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LAW OF LARGE NUMBERS FOR BRANCHING BROWNIAN MOTION

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ABSTRACT. Consider a supercritical Bellman-Harris process evolving from one particle. We superimpose on this process the additional structure of movement. A particle whose parent was at x at its time of birth moves until it dies according to a given Markov process X starting at x. The motions of different particles are assumed independent. In this paper we show that when the movement process X is standard Brownian the proportion of particles with position $\leq \sqrt{t}\, b$ and age $\leq a$ tends with probability 1 to $A(a)\Phi(b)$ where $A(\cdot)$ and $\Phi(\cdot)$ are the stable age distribution and standard normal distribution, respectively. We also extend this result to the case when the movement process is a Levy process.

1. Introduction

Let $\{Z(t); t \geq 0\}$ be a supercritical Bellman-Harris process evolving from one particle at time $t = 0$ whose lifetime distribution is G and offspring distribution is $\{p_k\}$. That is, the process starts at time 0 with one particle of age 0 and it dies at time λ and produces ξ offsprings where λ and ξ are independent random variables with distributions G and $\{p_k\}$ respectively. Then each particle dies and produces independently of each other in the same way as its parent, an so on. We superimpose on this process the additional structure of movement. A particle whose parent was at x at its time of birth moves until it dies according to a Markov process starting at x. The motions of different particles are assumed independent. If the movement process is a Brownian motion the process

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is called *branching Brownian motion*, whereas we call it a *branching Levy process* for a Levy movement process.

For any family tree ω, let $Z(t,a,b,\omega)$ be the number of particles living at time t which are of age at most a with position $\le b$ and let $A(t,a,\omega) = Z(t,a,\infty,\omega)/Z(t,\omega)$, where $Z(t,\omega) \equiv Z(t,\infty,\infty,\omega)$. Then under '$j\log j$' condition $A(t,a,\omega)$ converges to $A(a)$ the stable age distribution with probability 1 (see Athreya and Kaplan (1976)). If the underlying movement process is Brownian then it is known (see Asmussen and Kaplan (1976)) that under finite second moment condition on the offspring law $Z(t,\infty,\sqrt{t}\ b,\omega)/Z(t,\omega) \xrightarrow{\text{a.s.}} \Phi(b)$ where $\Phi(b) = (2\pi)^{-1/2}\int_{-\infty}^{b} e^{-\frac{x^2}{2}}dx$. Thus one would expect the proportion of particles with proposition $\le \sqrt{t}\ b$ who are younger than or equal to a tends to $A(a)\Phi(b)$. Indeed this essentially turns out to be the case here under '$j\log j$' condition. Furthermore, we can extend this result to branching Levy processes.

2. Statement of results

We make the following assumptions throughout. Sometimes they will appear in lemmas and theorems explicitly and sometimes not, but they will always be in force.

(A 1) $p_0 = 0$,
(A 2) $1 < \mu \equiv \sum_{j=0}^{\infty} jp_j < \infty$,
(A 3) $\sum_{j=1}^{\infty}(j\log j)p_j < \infty$.

The assumption (A 1) is primarily for convenience of exposition. Otherwise one has to keep qualifying "on the set of explosion". (A 3) guarantees (see Athreya and Ney (1972)) the existence of random variable W such that

$$\lim_{t\to\infty} e^{-\alpha t}Z(t) = W, \quad \text{and} \quad P(W>0) = 1, \tag{1}$$

where $\alpha = \alpha(\mu, G)$ is the Malthusian parameter for μ and G defined by the root of the equation $\mu\int_0^\infty e^{-\alpha t}dG(t) = 1$.

THEOREM 1. *Let the underlying movement process be a standard Brownian motion. Then for $a \in R^+$, $b \in R$,*

$$H_t(a,\sqrt{t}b,\omega) \equiv \frac{Z(t,a,\sqrt{t}b,\omega)}{Z(t,\omega)} \xrightarrow{\text{a.s.}} A(a)\Phi(b) \quad as \quad t\to\infty,$$

where $\Phi(b) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b} e^{-\frac{x^2}{2}}dx$, *and* $A(a) = \dfrac{\int_0^a e^{-\alpha u}(1-G(u))du}{\int_0^\infty e^{-\alpha u}(1-G(u))du}$ *is the stable age distribution.*

Now consider a branching Levy process. Let $\{X(t); t \geq 0\}$ be a underlying Markov process such that $X(0) = 0$ a.s. Suppose that $\{X(t); t \geq 0\}$ is stationary with independent increments and that for some measurable functions $a(t)$ and $m(t)$

$$Y(t) \equiv \frac{X(t) - m(t)}{a(t)} \xrightarrow{\text{d}} Y \quad \text{as} \quad t \to \infty, \tag{2}$$

where $P(Y \leq x) = F(x)$ is a nondegenerate and continuous distribution. Then we have

THEOREM 2. *Suppose* (2) *holds with* $a(t) = t^c L_1(t)$ *and* $m(t) = t^d L_2(t)$, *where* $c > d \geq 0$, *and* L_1, L_2 *are slowly varying functions at infinity such that*

$$\limsup_{t\to\infty} \left|\frac{L_2(t)}{L_1(t)}\right| < \infty.$$

If $E(|Y|^u) < \infty$ *for some* $u > 1/c$, *then for any* $a \in R^+$, $b \in R$,

$$H_t(a, a(t)b + m(t), \omega) \equiv \frac{Z(t, a, a(t)b + m(t), \omega)}{Z(t, \omega)} \xrightarrow{\text{a.s.}} A(a)F(b)$$

$$\text{as} \quad t \to \infty.$$

3. Preliminary results

In the proofs to come we make use of the following lemmas. The first one can be found in Nerman (1981).

LEMMA 1. *Let* $N = \sup_{t\geq 0}\{e^{-\alpha t}Z(t)\}$ *with* α *the Malthusian parameter. If* $\sum_{j=1}^{\infty}(j \log j)p_j < \infty$, *then* $E(N) < \infty$.

We add a superscript a to random variables and their moments to indicate the case when P is supported by those ω's which start with one particle of age a. Then we have the following

COROLLARY 1. *Put* $M = \sup_{s\geq 0}\sup_{a\geq 0}\{e^{-\alpha s}Z^a(s)\}$. *If* $\sum_{j=1}^{\infty}(j \log j)p_j < \infty$, *then* $E(M) < \infty$.

Proof. We first note that

$$Z^a(s) = I(\lambda^a > s) + \sum_{j=1}^{\xi} Z_j(s - \lambda^a) \tag{3}$$

where $\{Z_j(s); s \geq 0\}$, $j = 1, 2, \cdots$, are i.i.d. with $\{Z(s); s \geq 0\}$. So we have

$$\begin{aligned} e^{-\alpha s} Z^a(s) &= e^{-\alpha s} I(\lambda^a > s) + \sum_{j=1}^{\xi} e^{-\alpha(s-\lambda^a)} Z_j(s - \lambda^a) e^{-\alpha\lambda^a} \\ &\leq 1 + \sum_{j=1}^{\xi} e^{-\alpha(s-\lambda^a)} Z_j(s - \lambda^a) \\ &\leq 1 + \sum_{j=1}^{\xi} M_j, \end{aligned}$$

where $M_j \equiv \sup_{s \geq 0} e^{-\alpha s} Z_j(s)$. Then $M \leq 1 + \sum_{j=1}^{\xi} M_j$. Since $\sum_{j=1}^{\infty}$ $(j \log j) p_j < \infty$, $E(M_1) < \infty$ (Lemma 1) and hence by the independence of $\{M_j\}$ and ξ, $E(M) \leq 1 + \mu E(M_1) < \infty$. □

The following two lemmas are in Athreya and Kaplan (1976).

LEMMA 2. *Let* $V(y) = \mu \int_0^\infty e^{-\alpha t} G^y(dt)$ *where* $G^y(t) = \frac{G(t+y) - G(y)}{1 - G(y)}$ *and let* $n_1 = \int_0^\infty e^{-\alpha t}(1 - G(t))dt / \mu \int_0^\infty t e^{-\alpha t} G(dt)$. *Define* $m^y(s, a) = E(Z^y(s, a, \infty))$, $m^y(s) = E(Z^y(s, \infty, \infty))$, *then for any* $a \geq 0$

$$\sup_{y \geq 0} (|m^y(s, a) e^{-\alpha s} - n_1 V(y) A(a)|, \ |m^y(s) e^{-\alpha s} - n_1 V(y)|) \to 0$$

$$\text{as} \quad s \to \infty.$$

LEMMA 3. *Let* $V_t(\omega) = \sum_{j=1}^{Z(t,\omega)} V(a_j(t, \omega))$, *where* $\{a_j(t, \omega); j = 1, \cdots, Z(t, \omega)\}$ *is the age-chart at time* t. *Suppose* $\sum_{j=1}^{\infty} (j \log j) p_j < \infty$. *Then for every* $\delta > 0$,

$$\lim_{n \to \infty} \frac{V_{n\delta}(\omega)}{Z(n\delta, \omega)} = n_1^{-1} \quad \text{a.s.}$$

The proof of the following lemma can be found in Athreya and Kang (1998).

LEMMA 4. *Let $\{\mathcal{F}_n\}_0^\infty$ be a filtration contained in $(\Omega, \mathcal{B}, P)$. Let $\{X_{ni}; n, i = 1, 2, \cdots\}$ be a double array of random variables such that for each n, conditioned on $\mathcal{F}_n$ the sequence $\{X_{ni}; i = 1, 2, \cdots\}$ are independent w.p.1. Let $\{N_n; n = 1, 2, \cdots\}$ be a nondecreasing sequence of nonnegative integer valued random variables such that for each n N_n is $\mathcal{F}_n$-measurable. Assume*

(i) *that there exists a random probability measure Q on $[0, \infty)$ such that for some constant $0 < C < \infty$,*

$$\sup_{n,i} P(|X_{ni}| > t | \mathcal{F}_n) \leq CQ(t, \infty) \quad \textit{for all } t > 0 \ \text{ w.p.1},$$

(ii) *that* $\int_0^\infty Q(t, \infty) dt < \infty$ w.p.1, *and*

(iii) *that* $\liminf_n \dfrac{N_{n+1}}{N_n} > 1$ w.p.1.

Then, $\displaystyle \frac{1}{N_n} \sum_{i=1}^{N_n} (X_{ni} - E(X_{ni})) \to 0$ w.p.1.

4. Proof of Theorem 1

We begin with the following representation appealing to the additive property of branching processes,

$$(4) \qquad Z((t+s), a, \sqrt{(t+s)}\, b, \omega) = \sum_{j=1}^{Z(t,\omega)} Z_{x_j(t,\omega)}^{a_j(t,\omega)}(s, a, \sqrt{(t+s)}\, b, \omega)$$

where $\{(a_j(t,\omega),\ x_j(t,\omega)); j = 1, 2, \cdots\}$ is the (age, position)-chart at time t and $Z_{x_j(t,\omega)}^{a_j(t,\omega)}(s, a, x, \omega)$ is the number of particles at time $(t+s)$ whose age $\leq a$ and whose position is $\leq x$ in the line of descent initiated by the particle of age $a_j(t,\omega)$ and position $x_j(t,\omega)$ at time t. With abuse of notation we write (4) as (suppressing ω and (t,ω)),

$$(5) \qquad Z((t+s), a, \sqrt{(t+s)}\, b) = \sum_{j=1}^{Z(t)} Z_{x_j}^{a_j}(s, a, \sqrt{(t+s)}\, b)$$

Let $\mathcal{F}_t$ be a σ-algebra containing all the information up to time t. Noting that (see Asmussen and Kaplan (1976))

$$\begin{aligned} E(Z^{a_j}_{x_j}(s, a, \sqrt{(t+s)}\, b)|\mathcal{F}_t) &= E(Z^{a_j}(s, a, \sqrt{(t+s)}\, b - x_j)|\mathcal{F}_t) \\ &= m^{a_j}(s,a)\Phi\left(\frac{\sqrt{(t+s)}\, b - x_j}{\sqrt{s}}\right), \end{aligned}$$

we decompose (5) as follows;

$$\begin{aligned} &Z((t+s), a, \sqrt{(t+s)}\, b) \\ &= \sum_{j=1}^{Z(t)} \left\{ Z^{a_j}_{x_j}(s, a, \sqrt{(t+s)}\, b) - m^{a_j}(s,a)\Phi\left(\frac{\sqrt{(t+s)}\, b - x_j}{\sqrt{s}}\right)\right\} \\ &\quad + \sum_{j=1}^{Z(t)} \left\{ m^{a_j}(s,a)\Phi\left(\frac{\sqrt{(t+s)}\, b - x_j}{\sqrt{s}}\right) - n_1 e^{\alpha s} V(a_j) A(a)\Phi(b)\right\} . \\ &\quad + n_1 A(a)\Phi(b) e^{\alpha s} V_t. \end{aligned}$$

So

$$\begin{aligned} H_{(t+s)}(a, \sqrt{(t+s)}\, b) &\equiv \frac{Z((t+s), a, \sqrt{(t+s)}\, b)}{Z((t+s))} \\ &= \frac{a_t(s,a,b) + b_t(s,a,b) + c_t A(a)\Phi(b)}{a_t(s,\infty,\infty) + b_t(s,\infty,\infty) + c_t}, \end{aligned}$$

where

$$\begin{aligned} a_t(s,a,b) &= \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} \Bigg\{ e^{-\alpha s} Z^{a_j}_{x_j}(s, a, \sqrt{(t+s)}\, b) \\ &\qquad - e^{-\alpha s} m^{a_j}(s,a)\Phi\left(\frac{\sqrt{(t+s)}\, b - x_j}{\sqrt{s}}\right)\Bigg\}, \\ b_t(s,a,b) &= \frac{1}{Z(t)} \sum_{j=1}^{Z(t)} \Bigg\{ e^{-\alpha s} m^{a_j}(s,a)\Phi\left(\frac{\sqrt{(t+s)}\, b - x_j}{\sqrt{s}}\right) \\ &\qquad - n_1 V(a_j) A(a)\Phi(b)\Bigg\}, \\ c_t &= \frac{n_1}{Z(t)} V_t. \end{aligned}$$

Following Athreya and Kaplan (1978) we first discretize the process, i.e., for $\delta > 0$ let $t_n = n\delta$ and $s_n = s(t_n)$ and consider

$$H_{n\delta+s_n}(a, \sqrt{n\delta + s_n}\, b) = \frac{a_{n\delta}(s_n, a, b) + b_{n\delta}(s_n, a, b) + c_{n\delta}A(a)\Phi(b)}{a_{n\delta}(s_n, \infty, \infty) + b_{n\delta}(s_n, \infty, \infty) + c_{n\delta}}.$$

Note that $c_{n\delta} \xrightarrow{\text{a.s.}} 1$ as $n \to \infty$ (Lemma 3). Since $e^{-\alpha s} Z^{a_j}_{x_j}(s, a, \sqrt{(t+s)}\, b) \overset{s}{\leq} M \equiv \sup_{s\geq 0} \sup_{a\geq 0}\{e^{-\alpha s} Z^a(s)\}$, and $E(M) < \infty$, we can see easily that $a_{n\delta}(s_n, a, b) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$ for any choice of sequence s_n such that $s_n \to \infty$ as $n \to \infty$ from Lemma 4.

Now we show that for $s_n = (n\delta)^3 - n\delta$

$$b_{n\delta}(s_n, a, b) \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty. \tag{6}$$

Let $\{(a_j, x_j); j = 1, 2, \cdots\}$ be the (age, position)-chart at time $n\delta$. For $j = 1 \cdots, Z(n\delta)$, put $I_{nj} = I(|\frac{x_j}{\sqrt{n\delta}}| \leq \sqrt{n\delta})$, and $J_{nj} = 1 - I_{nj}$. Then

$$b_{n\delta}(s_n, a, b) = b^1_{n\delta}(s_n, a, b) + b^2_{n\delta}(s_n, a, b) + b^3_{n\delta}(s_n, a, b),$$

where

$$\begin{aligned}
&b^1_{n\delta}(s_n, a, b) \\
&= \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} \left\{ \Phi\left(\sqrt{1 + \frac{n\delta}{s_n}}\, b - \frac{x_j}{\sqrt{n\delta}} \sqrt{\frac{n\delta}{s_n}} \right) - \Phi(b) \right\} \\
&\qquad \cdot e^{-\alpha s_n} m^{a_j}(s_n, a) I_{nj}, \\
&b^2_{n\delta}(s_n, a, b) \\
&= \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} \left\{ \Phi\left(\sqrt{1 + \frac{n\delta}{s_n}}\, b - \frac{x_j}{\sqrt{n\delta}} \sqrt{\frac{n\delta}{s_n}} \right) - \Phi(b) \right\} \\
&\qquad \cdot e^{-\alpha s_n} m^{a_j}(s_n, a) J_{nj}, \\
&b^3_{n\delta}(s_n, a, b) \\
&= \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} \{e^{-\alpha s_n} m^{a_j}(s_n, a) - n_1 V(a_j) A(a)\} \Phi(b).
\end{aligned}$$

By the continuity of Φ and Corollary 1, it is easy to see that $b^1_{n\delta}(s_n, a, b) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. On the other hand, for any $\varepsilon > 0$,

$$\begin{aligned} P(|b^2_{n\delta}(s_n, a, b)| > \varepsilon|\mathcal{G}_{n\delta}) &\leq \frac{E(M)}{\varepsilon Z(n\delta)} \sum_{j=1}^{Z(n\delta)} 2E(J_{nj}) \\ &\leq \frac{2E(M)}{\varepsilon} 2(1 - \Phi(\sqrt{n\delta})) \\ &\leq \frac{4E(M)}{\varepsilon\sqrt{2\pi}} e^{-\frac{n\delta}{2}}, \end{aligned}$$

where $\mathcal{G}_t = \sigma(Z(s); s \leq t)$ is the σ-algebra generated by the usual Bellman-Harris process with no informations about positions. So $\sum_{n=0}^{\infty} P(|b^2_{n\delta}(s_n, a, b)| > \varepsilon|\mathcal{G}_{n\delta}) < \infty$ and by the conditional Borel-Cantelli lemma $b^2_{n\delta}(s_n, a, b) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. Finally we have $b^3_{n\delta}(s_n, a, b) \to 0$ as $n \to \infty$ directly from Lemma 2. So we have proved that

$$H_{(n\delta)^3}(a, (n\delta)^{3/2} b) \xrightarrow{\text{a.s.}} A(a)\Phi(b) \quad \text{as} \quad n \to \infty. \tag{7}$$

To prove $H_{n\delta}(a, \sqrt{n\delta}\, b) \xrightarrow{\text{a.s.}} A(a)\Phi(b)$ as $n \to \infty$ we adopt the method used in Athreya and Kaplan (1978).

Let $\delta_0 = \delta^{1/3}$. For any $n \geq 1$, there exists an integer $m_n \geq 0$ such that $m_n^3 \leq n < (m_n + 1)^3$. Put $k_n = (m_n - 1)^3$ then $3(m_n - 1)^2 \leq n - k_n < 6(m_n + 1)^2$. So as $n \to \infty$, $k_n \to \infty$ as well and further

$$0 \leq \frac{\sqrt{n} - \sqrt{k_n}}{\sqrt{n - k_n}} = \frac{\sqrt{n - k_n}}{\sqrt{n} + \sqrt{k_n}} \leq \frac{6(m_n + 1)}{m_n^{3/2}} \to 0 \quad \text{as} \quad n \to \infty, \tag{8}$$

$$\frac{\sqrt{k_n}}{\sqrt{n - k_n}} \geq \frac{(m_n - 1)^{3/2}}{6(m_n + 1)} \to \infty \quad \text{as} \quad n \to \infty. \tag{9}$$

Fix $\varepsilon > 0$ and define $B_1 = (-\infty, b - \varepsilon]$, $B_2 = [b + \varepsilon, \infty)$, and $B_3 = (b - \varepsilon, b + \varepsilon)$. Then

$$\begin{aligned} &H_{n\delta}(a, \sqrt{n\delta}\, b) \\ &= \frac{a_{k_n\delta}((n - k_n)\delta, a, b) + \sum_{i=1}^{3} d^i_{k_n\delta}((n - k_n)\delta, a, b)}{a_{k_n\delta}((n - k_n)\delta, \infty, \infty) + b_{k_n\delta}((n - k_n)\delta, \infty, \infty) + c_{k_n\delta}}, \end{aligned}$$

where

$$d^i_{k_n\delta}((n-k_n)\delta, a, b)$$
$$= \frac{1}{Z(k_n\delta)} \sum_{j=1}^{Z(k_n\delta)} e^{-\alpha(n-k_n)\delta} m^{a_j}((n-k_n)\delta, a)\Phi\left(\frac{\sqrt{k_n\delta}\, b - x_j}{\sqrt{(n-k_n)\delta}}\right) I_{\sqrt{k_n\delta}B_i}(x_j),$$

and $\{(a_j, x_j); j = 1, 2, \cdots\}$ is the (age, position)-chart at time $k_n\delta$. We already know that

1. $\max\{a_{k_n\delta}((n-k_n)\delta, a, b), a_{k_n\delta}((n-k_n)\delta, \infty, \infty)\} \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$ (Lemma 4),
2. $b_{k_n\delta}((n-k_n)\delta, \infty, \infty) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$,
3. $c_{k_n\delta} \xrightarrow{\text{a.s.}} 1$ as $n \to \infty$ (Lemma 3).

Furthermore, since $k_n\delta = ((m_n - 1)\delta_0)^3$, (7) with $a = \infty$ implies

$$\text{(10)} \qquad \begin{aligned} d^3_{k_n\delta}((n-k_n)\delta, a, b) &\le \frac{E(M)}{Z(k_n\delta)} \sum_{j=1}^{Z(k_n\delta)} I_{\sqrt{k_n\delta}B_3}(x_j) \\ &\xrightarrow{\text{a.s.}} E(M)\Phi(B_3) \quad \text{as} \quad n \to \infty, \end{aligned}$$

where $\Phi(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-y^2/2} dy$. Note that if $x_j \in \sqrt{k_n\delta}B_1$, from (8) and (9)

$$\begin{aligned} \frac{\sqrt{n\delta}b - x_j}{\sqrt{(n-k_n)\delta}} &= \frac{\sqrt{n\delta} - \sqrt{k_n\delta}}{\sqrt{(n-k_n)\delta}} b + \frac{\sqrt{k_n\delta}\, b - x_j}{\sqrt{(n-k_n)\delta}} \\ &\ge \frac{\sqrt{n} - \sqrt{k_n}}{\sqrt{(n-k_n)}} b + \frac{\sqrt{k_n}\varepsilon}{\sqrt{(n-k_n)}} \to \infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

So

$$\limsup_{n\to\infty}\left|\frac{1}{Z(k_n\delta)}\sum_{j=1}^{Z(k_n\delta)} e^{-\alpha(n-k_n)\delta}m^{a_j}((n-k_n)\delta,a)I_{\sqrt{k_n\delta}B_1}(x_j) - d^1_{k_n\delta}((n-k_n)\delta,a,b)\right|$$

$$= \limsup_{n\to\infty}\frac{1}{Z(k_n\delta)}\sum_{j=1}^{Z(k_n\delta)} e^{-\alpha(n-k_n)\delta}m^{a_j}((n-k_n)\delta,a) \left(1-\Phi\left(\frac{\sqrt{k_n\delta}\,b-x_j}{\sqrt{(n-k_n)\delta}}\right)\right)I_{\sqrt{k_n\delta}B_1}(x_j)$$

$$\leq E(M)\left(1-\Phi\left(\frac{\sqrt{k_n}\varepsilon}{\sqrt{(n-k_n)}}\right)\right)\to 0 \quad \text{a.s.} \quad \text{as} \quad n\to\infty \quad \text{by (9)}.$$

Hence

$$\begin{aligned}
&\lim_{n\to\infty} d^1_{k_n\delta}((n-k_n)\delta,a,b)\\
&= \lim_{n\to\infty}\frac{1}{Z(k_n\delta)}\sum_{j=1}^{Z(k_n\delta)} e^{-\alpha(n-k_n)\delta}m^{a_j}((n-k_n)\delta,a)I_{\sqrt{k_n\delta}B_1}(x_j)\\
(11) &= \int_{R^+\times R} e^{-\alpha(n-k_n)\delta}m^{y}((n-k_n)\delta,a)I_{B_1}(x)dH_{k_n\delta}(y,\sqrt{k_n\delta}\,x)\\
&\xrightarrow{\text{a.s.}} \int_{-\infty}^{\infty}\int_0^{\infty} n_1V(y)A(a)dA(y)I_{B_1}(x)d\Phi(x) \quad \text{by Lemma 2 and (7)}\\
&= n_1A(a)\Phi(b-\varepsilon)\int_0^{\infty}V(y)dA(y)\\
&= A(a)\Phi(b-\varepsilon).
\end{aligned}$$

On the other hand, if $x_j\in\sqrt{k_n\delta}B_2$,

$$\begin{aligned}
\frac{\sqrt{n\delta}\,b-x_j}{\sqrt{(n-k_n)\delta}} &= \frac{\sqrt{n\delta}-\sqrt{k_n\delta}}{\sqrt{(n-k_n)\delta}}b+\frac{\sqrt{k_n\delta}b-x_j}{\sqrt{(n-k_n)\delta}}\\
&\leq \frac{\sqrt{n}-\sqrt{k_n}}{\sqrt{(n-k_n)}}b-\frac{\sqrt{k_n}\varepsilon}{\sqrt{(n-k_n)}} \to -\infty \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Hence $\limsup_{n\to\infty} \sup_{x_j \in \sqrt{k_n\delta}B_2} \dfrac{\sqrt{n\delta}\, b - x_j}{\sqrt{(n-k_n)\delta}} = -\infty$, and so

$$d^2_{k_n\delta}((n-k_n)\delta, a, b) \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

So recalling Lemma 2 and 3 we have

$$\begin{aligned}
&\limsup_{n\to\infty} |H_{n\delta}(a, \sqrt{n\delta} b) - A(a)\Phi(b-\varepsilon)| \\
&\quad \le \limsup_{n\to\infty} |d^1_{k_n\delta}((n-k_n)\delta, a, b) - A(a)\Phi(b-\varepsilon)| \\
&\qquad + \limsup_{n\to\infty} |d^2_{k_n\delta}((n-k_n)\delta, a, b) + \limsup_{n\to\infty} |b^3_{k_n\delta}((n-k_n)\delta, a, b)| \\
&\quad \le E(M)\Phi(B_3) \quad \text{by (10) (11) and (12).}
\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we get

$$\limsup_{n\to\infty} |H_{n\delta}(a, \sqrt{n\delta}\, b) - A(a)\Phi(b)| = 0 \qquad \text{a.s.} \tag{13}$$

Now we prove that $\lim_{t\to\infty} H_t(a, \sqrt{t}\, b) = A(a)\Phi(b)$ a.s. Let $\varepsilon > 0$ and $\delta > 0$ be fixed. Let $n\delta \le t < (n+1)\delta$ and define

$$\delta_j = \begin{cases} 1 & \text{if } j\text{th particle at time } n\delta \text{ doesn't split until } (n+1)\delta \\ & \text{and the particle doesn't cover a distant} > \sqrt{n\delta}\,\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{(a_j, x_j); j = 1, \cdots, Z(n\delta)\}$ be the (age, position)-chart at time $n\delta$. Since the lifetime and the movement of a particle are independent,

$$\begin{aligned}
E(\delta_j|\mathcal{F}_{n\delta}) &= P(\delta_j = 1|\mathcal{F}_{n\delta}) = P(\lambda^{a_j} > \delta|\mathcal{F}_{n\delta})P(\bar{\xi}(\delta) \le \sqrt{n\delta}\,\varepsilon) \\
&= (1 - G^{a_j}(\delta))P(\bar{\xi}(\delta) \le \sqrt{n\delta}\,\varepsilon),
\end{aligned}$$

where $\bar{\xi}(\delta) = \sup_{0\le t\le\delta} |B_0(t)|$ with $\{B_0(t); t \ge 0\}$ a standard Brownian motion starting at 0. It is easy to see the following inequality from the definition of δ_j,

$$Z(t, a, \sqrt{t}\, b) \ge \sum_{j=1}^{Z(n\delta)} I(a_j + \delta \le a) I(x_j \le \sqrt{n\delta}(b-\varepsilon))\delta_j.$$

So

$$\begin{aligned}\frac{Z(t,a,\sqrt{t}\,b)}{Z(t)} &\geq \frac{Z(n\delta)}{Z(t)}\frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} I(a_j \leq a-\delta)I(x_j \leq \sqrt{n\delta}\,(b-\varepsilon))\delta_j \\ &= \frac{Z(n\delta)}{Z(t)}\{A(n\delta,a,b) + P(\bar{\xi}(\delta) \leq \sqrt{n\delta}\,\varepsilon)B(n\delta,a,b)\},\end{aligned}$$

where

$$\begin{aligned}A(n\delta,a,b) &= \frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} I(a_j \leq a-\delta)I(x_j \leq \sqrt{n\delta}(b-\varepsilon)) \\ &\qquad \cdot\{(\delta_j - (1-G^{a_j}(\delta))P(\bar{\xi}(\delta) \leq \sqrt{n\delta}\,\varepsilon)\} \\ B(n\delta,a,b) &= \frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} I(a_j \leq a-\delta)I(x_j \leq \sqrt{n\delta}(b-\varepsilon))(1-G^{a_j}(\delta)).\end{aligned}$$

Since $E(I(a_j \leq a-\delta)I(x_j \leq \sqrt{n\delta}(b-\varepsilon))\{\delta_j - (1-G^{a_j}(\delta))P(\bar{\xi}(\delta) \leq \sqrt{n\delta}\varepsilon)\}) = 0$ we apply Lemma 4 to get $A(n\delta,a,b) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. On the other hand

$$\begin{aligned}&B(n\delta,a,b) \\ &= \frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} I(a_j \leq a-\delta)I(x_j \leq \sqrt{n\delta}\,(b-\varepsilon)) \\ &\quad - \frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} I(a_j \leq a-\delta)I(x_j \leq \sqrt{n\delta}\,(b-\varepsilon))G^{a_j}(\delta) \\ &\geq \frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} I(a_j \leq a-\delta)I(x_j \leq \sqrt{n\delta}\,(b-\varepsilon)) - \frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} G^{a_j}(\delta).\end{aligned}$$

Note that $G_\delta(a) \equiv G^a(\delta)$ is bounded and continuous except on a countable set. So

$$\begin{aligned}\frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} G^{a_j}(\delta) &= \int_0^\infty G_\delta(u)A(du,n\delta) \\ &\xrightarrow{\text{a.s.}} \int_0^\infty G_\delta(u)A(du) \quad \text{as} \quad n \to \infty. \qquad (14)\end{aligned}$$

Since $P(\bar{\xi}(\delta) \leq \sqrt{n\delta}\,\varepsilon) \to 1$ as $n \to \infty$, (13) and (14) imply together that

$$\liminf_{n\to\infty} B(n\delta, a, b) \geq A(a-\delta)\Phi(b-\varepsilon) - \int_0^\infty G_\delta(u)A(du) \quad \text{a.s.}$$

Further from (1) we have

$$\begin{aligned}\liminf_{n\to\infty} \frac{Z(n\delta)}{Z(t)} &\geq \liminf_{n\to\infty} \frac{Z(n\delta)}{Z((n+1)\delta)} \\ &= \liminf_{n\to\infty} \frac{Z(n\delta)e^{-\alpha n\delta}}{Z((n+1)\delta)e^{-\alpha(n+1)\delta}} e^{-\alpha\delta} = e^{-\alpha\delta} \quad \text{a.s.}\end{aligned}$$

Hence
(15)

$$\liminf_{t\to\infty} \frac{Z(t, a, \sqrt{t}\,b)}{Z(t)} \geq e^{-\alpha\delta}(A(a-\delta)\Phi(b-\varepsilon) - \int_0^\infty G_\delta(u)A(du)) \quad \text{a.s.}$$

Since $G_\delta(u) \to 0$ a.e. as $\delta \to 0$, we see $\int_0^\infty G_\delta(u)A(du) \to 0$ as $\delta \to 0$ by the dominated convergence theorem. Letting $\delta \downarrow 0$ and then letting $\varepsilon \downarrow 0$, we get from (15) that

$$\liminf_{t\to\infty} \frac{Z(t, a, \sqrt{t}\,b)}{Z(t)} \geq A(a)\Phi(b).$$

For the other direction we have the following inequality

$$Z(t) - Z(t, a, \sqrt{t}\,b) \geq \sum_{j=1}^{Z(n\delta)} I(a_j \leq a-\delta) I(x_j \geq \sqrt{(n+1)\delta}\,(b+\varepsilon))\delta_j$$

and so

$$1 - \frac{Z(t, a, \sqrt{t}\,b)}{Z(t)} \geq \frac{Z(n\delta)}{Z(t)}\{A'((n\delta, a, b) + P(\bar{\xi}(\delta) \leq \sqrt{n\delta}\,\varepsilon)B'(n\delta, a, b)\},$$

where

$$\begin{aligned}A'(n\delta, a, b) &= \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a-\delta) I(x_j > \sqrt{(n+1)\delta}\,(b+\varepsilon)) \\ &\quad \cdot\{\delta_j - (1 - G^{a_j}(\delta))P(\bar{\xi}(\delta) \leq \sqrt{n\delta}\,\varepsilon)\} \\ B'(n\delta, a, b) &= \frac{1}{Z(n\delta)} \sum_{j=1}^{Z(n\delta)} I(a_j \leq a-\delta) I(x_j > \sqrt{(n+1)\delta}\,(b+\varepsilon)) \\ &\quad \cdot(1 - G^{a_j}(\delta)).\end{aligned}$$

The same arguments as above establish $A'(n\delta, a, b) \xrightarrow{\text{a.s.}} 0$ and

$$\liminf_{n\to\infty} B'(n\delta, a, b) \geq (1 - A(a-\delta)\Phi(b+\varepsilon)) - \int_0^\infty G_\delta(u)A(du).$$

So

$$\liminf_{t\to\infty}(1 - H_t(a, \sqrt{t}\,b)) \geq e^{-\alpha\delta}(1 - A(a-\delta)\Phi(b+\varepsilon) - \int_0^\infty G_\delta(u)A(du)).$$

Letting $\delta \downarrow 0$ and then letting $\varepsilon \downarrow 0$, we get

$$\liminf_{t\to\infty}(1 - H_t(a, \sqrt{t}\,b)) \geq 1 - A(a)\Phi(b).$$

So we have completed the proof of Theorem 1.

5. Proof of Theorem 2

In this case we have the following representation

$$Z((t+s), a, a((t+s))b + m((t+s))) = \sum_{j=1}^{Z(t)} Z_{x_j}^{a_j}(s, a, a((t+s))b + m((t+s)))$$

where $\{(a_j, x_j); j = 1, 2, \cdots\}$ and $Z_{x_j}^{a_j}(s, a, x)$ are as defined in section 4. Now put

$$y_j(t, s, b) = \frac{a((t+s))b + m((t+s)) - x_j - m(s)}{a(s)}, \quad j = 1, \cdots, Z(t),$$

then we have the following decomposition

$$
\begin{aligned}
&Z((t+s),a,a((t+s))b+m((t+s)))\\
&= \sum_{j=1}^{Z(t)}\{Z_{x_j}^{a_j}(s,a,a((t+s))b+m((t+s)))\\
&\qquad -m^{a_j}(s,a)P(Y(s)\le y_j(t,s,b))\}\\
&\quad + \sum_{j=1}^{Z(t)} m^{a_j}(s,a)\{P(Y(s)\le y_j(t,s,b)) - F(y_j(t,s,b))\}\\
&\quad + \sum_{j=1}^{Z(t)}\{m^{a_j}(s,a)F(y_j(t,s,b)) - e^{\alpha s}n_1V(a_j)A(a)F(b)\}\\
&\quad + e^{\alpha s}n_1A(a)F(b)V_t.
\end{aligned}
$$

So we can write

$$
\begin{aligned}
&H_{(t+s)}(a,a((t+s))b+m((t+s)))\\
&= \frac{a_t(s,a,b)+b_t(s,a,b)+c_t(s,a,b)+d_tA(a)F(b)}{a_t(s,\infty,\infty)+c_t(s,\infty,\infty)+d_t},
\end{aligned}
$$

where

$$
\begin{aligned}
a_t(s,a,b) &= \frac{1}{Z(t)}\sum_{j=1}^{Z(t)}\{e^{-\alpha s}Z_{x_j}^{a_j}(s,a,a((t+s))b+m((t+s)))\\
&\qquad -e^{-\alpha s}m^{a_j}(s,a)P(Y(s)\le y_j(t,s,b))\},\\
b_t(s,a,b) &= \frac{1}{Z(t)}\sum_{j=1}^{Z(t)} e^{-\alpha s}m^{a_j}(s,a)\{P(Y(s)\le y_j(t,s,b))\\
&\qquad -F(y_j(t,s,b))\},\\
c_t(t,s,b) &= \frac{1}{Z(t)}\sum_{j=1}^{Z(t)}\{e^{-\alpha s}m^{a_j}(s,a)F(y_j(t,s,b)) - n_1A(a)F(b)V(a_j)\},\\
d_t &= \frac{n_1}{Z(t)}V_t.
\end{aligned}
$$

Again from Lemma 4 we can see that for any sequence s_n

$$
a_{n\delta}(s_n,a,b)\xrightarrow{\text{a.s.}}0,\quad a_{n\delta}(s_n,\infty,\infty)\xrightarrow{\text{a.s.}}0\quad\text{as}\quad n\to\infty
$$

and it is immediate from Polya's theorem and (2) that for any choice of sequence s_n,

$$\max\{b_{n\delta}(s_n,a,b),\ b_{n\delta}(s_n,\infty,\infty)\} \le E(M)\sup_x |P(Y(s_n)\le x)-F(x)| \to 0 \quad \text{as} \quad n\to\infty.$$

Now we show that for $s_n=(n\delta)^3-n\delta$, $c_{n\delta}(s_n,a,b)\xrightarrow{\text{a.s.}} 0$ as $n\to\infty$. Note that

(17)

$$\begin{aligned}
&\left|\frac{m(n\delta+s_n)-m(s_n)-m(n\delta)}{a(s_n)}\right| \\
&= \left|\frac{(n\delta)^{3d}L_2((n\delta)^3)-((n\delta)^3-n\delta)^dL_2((n\delta)^3-n\delta)-(n\delta)^dL_2(n\delta)}{((n\delta)^3-n\delta)^cL_1((n\delta)^3-n\delta)}\right| \\
&= \left|\frac{\frac{L_2((n\delta)^3)}{L_1((n\delta)^3-n\delta)}-(1-\frac{1}{(n\delta)^2})^d\frac{L_2((n\delta)^3-n\delta)}{L_1((n\delta)^3-n\delta)}-\frac{1}{(n\delta)^{2d}}\frac{L_2(n\delta)}{L_1((n\delta)^3-n\delta)}}{(1-\frac{1}{(n\delta)^2})^d((n\delta)^3-n\delta)^{c-d}}\right| \\
&\to 0 \quad \text{as} \quad n\to\infty
\end{aligned}$$

and that

$$(18)\qquad y_j(n\delta,s_n,b) = \frac{a((n\delta)^3)}{a(s_n)}b+\frac{m((n\delta)^3)-m(s_n)-m(n\delta)}{a(s_n)} -\frac{x_j-m(n\delta)}{a(n\delta)}\cdot\frac{a(n\delta)}{a(s_n)}.$$

Let $I_{nj}=I\{|\frac{x_j-m(n\delta)}{a(n\delta)}|\le a(n\delta)\}$ and let $J_{nj}=1-I_{nj}$ then

$$c_{n\delta}(s_n,a,b)=c^1_{n\delta}(s_n,a,b)+c^2_{n\delta}(s_n,a,b)+c^3_{n\delta}(s_n,a,b),$$

where

$$c^1_{n\delta}(s_n,a,b) = \frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} e^{-\alpha s_n}m^{a_j}(s_n,a)\{F(y_j(n\delta,s_n,b))-F(b)\}I_{nj},$$

$$c^2_{n\delta}(s_n,a,b) = \frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} e^{-\alpha s_n}m^{a_j}(s_n,a)\{F(y_j(n\delta,s_n,b))-F(b)\}J_{nj},$$

$$c^3_{n\delta}(s_n,a,b) = \frac{1}{Z(n\delta)}\sum_{j=1}^{Z(n\delta)} \{e^{-\alpha s_n}m^{a_j}(s_n,a)-n_1V(a_j)A(a)\}F(b).$$

From (17) and (18) we conclude that $c^1_{n\delta}(s_n, a, b) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. On the other hand, for any $\varepsilon > 0$,

$$\begin{aligned} P(|c^2_{n\delta}(s_n, a, b)| > \varepsilon | \mathcal{G}_{n\delta}) &\le \frac{2E(M)}{\varepsilon Z(n\delta)} E\left(\sum_{j=1}^{Z(n\delta)} J_{nj} | \mathcal{G}_{n\delta}\right) \\ &= \frac{2E(M)}{\varepsilon} P\left(\left|\frac{x_1 - m(n\delta)}{a(n\delta)}\right| \ge a(n\delta)\right) \\ &\le \frac{2E(M)}{\varepsilon (a(n\delta))^u} E\left(\left|\frac{x_1 - m(n\delta)}{a(n\delta)}\right|^u\right). \end{aligned}$$

Since $a(t) = t^c L_1(t)$ with $u > 1/c$ and since $E(|\frac{x_1 - m(n\delta)}{a(n\delta)}|^u) \to E(|Y^u|)$ as $n \to \infty$,

$$\sum_{n=0}^{\infty} P(|c^2_{n\delta}(s_n, a, b)| > \varepsilon | \mathcal{G}_{n\delta}) < \infty.$$

Hence $c^2_{n\delta}(s_n, a, b) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, by the conditional Borel-Cantelli lemma. Finally it can be easily shown from Lemma 2 that $c^3_{n\delta}(s_n, a, b) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. Since $d_{n\delta} \xrightarrow{\text{a.s.}} 1$ as $n \to \infty$ (Lemma 3) we have shown that for $a \in R^+,\ b \in R$,

$$H_{(n\delta)^3}(a, a((n\delta)^3)b + m((n\delta)^3)) \xrightarrow{\text{a.s.}} A(a)F(b) \quad \text{as} \quad n \to \infty.$$

The techniques used in the proof of Theorem 1 can be applied to prove

$$H_{n\delta}(a, a(n\delta)b + m(n\delta)) \xrightarrow{\text{a.s.}} A(a)F(b) \quad \text{as} \quad n \to \infty$$

with some modification. We use the notations $m_n, k_n, B_i,\ i = 1, 2, 3$ which are defined in the proof of Theorem 1 without any change, but we define for each $i = 1, 2, 3$

$$\begin{aligned} d^i_{k_n\delta}((n-k_n)\delta, a, b) &= \frac{1}{Z(k_n\delta)} \sum_{j=1}^{Z(k_n\delta)} e^{-\alpha(n-k_n)\delta} m^{a_j}((n-k_n)\delta, a) \\ &\quad \cdot F(y_j(k_n\delta, (n-k_n)\delta, b)) I_{a(k_n\delta)B_i + m(k_n\delta)}(x_j), \end{aligned}$$

where $\{(a_j, x_j); j = 1, \cdots, Z(k_n\delta)\}$ is the (age, position)-chart at time $k_n\delta$ and $\{y_j; j = 1, \cdots, Z(k_n\delta)\}$ is given by (16).So we have

$$\begin{aligned} &H_{n\delta}(a, a(n\delta)b + m(n\delta)) \\ &= \frac{a_{k_n\delta}((n-k_n)\delta, a, b) + b_{k_n\delta}((n-k_n)\delta, a, b) + \sum_1^3 d^i_{k_n\delta}((n-k_n)\delta, a, b)}{a_{k_n\delta}((n-k_n)\delta, \infty, \infty) + c_{k_n\delta}((n-k_n)\delta, \infty, \infty) + d_{k_n\delta}}. \end{aligned}$$

We already know that for any $\delta > 0$ and for any $a \in R^+ \cup \{\infty\}$, $b \in R \cup \{\infty\}$

$$a_{k_n\delta}((n-k_n)\delta, a, b) \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty,$$
$$b_{k_n\delta}((n-k_n)\delta, a, b) \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, since $k_n\delta = ((m_n - 1)\delta^{1/3})^3$ $c_{k_n\delta}((n-k_n)\delta, \infty, \infty) \xrightarrow{\text{a.s.}} 0$, and $d_{k_n\delta} \to 1$ (Lemma 3) as $n \to \infty$. With some notational change the arguments in the proof of Theorem 1 give us

$$\limsup_{n\to\infty} |d^3_{k_n\delta}((n-k_n)\delta, a, b)| \leq E(M)(F(b+\varepsilon) - F(b-\varepsilon)),$$
$$\limsup_{n\to\infty} |d^1_{k_n\delta}((n-k_n)\delta, a, b) - A(a)F(b-\varepsilon)| = 0,$$
$$\limsup_{n\to\infty} |d^2_{k_n\delta}((n-k_n)\delta, a, b)| = 0.$$

Letting $\varepsilon \downarrow 0$ for $a \in R^+$, $b \in R$ we have

$$H_{n\delta}(a, a(n\delta)b + m(n\delta)) \xrightarrow{\text{a.s.}} A(a)F(b) \quad \text{as} \quad n \to \infty.$$

Now the proof of Theorem 2 can be completed with the exactly same lines as that of Theorem 1 and so it is omitted.

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