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U; \mathbf{H}_0^{-1}(\Omega) \times L_0^2(\Omega))$ may be defined as follows: $\mathcal{G}'(\mathbf{u})(\mathbf{y}, r, \mathbf{s}) = (\tilde{\mathbf{f}}, \tilde{z})$ if and only if

$$\begin{cases} \nu a(\mathbf{y}, \mathbf{v}) + c(\mathbf{y}, \mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{y}, \mathbf{v}) + \mathbf{s} \cdot (c(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \bar{\mathbf{u}}, \mathbf{v})) \\ \qquad\qquad + b(\mathbf{v}, r) = \langle \tilde{\mathbf{f}}, \mathbf{v} \rangle - \nu \mathbf{s} \cdot a(\bar{\mathbf{u}}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ b(\mathbf{y}, q) = \langle \tilde{z}, q \rangle, \qquad \forall q \in L_0^2(\Omega), \end{cases}$$

where $\langle \tilde{\mathbf{f}}, \mathbf{v} \rangle = -a(\bar{\mathbf{u}}^{(0)}, \mathbf{v})$. It follows from [13] that if the $\mathcal{G}'(\mathbf{u})$, which is the Fréchet derivative of $\mathcal{G}$ at $(\mathbf{w}, \mathbf{t})$ is surjective, then the regular point condition is satisfied and hence there exists a Lagrange multiplier $(\mathbf{d}, \phi) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ satisfying the Euler equations

$$\mathcal{J}'(\mathbf{u})(\mathbf{y}, r, \mathbf{s}) + \langle \mathcal{G}'(\mathbf{u})(\mathbf{y}, r, \mathbf{s}), (\mathbf{d}, \phi) \rangle = 0,$$

for all $(\mathbf{y}, r, \mathbf{s}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U$.

But, by the similar technique in [5] and [9], we can easily have the following results.

PROPOSITION 2.4. *For* $\mathbf{u} \in \mathbf{H}^1(\Omega)$, *the operator* $\mathcal{G}'(\mathbf{u})$ *from* $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times U$ *into* $\mathbf{H}_0^{-1}(\Omega) \times L_0^2(\Omega)$ *is surjective.*

Now, we obtain the first-order necessary conditions for optimality:

$$
\left\{
\begin{array}{ll}
\nu a(\mathbf{w}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \bar{\mathbf{f}}, \mathbf{v} \rangle, & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\
b(\mathbf{w}, q) = 0, \qquad \forall q \in L_0^2(\Omega), &
\end{array}
\right.
\tag{2.18}
$$

$$
\left\{
\begin{array}{l}
\nu a(\mathbf{d}, \mathbf{e}) + c(\mathbf{e}, \mathbf{u}, \mathbf{d}) + c(\mathbf{u}, \mathbf{e}, \mathbf{d}) + b(\mathbf{e}, \phi) \\
\qquad\qquad\qquad + \langle \nabla \times (\nabla \times \mathbf{u}), \mathbf{e} \rangle = 0, \quad \forall \mathbf{e} \in \mathbf{H}_0^1(\Omega), \\
b(\mathbf{d}, r) = 0, \qquad \forall r \in L_0^2(\Omega),
\end{array}
\right.
\tag{2.19}
$$

$$
\left\{
\begin{array}{l}
\nu a(\bar{\mathbf{u}}^{(i)}, \mathbf{d}) + c(\bar{\mathbf{u}}^{(i)}, \mathbf{u}, \mathbf{d}) + c(\mathbf{u}, \bar{\mathbf{u}}^{(i)}, \mathbf{d}) \\
\qquad\qquad\qquad + \langle \nabla \times (\nabla \times \mathbf{u}), \bar{\mathbf{u}}^{(i)} \rangle = 0, \quad i = 1, \ldots, m.
\end{array}
\right.
\tag{2.20}
$$

3. Finite element approximations

In this section we investigate a finite element discretization of the optimality system and an estimation of the approximation error. First, we choose a family of finite dimensional subspaces $\mathbf{V}^h \subset \mathbf{H}^1(\Omega)$, $S^h \subset L^2(\Omega)$. We let $\mathbf{V}_0^h = \mathbf{V}^h \cap \mathbf{H}_0^1(\Omega)$ and $S_0^h = S^h \cap L_0^2(\Omega)$.

We may choose any pair of subspaces $\mathbf{V}^h$ and S^h that can be used for finding finite element approximations of solutions of the Navier-Stokes equations. Thus concerning these subspaces we make the following standard assumptions which are exactly those employed in well-known finite element methods for the Navier-Stokes equations. First, we have the approximation properties: there exist an integer k and a constant C, independent of h, $\mathbf{v}$ and q, such that for each $m = 1, 2, \ldots, k$,

$$
\inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{v} - \mathbf{v}^h\|_1 \le C h^m \|\mathbf{v}\|_{m+1}, \quad \forall \mathbf{v} \in \mathbf{H}^{m+1}(\Omega),
\tag{3.21}
$$

and

$$
\inf_{q^h \in S_0^h} \|q - q^h\|_0 \le C h^m \|q\|_m, \quad \forall q \in H^m(\Omega) \cap L_0^2(\Omega),
\tag{3.22}
$$

next, we assume the *inf-sup condition*, or *Ladyzhenskaya-Babuska-Brezzi condition*: there exists a constant C, independent of h, such that

$$\inf_{0\neq q^h\in S_0^h}\ \sup_{\mathbf{0}\neq\mathbf{v}^h\in\mathbf{V}^h}\frac{b(\mathbf{v}^h,q^h)}{\|\mathbf{v}^h\|_1\|q^h\|_0}\ \geq\ C. \tag{3.23}$$

This condition assures the stability of finite element discretizations of the Navier-Stokes equations.

Once the approximating subspaces have been chosen, we seek $\mathbf{u}^h\in\mathbf{V}^h$, $p^h\in S_0^h$, $\mathbf{d}^h\in\mathbf{V}^h$, and $\phi^h\in S_0^h$ such that

$$\nu a(\mathbf{w}^h,\mathbf{v}^h)+c(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h)+b(\mathbf{v}^h,p^h)=\langle\bar{\mathbf{f}},\mathbf{v}^h\rangle,\quad\forall\mathbf{v}^h\in\mathbf{V}^h, \tag{3.24}$$

$$b(\mathbf{w}^h,q^h)=0,\qquad\forall q^h\in S_0^h, \tag{3.25}$$

$$\begin{aligned}\nu a(\mathbf{d}^h,\mathbf{e}^h)+c(\mathbf{e}^h,\mathbf{u}^h,\mathbf{d}^h)+c(\mathbf{u}^h,\mathbf{e}^h,\mathbf{d}^h)+b(\mathbf{e}^h,\phi^h)&\\ +\langle\nabla\times(\nabla\times\mathbf{u}),\mathbf{e}\rangle=0,\quad\forall\mathbf{e}^h\in\mathbf{V}^h,&\end{aligned} \tag{3.26}$$

$$b(\mathbf{d}^h,r^h)=0,\qquad\forall r^h\in S_0^h, \tag{3.27}$$

$$\begin{aligned}\nu a(\bar{\mathbf{u}}^{(i)},\mathbf{d}^h)+c(\bar{\mathbf{u}}^{(i)},\mathbf{u}^h,\mathbf{d}^h)+c(\mathbf{u}^h,\bar{\mathbf{u}}^{(i)},\mathbf{d}^h)&\\ +\langle\nabla\times(\nabla\times\mathbf{u}),\bar{\mathbf{u}}^{(i)}\rangle=0,\quad i=1,\ldots,m.&\end{aligned} \tag{3.28}$$

We concern ourselves with questions related to the accuracy of finite element approximations in this section. The error estimate makes use of the results of [3] and [6] concerning the approximation of a class of nonlinear problems. Here for the sake of completeness, we will state the relevant results specialized to our needs.

The nonlinear problems considered in [3] and [6] are of the type

$$F(\lambda,\psi)\equiv\psi+T\,G(\lambda,\psi)=0 \tag{3.29}$$

where $T\in\mathcal{L}(Y;X)$, G is a C^2 mapping from $\Lambda\times X$ into Y, where X and Y are Banach spaces and Λ is a compact interval of R. We say that $\{(\lambda,\psi(\lambda)):\lambda\in\Lambda\}$ is a branch of solutions of (3.29) if $\lambda\to\psi(\lambda)$ is a continuous function from Λ into X such that $F(\lambda,\psi(\lambda))=0$. The branch is called a nonsingular branch if we also have that $D_\psi F(\lambda,\psi(\lambda))$ is an isomorphism from X into X for all $\lambda\in\Lambda$. ($D_\psi F(\cdot,\cdot)$ denotes the Frechet derivative of $F(\cdot,\cdot)$ with respect to the second argument).

Approximations are defined by introducing a subspace $X^h\subset X$ and an approximating operator $T^h\in\mathcal{L}(Y;X^h)$. Then, we seek $\psi^h\in X^h$ such that

$$F^h(\lambda,\psi^h)\equiv\psi^h+T^hG(\lambda,\psi^h)=0. \tag{3.30}$$

We will assume that there exists another Banach space Z, contained in Y, with continuous imbedding, such that

$$D_\psi G(\lambda, \psi) \in \mathcal{L}(X; Z), \quad \forall \lambda \in \Lambda \text{ and } \forall \psi \in X. \tag{3.31}$$

Concerning the operator T^h, we assume the approximation properties

$$\lim_{h \to 0} \|(T^h - T)y\|_X = 0, \qquad \forall y \in Y \tag{3.32}$$

and

$$\lim_{h \to 0} \|T^h - T\|_{\mathcal{L}(Z;X)} = 0. \tag{3.33}$$

Note that (3.31) and (3.33) imply that the operator $D_\psi G(\lambda, \psi) \in \mathcal{L}(X; X)$ is compact. Moreover, (3.33) follows from (3.32) whenever the imbedding $Z \subset Y$ is compact.

Now we can state the first result of [3] and [6] that used in the sequel.

THEOREM 3.1. *Let X and Y be Banach spaces and Λ a compact subset of R. Assume that G is a C^2 mapping from $\Lambda \times X$ into Y and that D^2G is bounded on all bounded sets of $\Lambda \times X$. (D^2G represents second Fréchet derivative of G) Assume that* (3.31)-(3.33) *hold and $\{(\lambda, \psi(\lambda)); \lambda \in \Lambda\}$ is a branch of nonsingular solutions of* (3.29). *Then, there exists a neighborhood $\mathcal{O}$ of the origin in X and for $h \le h_0$ small enough, a unique C^2 function $\lambda \in \Lambda \to \psi^h(\lambda) \in X^h$ such that $\{(\lambda, \psi^h(\lambda)); \lambda \in \Lambda\}$ is a branch of nonsingular solutions of* (3.30) *and $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$. Moreover, there exists a constant $C > 0$, independent of h and λ, such that*

$$\|\psi^h(\lambda) - \psi(\lambda)\|_X \le C \|(T^h - T)G(\lambda, \psi(\lambda))\|_X, \quad \forall \lambda \in \Lambda. \tag{3.34}$$

For the second result, we have to introduce two other Banach spaces H and W, such that $W \subset X \subset H$, with continuous imbeddings, and assume that

$$\begin{aligned} &\text{for all } w \in W, \text{ the operator } D_\psi G(\lambda, w) \text{ may be} \\ &\text{extended as a linear operator of } \mathcal{L}(H; Y), \end{aligned} \tag{3.35}$$

and the mapping $w \to D_\psi G(\lambda, w)$ is continuous from W onto $\mathcal{L}(H; Y)$.

We also suppose that

$$\lim_{h \to 0} \|T^h - T\|_{\mathcal{L}(Y;H)} = 0. \tag{3.36}$$

Then we may state the following additional result.

THEOREM 3.2. *Assume the hypotheses of* Theorem 3.1 *and also assume that* (3.35) *and* (3.36) *hold. Assume in addition that*

$$\text{(3.37)}\qquad \begin{array}{c}\text{for each } \lambda \in \Lambda, \psi(\lambda) \in W \text{ and the function}\\ \lambda \to \psi(\lambda) \text{ is continuous from } \Lambda \text{ into } W\end{array}$$

and

$$\text{(3.38)}\quad \text{for each } \lambda \in \Lambda, D_\psi F(\lambda, \psi(\lambda)) \text{ is an isomorphism of } H.$$

Then, for $h \le h_1$ sufficiently small, there exists a constant C, independent of h and λ, such that

$$\text{(3.39)}\qquad \begin{aligned}\|\psi^h(\lambda) - \psi(\lambda)\|_H \;&\le\; C\|(T^h - T)G(\lambda, \psi(\lambda))\|_H \\ &\qquad + \|\psi^h(\lambda) - \psi(\lambda)\|_X^2, \quad \forall \lambda \in \Lambda.\end{aligned}$$

We begin by recasting the optimality system (2.18)-(2.20) and its discretization (3.24)-(3.28) into a form that fits into the framework. Let $\lambda = 1/\nu$; thus λ is the Reynolds number. Let

$$\begin{aligned}&X = \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^1(\Omega) \times L_0^2(\Omega),\\ &Y = \mathbf{H}^{-1}(\Omega) \times \mathbf{H}^{-1}(\Omega),\\ &Z = L^{3/2}(\Omega) \times L^{3/2}(\Omega),\\ &X^h = \mathbf{V}^h \times S_0^h \times \mathbf{V}^h \times S_0^h.\end{aligned}$$

Note that $Z \subset Y$ with a compact imbedding.

Let the operator $T \in \mathcal{L}(Y : X)$ be defined as the following: $T(\zeta, \eta) = (\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{d}}, \tilde{\phi})$ for $(\zeta, \eta) \in Y$ and $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{d}}, \tilde{\phi}) \in X$ if and only if

$$\text{(3.40)}\qquad a(\tilde{\mathbf{w}}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}) = \langle \zeta, \mathbf{v}\rangle, \ \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

$$\text{(3.41)}\qquad b(\tilde{\mathbf{w}}, q) = 0, \qquad \forall q \in L_0^2(\Omega),$$

$$\text{(3.42)}\qquad a(\tilde{\mathbf{d}}, \mathbf{e}) + b(\mathbf{e}, \tilde{\phi}) = \langle \eta, \mathbf{e}\rangle, \ \forall \mathbf{e} \in \mathbf{H}^1(\Omega),$$

$$\text{(3.43)}\qquad b(\tilde{\mathbf{d}}, r) = 0, \qquad \forall r \in L_0^2(\Omega),$$

$$\text{(3.44)}\qquad a(\bar{\mathbf{u}}^{(i)}, \tilde{\mathbf{d}}) = \langle \eta, \tilde{\mathbf{d}}\rangle, \quad 1 \le i \le m.$$

Note that this system is weakly coupled. Analogously, the operator $T^h \in \mathcal{L}(Y : X^h)$ is defined as follows: $T^h(\zeta, \eta) = (\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{\mathbf{d}}^h, \tilde{\phi}^h)$ for

$(\zeta,\eta) \in Y$ and $(\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{\mathbf{d}}^h, \tilde{\phi}^h) \in X^h$ if and only if

$$a(\tilde{\mathbf{w}}^h, \mathbf{v}^h) + b(\mathbf{v}^h, \tilde{p}^h) = \langle \zeta^h, \mathbf{v}^h \rangle, \ \forall \mathbf{v}^h \in \mathbf{V}^h, \tag{3.45}$$

$$b(\tilde{\mathbf{w}}^h, q^h) = 0, \qquad \forall q^h \in S_0^h(\Omega), \tag{3.46}$$

$$a(\tilde{\mathbf{d}}^h, \mathbf{e}^h) + b(\mathbf{e}^h, \tilde{\phi}^h) = \langle \eta^h, \mathbf{e}^h \rangle, \ \forall \mathbf{e}^h \in \mathbf{V}^h, \tag{3.47}$$

$$b(\tilde{\mathbf{d}}^h, r^h) = 0, \qquad \forall r^h \in S_0^h(\Omega), \tag{3.48}$$

$$a(\bar{\mathbf{u}}^{(i)}, \tilde{\mathbf{d}}^h) = \langle \eta^h, \tilde{\mathbf{d}}^h \rangle, \quad 1 \le i \le m. \tag{3.49}$$

This system is weakly coupled as the system (3.40)-(3.44).

Let Λ denote a compact subset of R. Next we define the nonlinear mapping $G : \Lambda \times X \to Y$ as follows: $G(\lambda, (\mathbf{u}, p, \mathbf{d}, \phi)) = (\zeta, \eta)$ for $\lambda \in \Lambda$, $(\mathbf{u}, p, \mathbf{d}, \phi) \in X$ and $(\zeta, \eta) \in Y$ if and only if

$$\langle \zeta, \mathbf{v} \rangle = \lambda\, c(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda\, \langle \bar{\mathbf{f}}, \mathbf{v} \rangle, \ \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

$$\langle \eta, \mathbf{e} \rangle = -\lambda\, c(\mathbf{e}, \mathbf{u}, \mathbf{d}) - \lambda\, c(\mathbf{u}, \mathbf{e}, \mathbf{d}) - \langle \nabla \times (\nabla \times \mathbf{u}), \mathbf{e} \rangle, \ \forall \mathbf{e} \in \mathbf{H}^1(\Omega).$$

It is easily seen that the optimality system (2.18)-(2.20) is equivalent to

$$(\mathbf{u}, \lambda\, p, \mathbf{d}, \lambda\, \phi) + T\, G(\lambda, (\mathbf{u}, \lambda\, p, \mathbf{d}, \lambda\, \phi)) = 0 \tag{3.50}$$

and that the discrete optimality system (3.24)-(3.28) is equivalent to

$$(\mathbf{u}^h, \lambda\, p^h, \mathbf{d}^h, \lambda\, \phi^h) + T^h\, G(\lambda, (\mathbf{u}^h, \lambda\, p^h, \mathbf{d}^h, \lambda\, \phi^h)) = 0. \tag{3.51}$$

Thus we have recast our continuous and discrete optimality problems into a form that enables us to apply Theorem 3.1 and 3.2.

REMARK. It can be shown that for almost all values of the Reynolds number, i.e., for almost all data and values of the viscosity ν, that the optimality system (2.18)-(2.20), or equivalently, of (3.50), is nonsingular, i.e., is locally unique. Thus, it is reasonable to assume that the optimality system has branches of nonsingular solutions.

In order to apply the previous theorems, we need to estimate the approximation properties of the operator T^h.

PROPOSITION 3.3. *The problem* (3.40)-(3.44) *has a unique solution belonging to* X. *Assume that* (3.21)-(3.23) *hold. Then, the problem* (3.45)-(3.49) *has a unique solution belonging to* X^h. *Let* $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{d}}, \tilde{\phi})$ *and* $(\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{\mathbf{d}}^h, \tilde{\phi}^h)$ *denote the solutions of* (3.40)-(3.44) *and* (3.45)-(3.49), *respectively. Then, we also have that*

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h\|_1 + \|\tilde{p} - \tilde{p}^h\|_0 + \|\tilde{\mathbf{d}} - \tilde{\mathbf{d}}^h\|_1 + \|\tilde{\phi} - \tilde{\phi}^h\|_0 \to 0 \tag{3.52}$$

as $h \to 0$. If, in addition, $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{d}}, \tilde{\psi}) \in \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega) \times \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega)$, then there exists a constant C, independent of h, such that

$$(3.53) \quad \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h\|_1 + \|\tilde{p} - \tilde{p}^h\|_0 + \|\tilde{\mathbf{d}} - \tilde{\mathbf{d}}^h\|_1 + \|\tilde{\phi} - \tilde{\phi}^h\|_0 \leq C\, h^m(\|\tilde{\mathbf{u}}\|_{m+1} + \|\tilde{p}\|_m + \|\tilde{\mathbf{d}}\|_{m+1} + \|\tilde{\phi}\|_m).$$

Proof. First, it is well known [6] that the two Stokes problems (3.40)-(3.41) and (3.42)-(3.43) have a unique solution $(\tilde{\mathbf{u}}, \tilde{p})$ and $(\tilde{\mathbf{d}}, \tilde{\phi})$ belonging to $\mathbf{H}^1(\Omega) \times L_0^2(\Omega)$, respectively. Also, the discrete Stokes problems (3.45)-(3.46) and (3.47)-(3.48) have a unique solution $(\tilde{\mathbf{u}}^h, \tilde{p}^h)$ and $(\tilde{\mathbf{d}}^h, \tilde{\phi}^h)$ belonging to $\mathbf{V}^h \times S_0^h(\Omega)$, respectively. Moreover, we have that

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h\|_1 + \|\tilde{p} - \tilde{p}^h\|_0 \to 0$$

and

$$\|\tilde{\mathbf{d}} - \tilde{\mathbf{d}}^h\|_1 + \|\tilde{\phi} - \tilde{\phi}^h\|_0 \to 0$$

as $h \to 0$, and if in addition $(\tilde{\mathbf{u}}, \tilde{p}) \in \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cup L_0^2(\Omega)$ and $(\tilde{\mathbf{d}}, \tilde{\phi}) \in \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cup L_0^2(\Omega)$, we have that

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h\|_1 + \|\tilde{p} - \tilde{p}^h\|_0 \leq Ch^m(\|\tilde{\mathbf{u}}\|_{m+1} + \|\tilde{p}\|_m)$$

and

$$\|\tilde{\mathbf{d}} - \tilde{\mathbf{d}}^h\|_1 + \|\tilde{\phi} - \tilde{\phi}^h\|_0 \leq Ch^m(\|\tilde{\mathbf{d}}\|_{m+1} + \|\tilde{\phi}\|_m).$$

□

Using this proposition and Theorem 3.1, we are led to the following result.

THEOREM 3.4. *Assume that Λ is a compact interval of R and that there exists a branch $\{(\lambda, \psi(\lambda) := (\mathbf{u}, p, \mathbf{d}, \phi)) \in \Lambda \times X\}$ of nonsingular solutions of the optimality system (2.18)-(2.20). Assume that the finite element spaces $\mathbf{V}^h$ and S^h satisfy the conditions (3.21)-(3.23). Then, there exists a neighborhood $\mathcal{O}$ of the origin in X and, for $h \leq h_0$ small enough, a unique branch $\{(\lambda, \psi^h(\lambda) := (\mathbf{u}^h, p^h, \mathbf{d}^h, \phi^h)) \in \Lambda \times X^h\}$ of solutions of the discrete optimality system (3.24)-(3.28) such that $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$. Moreover,*

$$(3.54) \quad \|\psi^h(\lambda) - \psi(\lambda)\|_X = \|\mathbf{u}^h - \mathbf{u}\|_1 + \|p^h - p\|_0 + \|\mathbf{d}^h - \mathbf{d}\|_1 + \|\phi^h - \phi\|_0 \;\to\; 0$$

as $h \to 0$, uniformly in $\lambda \in \Lambda$. If, in addition, $(\mathbf{u}, p, \mathbf{d}, \phi) \in \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega) \times \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega)$ for $\lambda \in \Lambda$, then there exists a constant C, independent of h, such that

$$(3.55) \quad \|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 + \|\mathbf{d} - \mathbf{d}^h\|_1 + \|\phi - \phi^h\|_0 \\ \leq C h^m (\|\mathbf{u}(\lambda)\|_{m+1} + \|p(\lambda)\|_m + \|\mathbf{d}(\lambda)\|_{m+1} + \|\phi(\lambda)\|_m)$$

uniformly in $\lambda \in \Lambda$.

Proof. Clearly, G is a C^∞ polynomial map from $R \times X$ into Y. Therefore, using (1.6)-(1.8), it is easily shown that $D^2G(\lambda, \cdot)$ is bounded on all bounded sets of X. Now, given $(\mathbf{u}, p, \mathbf{d}, \phi) \in X$, a direct computation yields that $(\tilde{\zeta}, \tilde{\eta}) \in Y$ satisfies

$$(\tilde{\zeta}, \tilde{\eta}) = D_\psi G(\lambda, (\mathbf{u}, p, \mathbf{d}, \phi))\,(\mathbf{v}, q, \mathbf{e}, r)$$

for $(\mathbf{v}, q, \mathbf{e}, r) \in X$ if and only if

$$\langle \zeta, \bar{\mathbf{v}} \rangle = \lambda\, c(\mathbf{u}, \mathbf{v}, \bar{\mathbf{v}}) + \lambda\, c(\mathbf{v}, \mathbf{u}, \bar{\mathbf{v}}), \ \forall \bar{\mathbf{v}} \in \mathbf{H}^1(\Omega)$$
$$\langle \eta, \bar{\mathbf{e}} \rangle = -\lambda\, c(\bar{\mathbf{e}}, \mathbf{v}, \mathbf{d}) - \lambda\, c(\bar{\mathbf{e}}, \mathbf{u}, \mathbf{e}) - \lambda\, c(\mathbf{v}, \bar{\mathbf{e}}, \mathbf{d}) \\ -\lambda\, c(\mathbf{u}, \bar{\mathbf{e}}, \mathbf{e}) - \mathcal{J}''(\mathbf{u})(\bar{\mathbf{e}}), \ \forall \mathbf{e} \in \mathbf{H}^1(\Omega).$$

Thus, it follows from (1.6)-(1.8) that $D_\psi G(\lambda, (\mathbf{u}, p, \mathbf{d}, \phi)) \in \mathcal{L}(X, Y)$. On the other hand, since $(\mathbf{u}, p, \mathbf{d}, \phi) \in X$ and $(\mathbf{v}, q, \mathbf{e}, r) \in X$, by the Sobolev imbedding theorem, $\mathbf{u}, \mathbf{v}, \mathbf{d}, \mathbf{e} \in L^6(\Omega)$ and $\partial \mathbf{u}/\partial x_j$, $\partial \mathbf{v}/\partial x_j$, $\partial \mathbf{u}/\partial x_j$ and $\partial \mathbf{u}/\partial x_j \in \mathbf{L}^2(\Omega)$ for $j = 1, 2$. Then it follows that $(\tilde{\zeta}, \tilde{\eta}) \in Z$ and that for $(\mathbf{u}, p, \mathbf{d}, \phi) \in X$

$$D_\psi G(\lambda, (\mathbf{u}, p, \mathbf{d}, \phi)) \in \mathcal{L}(X, Z).$$

Next, we turn to the approximation properties of the operator T^h. From Proposition 3.3, we have that (3.32) holds. Since the imbedding of Z into Y is compact, (3.33) follows from (3.32), and the (3.54) follows from Theorem 3.1. Also from Proposition 3.3, we may conclude that there exists a constant C, independent of h, such that

$$\|(T - T^h)G(\lambda, \psi(\lambda))\|_X \leq C h^m (\|\mathbf{u}\|_{m+1} + \|p\|_m + \|\mathbf{d}\|_{m+1} + \|\phi\|_m).$$

Then (3.55) follows from Theorem 3.1. □

Now, using Theorem 3.2 we derive an estimate for the error of $\mathbf{u}^h$ and $\mathbf{d}^h$ in the $L^2(\Omega)$-norm. Since $G(\lambda, \psi(\lambda))$ does not depend on p or ϕ, we redefine $X = \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ and $X^h = \mathbf{V}^h \times \mathbf{V}^h$, Y and Z remain as before.

THEOREM 3.5. *Assume the hypotheses of* Theorem 3.4. *Then there exists a constant C, independent of h such that*

$$\begin{aligned}&\|\mathbf{u}^h - \mathbf{u}\|_0 + \|\mathbf{d}^h - \mathbf{d}\|_0 \\ &\quad \le C\, h^{m+1/2}(\|\mathbf{u}(\lambda)\|_{m+1} + \|p(\lambda)\|_m + \|\mathbf{d}(\lambda)\|_{m+1} + \|\phi(\lambda)\|_m).\end{aligned}$$

Proof. We must verify that (3.35)-(3.38) hold in our setting; then the approximation properties (3.21) and the results of Theorem 3.2 and Theorem 3.4 easily leads to the conclusion.

In similar method with [9], we can verify (3.35)-(3.38). □

REMARK. By other means, it can be shown [12] that actually

$$\begin{aligned}&\|\mathbf{u}^h - \mathbf{u}\|_0 + \|\mathbf{d}^h - \mathbf{d}\|_0 \\ &\quad \le C\, h^{m+1}(\|\mathbf{u}(\lambda)\|_{m+1} + \|p(\lambda)\|_m + \|\mathbf{d}(\lambda)\|_{m+1} + \|\phi(\lambda)\|_m).\end{aligned}$$

4. Numerical results

The optimality system of equations (2.18)-(2.20) consists of three groups of equations: the state equations for $(\mathbf{w}, p)$, the adjoint state equation for $(\mathbf{d}, \phi, \mathbf{u})$, and the optimality condition for $(\mathbf{u}, \mathbf{d})$. We may construct iterative methods, i.e., to iterate among the three groups of equations so that at each iteration we are dealing with a smaller size system of equations. A simple gradient method is given as follows:

1) choose an initial $\mathbf{t}^{(0)}$;

2) for each $n \ge 1$,

solve for $(\mathbf{w}^{(n)}, p^{(n)})$ from the state equation with $\mathbf{t}^{(n-1)}$

(4.56)
$$\begin{cases} \nu a(\mathbf{w}^{(n)}, \mathbf{v}) + c(\mathbf{u}^{(n)}, \mathbf{u}^{(n)}, \mathbf{v}) + b(\mathbf{v}, p^{(n)}) = \langle \bar{\mathbf{f}}, \mathbf{v}\rangle, \ \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\mathbf{w}^{(n)}, q) = 0, \qquad \forall q \in L_0^2(\Omega), \end{cases}$$

and

solve $(\mathbf{d}^{(n)}, \pi^{(n)})$ from the adjoint state equation

(4.57)
$$\begin{cases} \nu a(\mathbf{d}^{(n)}, \mathbf{e}) + c(\mathbf{e}, \mathbf{u}^{(n)}, \mathbf{d}^{(n)}) + c(\mathbf{u}^{(n)}, \mathbf{e}, \ \mathbf{d}^{(n)}) + b(\mathbf{e}, \phi^{(n)}) \\ \qquad\qquad + \langle \nabla \times (\nabla \times \mathbf{u}^{(n)}), \mathbf{e}\rangle = 0, \quad \forall \mathbf{e} \in \mathbf{H}_0^1(\Omega), \\ b(\mathbf{d}^{(n)}, r) = 0, \qquad \forall r \in L_0^2(\Omega), \end{cases}$$

and

solve for $\mathbf{t}^{(n)}$ from the optimality condition

$$(4.58)\quad \begin{cases} \nu a(\bar{\mathbf{u}}^{(i)}, \mathbf{d}^{(n)}) + c(\bar{\mathbf{u}}^{(i)}, \mathbf{u}^{(n)}, \mathbf{d}^{(n)}) + c(\mathbf{u}^{(n)}, \bar{\mathbf{u}}^{(i)}, \mathbf{d}^{(n)}) \\ \qquad\qquad + \langle \nabla \times (\nabla \times \mathbf{u}^{(n)}), \bar{\mathbf{u}}^{(i)} \rangle = 0, \quad i = 1, \ldots, m \end{cases}$$

where $\mathbf{u}^{(n)} = \mathbf{w}^{(n)} + \bar{\mathbf{u}}^{(0)} + \mathbf{t} \cdot \bar{\mathbf{u}}$.

Now we discuss numerical solution of the optimal control problem. To carry out the computation we discretized the problem using the finite element method. We use the Taylor-Hood finite element, that is, the piecewise quadratic element for the velocity and the bilinear element for the pressure defined on the uniform parallelogram grid with horizontal mesh size 0.0625 and vertical mesh size 0.03125 for the finite element method.

Since the equations (4.56) is a nonlinear equations, we use the Newton's method based on exact Jacobian. Let us denote the finite element spaces by $\mathbf{V}^h \subset \mathbf{H}^1(\Omega)$ and $S^h \subset L^2(\Omega)$ for velocity and pressure, respectively. The approximation problem for (4.56)-(4.58) is given as follows:

i) Initialize $\mathbf{u}_h^{(0)}$, $\mathbf{t}^{(0)}$.

For each $n = 1, \ldots,$

ii) Solve state equation using previous solution $\mathbf{u}_h^{(n-1)}$.

Find $\mathbf{w}_h^{(n)} \in V_0^h$, $p_h^{(n)} \in S_h$ such that

(4.59)

$$\begin{cases} \nu a(\mathbf{w}^{(n)}, \mathbf{v}_h) + c(\mathbf{u}_h^{(n-1)}, \mathbf{u}_h^{(n)}, \mathbf{v}_h) + c(\mathbf{u}_h^{(n)} - \mathbf{u}_h^{(n-1)}, \mathbf{u}_h^{(n-1)}, \mathbf{v}_h) \\ \qquad\qquad + b(\mathbf{v}_h, p_h^{(n)}) = \langle \bar{\mathbf{f}}_h, \mathbf{v}_h \rangle, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{w}_h^{(n)}, q_h) = 0, \quad \forall q_h \in S_h. \end{cases}$$

iii) Solve adjoint equation using previous solution $\mathbf{u}_h^{(n)}$.

Find $\mathbf{d}_h^{(n)} \in V_0^h$, $\psi_h^{(n)} \in S_h$ such that

(4.60)

$$\begin{cases} \nu a(\mathbf{d}_h^{(n)}, \mathbf{e}_h) + c(\mathbf{e}_h, \mathbf{u}_h^{(n)}, \mathbf{d}_h^{(n)}) + c(\mathbf{u}_h^{(n)}, \mathbf{e}_h, \mathbf{d}_h^{(n)}) + b(\mathbf{e}_h, \psi_h^{(n)}) \\ \qquad\qquad + \langle \nabla \times (\nabla \times \mathbf{u}_h^{(n)}), \mathbf{e}_h \rangle = 0, \quad \forall \mathbf{e}_h \in \mathbf{V}_h, \\ b(\mathbf{d}_h^{(n)}, r_h) = 0, \quad \forall r_h \in S_h. \end{cases}$$

iv) Find $\mathbf{t}^{(n)} \in U$ such that

$$(4.61) \qquad \begin{cases} \nu a(\bar{\mathbf{u}}^{(i)}, \mathbf{d}_h^{(n)}) + c(\bar{\mathbf{u}}^{(i)}, \mathbf{u}_h^{(n)}, \mathbf{d}_h^{(n)}) + c(\mathbf{u}_h^{(n)}, \bar{\mathbf{u}}^{(i)}, \mathbf{d}_h^{(n)}) \\ \qquad + \langle \nabla \times (\nabla \times \mathbf{u}_h^{(n)}), \bar{\mathbf{u}}^{(i)} \rangle = 0, \quad \text{for } i = 1, 2 \end{cases}$$

where $\langle \bar{\mathbf{f}}, \mathbf{v} \rangle = -a(\mathbf{z}, \mathbf{v})$ and $\mathbf{z} = \bar{\mathbf{u}}^{(0)} + \mathbf{t} \cdot \bar{\mathbf{u}}$. At each Newton's iteration, we solve the linear system of equations by Gaussian eliminations for banded matrices. Since quadratic convergence of Newton's method is valid only within a contraction ball, we normally first perform a few (usually 3 or 4 times) simple successive iterations and then switch to the Newton's method. The simple successive iterations are defined by

(4.62)

$$\begin{cases} \nu a(\mathbf{w}^{(n)}, \mathbf{v}_h) + c(\mathbf{u}_h^{(n-1)}, \mathbf{u}_h^{(n)}, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^{(n)}) = \langle \bar{\mathbf{f}}_h, \mathbf{v}_h \rangle, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{w}_h^{(n)}, q_h) = 0, \quad \forall q_h \in S_h, \\ \nu a(\mathbf{d}_h^{(n)}, \mathbf{e}_h) + c(\mathbf{e}_h, \mathbf{u}_h^{(n)}, \mathbf{d}_h^{(n)}) + c(\mathbf{u}_h^{(n)}, \mathbf{e}_h, \mathbf{d}_h^{(n)}) + b(\mathbf{e}_h, \psi_h^{(n)}) \\ \qquad + \langle \nabla \times (\nabla \times \mathbf{u}_h^{(n)}), \mathbf{e}_h \rangle = 0, \quad \forall \mathbf{e}_h \in \mathbf{V}_h, \\ b(\mathbf{d}_h^{(n)}, r_h) = 0, \quad \forall r_h \in S_h, \\ \nu a(\bar{\mathbf{u}}^{(i)}, \mathbf{d}_h^{(n)}) + c(\bar{\mathbf{u}}^{(i)}, \mathbf{u}_h^{(n)}, \mathbf{d}_h^{(n)}) + c(\mathbf{u}_h^{(n)}, \bar{\mathbf{u}}^{(i)}, \mathbf{d}_h^{(n)}) \\ \qquad + \langle \nabla \times (\nabla \times \mathbf{u}_h^{(n)}), \bar{\mathbf{u}}^{(i)} \rangle = 0, \quad \text{for } i = 1, 2. \end{cases}$$

In the case of the uncontrolled Navier-Stokes equations, the solution is unique for a small Reynold numbers and the simple successive approximations converges globally and linearly (See [7]).

In our computation, we take the Reynolds number to be $100(\nu = 1/100)$ and the three holes at $(0.875, 0)$, $(1.125, 0.5)$ and $(2, 0.375)$ with the size 0.0625 and $\bar{\mathbf{u}}^{(0)}$,$\bar{\mathbf{u}}^{(1)}$ and $\bar{\mathbf{u}}^{(2)}$ to be the solutions of auxiliary Stokes problems. We obtain the optimal control $t_1 = -2.018$(suction), $t_2 = -1.480$(suction) and $t_3 = -0.5901$(suction) after 10 Newton iterations.

Figure 3 gives the uncontrolled and controlled flows. Figure 4 shows the blow-up of the uncontrolled and controlled flows at the corner of the backward-facing-step. Finally, figure 5 shows the enlargement of the uncontrolled and controlled flows at the corner of the forward-facing-step. All computations in this paper were carried out on the SUN UltraSparc 2 workstation at Ajou University.

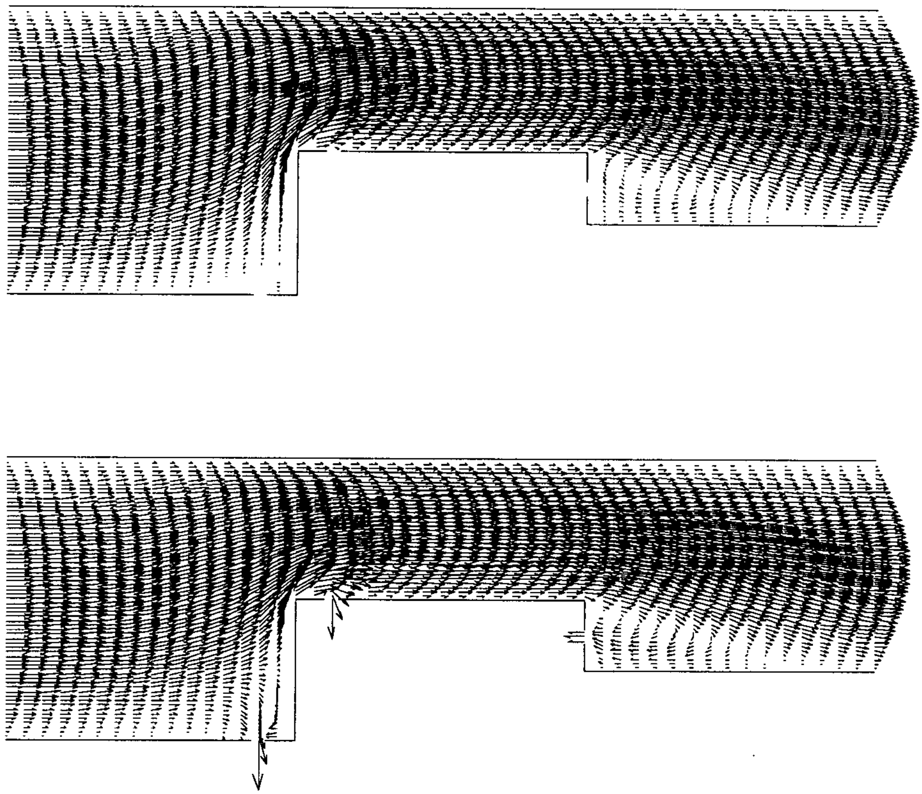

FIGURE 3. Uncontrolled channel flow (top) and Controlled channel flow (bottom) at $Re = 100$.

References

[1] I. Babuska, *The finite element method with Lagrange multipliers*, Numer. Math. **16** (1973), 179-192.

[2] I. Babuska and A. Aziz, *Survey lectures on the mathematical foundations of the finite element method, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* (Ed. by A. Aziz), Academic, New York, pp. 3-359, 1973.

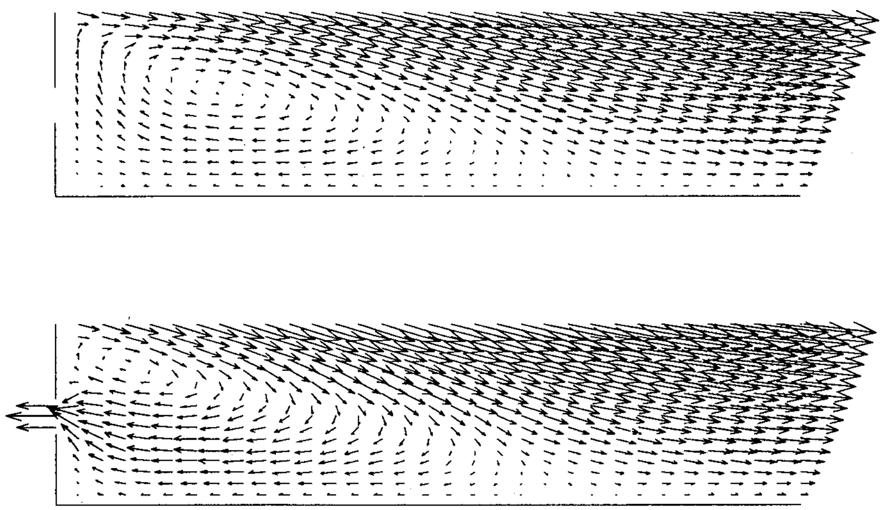

FIGURE 4. Partial enlargements (Backward-facing-step) of Fig 3.

[3] F. Brezzi, J. Rappaz and P. A. Raviart, *Finite-dimensional approximation of nonlinear problems. Part* I*: branches of nonsingular solutions*, Numer. Math. **36** (1980), 1-25.

[4] P. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[5] M. Desai an K. Ito, *Optimal controls of Navier-Stokes equations*, SIAM J. Control and Optimization **32** (1994), 1428-1446.

[6] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin, 1986.

[7] M. Gunzburger, *Finite Element Methods for Incompressible Viscous Flows: A Guide to Theory, Practice and Algorithms*, Academic, Boston, 1989.

[8] M. Gunzburger and L. Hou, *Treating inhomogeneous essential boundary conditions in finite element methods*, SIAM J. Num. Anal. **29** (1992), 390-424.

[9] M. Gunzburger, L. Hou and T. Svobodny, *Analysis and Finite Element Approximation of Optimal Control Problems for the Stationary Navier-Stokes Equations with Dirichlet Controls, Mod*, Math. Anal. Numer. **25** (1991), 711-748.

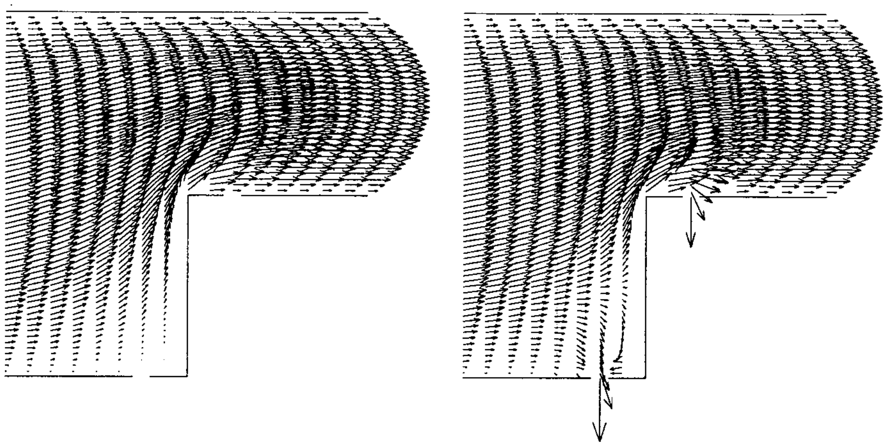

FIGURE 5. Partial enlargements (Forward-facing-step) of Fig 3.

[10] ———, *Boundary velocity control of incompressible flow with an application to viscous drag reduction*, SIAM J. Control and Optimization **30** (1992), 167-181.

[11] M. Gunzburger and H. Lee, *Feedback control of Karman vortex shedding*, J. of Appl. Mech. **63** (1996), 828-835.

[12] L. Hou, *Analysis and finite element approximation of some optimal control problem associated with the Navier-Stokes equations*, Ph. D. Thesis, Carnegie Mellon University, Pittsburgh, 1989.

[13] H. Maurer and J. Zowe, *First and Second Necessary and Sufficient Optimality Conditions for Infinite-dimensional Programming Problems*, Math Programming **16** (1979), 98-110.

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