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CODIMENSION REDUCTION FOR REAL SUBMANIFOLDS OF QUATERNIONIC PROJECTIVE SPACE

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ABSTRACT. In this paper we prove a reduction theorem of the codimension for real submanifold of quaternionic projective space as a quaternionic analogue corresponding to those in Cecil [4], Erbacher [5] and Okumura [9], and apply the theorem to quaternionic CR-submanifold of quaternionic projective space.

1. Introduction

In general, it is very hard to classify submanifolds immersed in a Riemannian manifold even though the ambient manifold is specified, and so the so-called codimension reduction problem is sometimes very important role in the theory of submanifolds.

The codimension reduction problem was investigated by Allendoerfer [1] in the case that the ambient manifold is a Euclidean space and by Erbacher [5] in the case that the ambient manifold is a real space form. On the other hand, as a complex analogue for submanifold of complex projective space, Cecil [4] proved a codimension reduction theorem for complex submanifold and Okumura [9] a theorem corresponding to those in [4] and [5] for real submanifold.

In this paper we prove a quaternionic analogue for real submanifold of quaternionic projective space which may correspond to those in [4], [5] and [9]. We mainly follow Okumura's method in his paper [9].

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2. Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting with tensor fields of type (1,1) over $\overline{M}$ satisfying the following conditions (a), (b) and (c):

(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$(2.1)\qquad \begin{cases} F^2 = -I, \ G^2 = -I, \ H^2 = -I, \\ FG = -GF = H, \ GH = -HG = F, \ HF = -FH = G. \end{cases}$$

(b) There is a Riemannian metric g which satisfies the Hermitian property with respect to all of F, G and H.

(c) For the Levi-Civita connection $\overline{\nabla}$ with respect to g

$$(2.2)\qquad \begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in $\overline{\mathcal{U}}$(cf. [6,7]).

For canonical local bases $\{F, G, H\}$ and $\{'F, 'G, 'H\}$ of V in coordinate neighborhoods $\overline{\mathcal{U}}$ and $'\overline{\mathcal{U}}$ respectively, it follows from (2.1) that in $'\overline{\mathcal{U}} \cap \overline{\mathcal{U}}$

$$\begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix} = (s_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \qquad (x,\ y = 1,\ 2,\ 3)$$

with differentiable functions s_{xy}, where the matrix $S = (s_{xy})$ is contained in $SO(3)$. As is well known, every quaternionic Kähler manifold is orientable(cf. [6,7]).

Let M be an n-dimensional submanifold isometrically immersed in $\overline{M}$ and let i the isometric immersion. Then, for any tangent vector field X and normal vector field ξ to M, we have the following decompositions in tangential and normal components (In what follows we will delete i and its differential i_* in our notation):

$$(2.4)\ FX = \phi X + u(X), \ GX = \psi X + v(X), \ HX = \theta X + w(X),$$

$$(2.5)\quad F\xi = -U_\xi + P_F\xi, \ G\xi = -V_\xi + P_G\xi, \ H\xi = -W_\xi + P_H\xi.$$

Then ϕ, ψ and θ are skew-symmetric endomorphisms acting on the tangent bundle TM, and P_F, P_G and P_H define those on the normal bundle $TM^\perp$. Also u, v and w are normal bundle valued 1-forms on TM. It is easily verified that

(2.6) $$g(X,U_\xi)=g(u(X),\xi),\quad g(X,V_\xi)=g(v(X),\xi),$$
$$g(X,W_\xi)=g(w(X),\xi)$$

for any $X\in TM$, $\xi\in TM^\perp$, where and in the sequel we denote the induced metric form that of $\overline{M}$ by the same letter g. Applying F to the first equation of (2.4) and using (2.1), (2.4) and (2.5), we have

$$\phi^2X=-X+U_{u(X)},\quad u(\phi X)=-P_Fu(X).$$

Similarly we have

(2.7)
$$\phi^2X=-X+U_{u(X)},\ \psi^2X=-X+V_{v(X)},\ \theta^2X=-X+W_{w(X)},$$
(2.8)
$$u(\phi X)=-P_Fu(X),\ v(\psi X)=-P_Gv(X),\ w(\theta X)=-P_Hw(X).$$

Next, applying G and H to the first equation of (2.4), respectively and using (2.1), (2.4) and (2.5), we have

$$\theta X+w(X)=-\psi(\phi X)-v(\phi X)+V_{u(X)}-P_Gu(X),$$
$$\psi X+v(X)=\theta(\phi X)+w(\phi X)-W_{u(x)}+P_Hu(X),$$

and consequently

(2.9) $$\psi\phi X=-\theta X+V_{u(x)},\quad v(\phi X)+P_Gu(X)=-w(X),$$
(2.10) $$\theta\phi X=\psi X+W_{u(x)},\quad w(\phi X)+P_Hu(X)=v(X).$$

Similarly we have from the other equations of (2.4)

(2.11) $$\phi\psi X=\theta X+U_{v(X)},\quad u(\psi X)+P_Fv(X)=w(X),$$
(2.12) $$\theta\psi X=-\phi X+W_{v(x)},\quad w(\psi X)+P_Hv(X)=-u(X),$$
(2.13) $$\phi\theta X=-\psi X+U_{w(x)},\quad u(\theta X)+P_Fw(X)=-v(X),$$
(2.14) $$\psi\theta X=\phi X+V_{w(x)},\quad v(\theta X)+P_Gw(X)=u(X).$$

By the quite similar method as above, we have from (2.5) that

$$P_F{}^2\xi = -\xi + u(U_\xi), \quad P_G{}^2\xi = -\xi + v(V_\xi), \quad P_H{}^2\xi = -\xi + w(W_\xi), \tag{2.15}$$

$$\phi(U_\xi) = -U_{P_F\xi}, \ \psi(V_\xi) = -V_{P_G\xi}, \ \theta(W_\xi) = -W_{P_H\xi}, \tag{2.16}$$

$$W_\xi = -V_{P_F\xi} - \psi U_\xi, \quad P_G P_F\xi = -P_H\xi + v(U_\xi), \tag{2.17}$$

$$V_\xi = W_{P_F\xi} + \theta U_\xi, \quad P_H P_F\xi = P_G\xi + w(U_\xi), \tag{2.18}$$

$$W_\xi = U_{P_G\xi} + \phi V_\xi, \quad P_F P_G\xi = P_H\xi + u(V_\xi), \tag{2.19}$$

$$U_\xi = -W_{P_G\xi} - \theta V_\xi, \quad P_H P_G\xi = -P_F\xi + w(V_\xi), \tag{2.20}$$

$$V_\xi = -U_{P_H\xi} - \phi W_\xi, \quad P_F P_H\xi = -P_G\xi + u(W_\xi), \tag{2.21}$$

$$U_\xi = V_{P_H\xi} + \psi W_\xi, \quad P_G P_H\xi = P_F\xi + v(W_\xi). \tag{2.22}$$

We denote by ∇ the Levi-Civita connection of M and by $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ to $TM^\perp$. Then they are related by the Gauss and Weingarten equations (In what follows we will again delete i and its differential i_* in our notation):

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.23}$$

$$\overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.24}$$

where h is the second fundamental form and A_ξ the shape operator with respect to the normal vector field ξ.

Differentiating the first equation of (2.4) covariantly and using (2.2), (2.4), (2.5), (2.23) and (2.24), we have

$$\begin{aligned} (\nabla_Y\phi)X &= r(Y)\psi X - q(Y)\theta X - U_{h(Y,X)} + A_{u(X)}Y, \\ (\overset{\star}{\nabla}_Y u)X &= r(Y)v(X) - q(Y)w(X) - h(Y, \phi X) + P_F h(Y, X), \end{aligned} \tag{2.25}$$

where $(\overset{\star}{\nabla}_Y u)X$ is defined by $(\overset{\star}{\nabla}_Y u)X = \nabla^\perp_Y u(X) - u(\nabla_Y X)$.

Similarly, from the other equations of (2.4), we have

$$\begin{aligned} (\nabla_Y\psi)X &= p(Y)\theta X - r(Y)\phi X - V_{h(Y,X)} + A_{v(X)}Y, \\ (\overset{\star}{\nabla}_Y v)X &= p(Y)w(X) - r(Y)u(X) - h(Y, \psi X) + P_G h(Y, X), \end{aligned} \tag{2.26}$$

$$(2.27)\quad \begin{aligned} &(\nabla_Y \theta)X = q(Y)\phi X - p(Y)\psi X - W_{h(Y,X)} + A_{w(X)}Y, \\ &(\overset{\star}{\nabla}_Y w)X = q(Y)u(X) - p(Y)v(X) - h(Y, \theta X) + P_H h(Y, X), \end{aligned}$$

where $(\overset{\star}{\nabla}_Y v)X = \nabla^{\perp}_Y v(X) - v(\nabla_Y X)$ and $(\overset{\star}{\nabla}_Y w)X = \nabla^{\perp}_Y w(X) - w(\nabla_Y X)$.

Next, differentiating the first equation of (2.5) covariantly and making use of (2.2), (2.4), (2.5), (2.23) and (2.24), we have

$$(2.28)\quad \begin{aligned} &\nabla_Y U_\xi = r(Y)V_\xi - q(Y)W_\xi + \phi A_\xi Y - A_{P_F \xi}Y + U_{\nabla^{\perp}_Y \xi}, \\ &(\nabla^{\perp}_Y P_F)\xi = r(Y)P_G \xi - q(Y)P_H \xi - u(A_\xi Y) + h(Y, U_\xi), \end{aligned}$$

where $(\nabla^{\perp}_Y P_F)\xi$ is defined by $(\nabla^{\perp}_Y P_F)\xi = \nabla^{\perp}_Y (P_F \xi) - P_F(\nabla^{\perp}_Y \xi)$.

Similarly, from the other equations of (2.5), we have

$$(2.29)\quad \begin{aligned} &\nabla_Y V_\xi = -r(Y)U_\xi + p(Y)W_\xi + \psi A_\xi Y - A_{P_G \xi}Y + V_{\nabla^{\perp}_Y \xi}, \\ &(\nabla^{\perp}_Y P_G)\xi = -r(Y)P_F \xi + p(Y)P_H \xi - v(A_\xi Y) + h(Y, V_\xi), \end{aligned}$$

$$(2.30)\quad \begin{aligned} &\nabla_Y W_\xi = q(Y)U_\xi - p(Y)V_\xi + \theta A_\xi Y - A_{P_H \xi}Y + W_{\nabla^{\perp}_Y \xi}, \\ &(\nabla^{\perp}_Y P_H)\xi = q(Y)P_F \xi - p(Y)P_G \xi - w(A_\xi Y) + h(Y, W_\xi), \end{aligned}$$

where $(\nabla^{\perp}_Y P_G)\xi = \nabla^{\perp}_Y (P_G \xi) - P_G(\nabla^{\perp}_Y \xi)$ and $(\nabla^{\perp}_Y P_H)\xi = \nabla^{\perp}_Y (P_H \xi) - p_H(\nabla^{\perp}_Y \xi)$.

3. Quaternionically holomorphic first normal space

Let $N_0(x) := \{\xi \in T_x M^{\perp} \mid A_\xi = 0\}$. The first normal space $N_1(x)$ is defined to be the orthogonal complement to $N_0(x)$ in $T_x M^{\perp}$. We put

$$H_0(x) := N_0(x) \cap FN_0(x) \cap GN_0(x) \cap HN_0(x).$$

Then $H_0(x)$ is the maximal quaternionically invariant(or briefly Q-invariant) subspace of $N_0(x)$, that is,

$$FH_0(x) \subset H_0(x), \; GH_0(x) \subset H_0(x), \; HH_0(x) \subset H_0(x).$$

Since F, G and H are isomorphisms, it is clear that

$$FH_0(x) = H_0(x), \quad GH_0(x) = H_0(x), \quad HH_0(x) = H_0(x).$$

Taking account of (2.5), we can easily verify

LEMMA 3.1. *For any $\xi \in H_0(x)$, we have*

$$A_\xi = 0 \quad \text{and} \quad U_\xi = V_\xi = W_\xi = 0.$$

DEFINITION. The *quaternionically holomorphic* (or *Q-holomorphic*) *first normal space* $H_1(x)$ is the orthogonal complement of $H_0(x)$ in $T_xM^\perp$.

By definition, it is clear that $N_1(x) \subset H_1(x)$ in $T_xM^\perp$. Moreover we have

LEMMA 3.2. *If M is a Q-invariant submanifold of a quaternionic Kähler manifold, then $H_1(x) = N_1(x)$.*

Proof. Since $H_1(x)$ and $N_1(x)$ are the orthogonal complements of $H_0(x)$ and $N_0(x)$, respectively, we have only to show that $H_0(x) = N_0(x)$. Since $T_xM^\perp$ is Q-invariant, it follows from (2.2), (2.23) and (2.24) that

$$\begin{aligned}\overline{\nabla}_X(F\xi) &= r(X)G\xi - q(X)H\xi + F(\nabla^\perp_X\xi - A_\xi X)\\ &= -A_{F\xi}X + \nabla^\perp_X(F\xi)\end{aligned}$$

and consequently $A_{F\xi}X = FA_\xi X$. Similarly we have $A_{G\xi}X = GA_\xi X$ and $A_{H\xi}X = HA_\xi X$. Thus, if $\xi \in N_0(x)$, then

$$A_{F\xi} = 0 \text{ and } \xi \in FN_0(x), \quad A_{G\xi} = 0 \text{ and } \xi \in GN_0(x),$$
$$A_{H\xi} = 0 \text{ and } \xi \in HN_0(x).$$

This shows that $\xi \in N_0(x)$ implies $\xi \in H_0(x)$, which completes the proof. □

LEMMA 3.3. *Let $H(x)$ be a Q-invariant subspace of $H_0(x)$ and $H_2(x)$ its orthogonal complement in $T_xM^\perp$. Then $T_xM \oplus H_2(x)$ is a Q-invariant subspace of $T_x\overline{M}$.*

Proof. We first note that

$$T_x\overline{M} = T_xM \oplus H_2(x) \oplus H(x).$$

Since $FH(x) = H(x)$, for any $\xi \in H(x)$ there exists $\eta \in H(x)$ such that $F\eta = \xi$. Now let $Z \in T_xM \oplus H_2(x)$. Then for any $\xi \in H(x)$, $\langle FZ, \xi\rangle = \langle Z, \eta\rangle = 0$. This means that $FZ \in T_xM \oplus H_2(x)$. By quite similar method, we can verify that $T_xM \oplus H_2(x)$ is Q-invariant. This completes the proof. □

Now we recall that an $(n+p+3)$-dimensional sphere S^{n+p+3} of radius 1 in a Euclidean $(n+p+4)$-space is a principal S^3-bundle over $QP^{\frac{n+p}{4}}$. Then the Hopf-fibration $\tilde{\pi} : S^{n+p+3} \to QP^{\frac{n+p}{4}}$ defines a Riemannian submersion. We construct the S^3-bundle over the submanifold M in such a way that the diagram

$$\begin{array}{ccc} \pi^{-1}(M) & \xrightarrow{\ \tilde{i}\ } & S^{n+p+3}(1) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ M & \xrightarrow{\ i\ } & QP^{\frac{n+p}{4}} \end{array}$$

is commutative (i, $\tilde{i}$ being the isometric immersions). We denote by X^* the horizontal lift of $X \in TM$ and by ξ^* that of the normal vector field $\xi \in TM^\perp$. We put

$$N'_0(x') = \{\xi' \in T_{x'}\pi^{-1}(M)^\perp \mid A'_{\xi'} = 0\}, \quad x' \in \pi^{-1}(M),$$

where $A'_{\xi'}$ denotes the shape operator with respect to the normal vector field ξ' to $\pi^{-1}(M)$. Then, as shown in [8], we have

$$N'_0(x') = \{\xi^* \mid A_\xi = 0,\ U_\xi = V_\xi = W_\xi = 0\}. \tag{3.1}$$

Applying the reduction theorem due to Erbacher [5], we prove

THEOREM 3.4. *Let M be an n-dimensional real submanifold of a real $(n+p)$-dimensional quaternionic projective space $QP^{\frac{n+p}{4}}$ and let $H(x)$ a Q-invariant subspace of $H_0(x)$. If the orthogonal complement $H_2(x)$ of $H(x)$ in $T_xM^\perp$ is invariant under parallel translation with respect to the normal connection and q is the constant dimension of H_2, then there exists a real $(n+q)$-dimensional totally geodesic quaternionic projective subspace $QP^{\frac{n+q}{4}}$ such that $M \subset QP^{\frac{n+q}{4}}$.*

Proof. Let $\xi \in H(x)$. Then $\xi \in H_0(x)$, which and Lemma 3.1 give

$$A_\xi = 0 \quad \text{and} \quad U_\xi = V_\xi = W_\xi = 0$$

and consequently $A'_{\xi^*} = 0$ because of (3.1). This means that for a point x' with $\pi(x') = x$

$$H(x)^* = \{\xi^* \mid \xi \in H(x)\} \subset N'_0(x').$$

Hence the orthogonal complement $H_2(x)^* = \{\xi^* \mid \xi \in H_2(x)\}$ of $H(x)^*$ in $T_{x'}(\pi^{-1}(M))^\perp$ is a subspace of $T_{x'}(\pi^{-1}(M))^\perp$ such that $H'_1(x') \subset H_2(x)^*$. Since $H_2(x)$ is invariant under parallel translation with respect to the normal connection, so does $H(x)$. This shows that for any $\xi \in H(x)$, $\nabla^\perp_X \xi \in H(x)$, which and

$$\nabla'^\perp_{X^*}\xi^* = (\nabla^\perp_X\xi)^* \in H(x)^*, \quad \nabla'^\perp_{V_\xi}\xi^* = -(F\xi)^* \in H(x)^*,$$
$$\nabla'^\perp_{V_\eta}\xi^* = -(G\xi)^* \in H(x)^*, \quad \nabla'^\perp_{V_\zeta}\xi^* = -(H\xi)^* \in H(x)^*$$

imply that $H(x)^*$ is invariant under parallel translation with respect to the normal connection $\nabla'^\perp$ of $\pi^{-1}(M)$. From the reduction theorem ([5], p. 339), we know that there exists a totally geodesic submanifold S^{n+q+3} such that $\pi^{-1}(M) \subset S^{n+q+3}$. Let $\hat{\mathcal{U}}(x')$ be a neighborhood of a point x' with $\pi(x') = x$. Then the tangent space $T_{y'}(S^{n+q+3})$ of the totally geodesic submanifold at $y' \in \hat{\mathcal{U}}(x')$ is

$$T_{y'}(\pi^{-1}(M)) \oplus H_2(y)^* = (T_yM \oplus H_2(y))^* \oplus T_{y'}(\pi^{-1}(y)),$$

where $y = \pi(y')$. Since S^{n+q+3} is totally geodesic in S^{n+p+3}, the maximal integral submanifold S^3 of the distribution $y' \mapsto T_{y'}(\pi^{-1}(y))$ is a 3-dimensional great sphere of S^{n+p+3}. Hence the Hopf-fibration $S^{n+q+3} \to QP^{\frac{n+q}{4}}$ by S^3 is compatible with the Hopf-fibration $\tilde{\pi}$: $S^{n+p+3} \to QP^{\frac{n+p}{4}}$ and the tangent space of $QP^{\frac{n+q}{4}}$ at x is $T_xM \oplus H_2(x)$. Moreover, by Lemma 3.3, $QP^{\frac{n+q}{4}}$ is Q-invariant in $QP^{\frac{n+p}{4}}$. This completes the proof. □

For a Q-invariant submanifold, by Lemma 3.2 we see that $H_0(x) = N_0(x)$ at any x in M. Thus we have

COROLLARY 3.5. *Let M be a real n-dimensional Q-invariant submanifold of $QP^{\frac{n+p}{4}}$. Assume that a Q-invariant subspace of the first normal space $N_1(x)$ has constant dimension q and is invariant under parallel translation with respect to the normal connection. Then there exists a totally geodesic real $(n+q)$-dimensional quaternionic projective space $QP^{\frac{n+q}{4}}$ such that $M \subset QP^{\frac{n+q}{4}}$.*

4. Quaternionic CR-submanifolds

In this section, let M be an n-dimensional real submanifold of a quaternionic Kähler manifold, If there is a Q-invariant distribution $\mathcal{D}: x \mapsto \mathcal{D}_x \subset T_xM$ such that its complementary orthogonal distribution $\mathcal{D}^\perp : x \mapsto \mathcal{D}_x^\perp$ in TM is anti-quaternionic, that is,

$$F\mathcal{D}_x^\perp \subset T_xM^\perp, \quad G\mathcal{D}_x^\perp \subset T_xM^\perp, \quad H\mathcal{D}_x^\perp \subset T_xM^\perp,$$

then M is called a *quaternionic CR-submanifold* ([2,3]). In particular, if $dim\mathcal{D}_x = 0$ for any x in M, the quaternionic CR-submanifold is called an *anti-quaternionic submanifold* ([3,10]).

Let M be a quaternionic CR-submanifold of a quaternionic Kähler manifold $\overline{M}$. Then, by definition, the tangent space $T_x\overline{M}$ at x in M is decomposed as

$$T_x\overline{M} = T_xM \oplus F\mathcal{D}_x^\perp \oplus G\mathcal{D}_x^\perp \oplus H\mathcal{D}_x^\perp \oplus N_xM, \tag{4.1}$$

where N_xM is the orthogonal complement of $F\mathcal{D}_x^\perp \oplus G\mathcal{D}_x^\perp \oplus H\mathcal{D}_x^\perp$ in $T_xM^\perp$.

LEMMA 4.1. *N_xM is a Q-invariant, that is,*

$$FN_xM \subset N_xM, \quad GN_xM \subset N_xM, \quad HN_xM \subset N_xM.$$

Proof. Let $X \in T_xM \oplus F\mathcal{D}_x^\perp \oplus G\mathcal{D}_x^\perp \oplus H\mathcal{D}_x^\perp$ and $\xi \in N_xM$. Since X is decomposed as

$$X = X_1 + X_2 + FY_1 + GY_2 + HY_3$$

for some $X_1 \in \mathcal{D}_x$ and $X_2, Y_1, Y_2, Y_3 \in \mathcal{D}_x^\perp$, it is clear that

$$\begin{aligned}\langle X, F\xi\rangle &= -\langle FX, \xi\rangle \\ &= -\langle FX_1, \xi\rangle - \langle FX_2, \xi\rangle + \langle Y_1, \xi\rangle + \langle Y_2, \xi\rangle + \langle Y_3, \xi\rangle \\ &= 0.\end{aligned}$$

Similarly we have $\langle X, G\xi\rangle = \langle X, H\xi\rangle = 0$, and consequently

$$FN_xM \subset N_xM, \quad GN_xM \subset N_xM, \quad HN_xM \subset N_xM.$$

This completes the proof. □

LEMMA 4.2. *Assume that NM is invariant under parallel translation with respect to the normal connection. Then, for any $\xi \in NM$ and $\eta \in TM^\perp$,*

$$A_\xi U_\eta = 0, \quad A_\xi V_\eta = 0, \quad A_\xi W_\eta = 0.$$

Proof. By means of Lemma 4.1 and our assumption, it follows that for any $\xi \in NM$

$$\begin{aligned}&F\xi = P_F\xi, \quad G\xi = P_G\xi, \quad H\xi = P_H\xi, \quad \nabla^\perp_X\xi, \\ &F\nabla^\perp_X\xi = P_F\nabla^\perp_X\xi, \quad G\nabla^\perp_X\xi = P_G\nabla^\perp_X\xi, \quad H\nabla^\perp_X\xi = P_H\nabla^\perp_X\xi\end{aligned}$$

are all contained in NM. Differentiating the first three equations of those covariantly, we have

$$\begin{aligned}\overline{\nabla}_X(F\xi) &= \overline{\nabla}_X(P_F\xi) = -A_{P_F\xi}X + \nabla^\perp_X(P_F\xi), \\ \overline{\nabla}_X(G\xi) &= \overline{\nabla}_X(P_G\xi) = -A_{P_G\xi}X + \nabla^\perp_X(P_G\xi), \\ \overline{\nabla}_X(H\xi) &= \overline{\nabla}_X(P_H\xi) = -A_{P_H\xi}X + \nabla^\perp_X(P_H\xi).\end{aligned} \tag{4.2}$$

Also we have

$$\begin{aligned}\overline{\nabla}_X(F\xi) &= r(X)P_G\xi - q(X)P_H\xi - \phi A_\xi X - u(A_\xi X) + P_F\nabla^\perp_X\xi, \\ \overline{\nabla}_X(G\xi) &= p(X)P_H\xi - r(X)P_F\xi - \psi A_\xi X - v(A_\xi X) + P_G\nabla^\perp_X\xi, \\ \overline{\nabla}_X(H\xi) &= q(X)P_F\xi - p(X)P_G\xi - \theta A_\xi X - w(A_\xi X) + P_H\nabla^\perp_X\xi.\end{aligned} \tag{4.3}$$

We notice that $U_\zeta = V_\zeta = W_\zeta = 0$ for any $\zeta \in NM$. Consequently (2.6) implies

$$u(X),\ v(X),\ w(X) \in F\mathcal{D}^\perp \oplus G\mathcal{D}^\perp \oplus H\mathcal{D}^\perp$$

for any X in TM. Comparing the normal parts of (4.2) and (4.3), we have

$$u(A_\xi X) = 0, \quad v(A_\xi X) = 0, \quad w(A_\xi X) = 0.$$

Thus for any $\eta \in TM^\perp$

$$g(A_\xi U_\eta, X) = 0, \quad g(A_\xi V_\eta, X) = 0, \quad g(A_\xi W_\eta, X) = 0$$

and consequently $A_\xi U_\eta = A_\xi V_\eta = A_\xi W_\eta = 0$. This completes the proof. □

THEOREM 4.3. *Let M be an n-dimensional anti-quaternionic submanifold of $QP^{\frac{n+p}{4}}$. If NM is invariant under parallel translation with respect to the normal connection, then there exists a real $4n$-dimensional totally geodesic quaternionic projective space QP^n of $QP^{\frac{n+p}{4}}$ such that M is an anti-quaternionic submanifold of QP^n.*

Proof. Since M is anti-quaternionic, the tangential parts of (4.3) vanish identically. Comparing the tangential parts of (4.2) and (4.3), we have

$$A_{P_F\xi} = 0, \quad A_{P_G\xi} = 0, \quad A_{P_H\xi} = 0$$

for all ξ in NM. But, in an anti-quaternionic submanifold, P_F, P_G and P_H are all isomorphisms on NM and consequently $A_\xi = 0$ for any ξ in NM. Thus by means of Lemma 4.1 $NM \subset H_0M$.

Conversely, let $\xi \in H_0(x)$. Then for any $X, Y_1, Y_2, Y_3 \in T_xM$, we have

$$\langle \xi, X + FY_1 + GY_2 + HY_3 \rangle = 0$$

since $H_0(x)$ is Q-invariant. Thus ξ belongs to the orthogonal complement of $T_xM \oplus FT_xM \oplus GT_xM \oplus HT_xM$, that is, $\xi \in N_xM$. Hence $NM = H_0M$ and consequently $FT_xM \oplus GT_xM \oplus HT_xM$ is the Q-holomorphic first normal space. Applying Theorem 3.4, we conclude that there is a real $4n$-dimansional totally geodesic quaternionic projective space QP^n of $QP^{\frac{n+p}{4}}$ such that M is anti-quaternionic in QP^n.□

In [8] and [11] it was already proved that the normal connection of $\pi^{-1}(M)$ in S^{n+p+3} is flat if and only if the following conditions are satisfied on M:

$$\text{(4.4)} \quad \begin{aligned} &\text{(a)} \quad R^{\perp}(X,Y)\xi = -2g(\phi X,Y)P_F\xi - 2g(\psi X,Y)P_G\xi \\ &\qquad\qquad\qquad - 2g(\theta X,Y)P_H\xi, \end{aligned}$$

(b) The structure induced on the normal bundle is parallel (for the definition, see [11]).

In this sense the normal connection of M in $QP^{\frac{n+p}{4}}$ is said to be *lift flat* if the conditions (a) and (b) are valid.

LEMMA 4.4. *Let M be a quaternionic CR-submanifold of $QP^{\frac{n+p}{4}}$ with lift flat normal connection. Then $A_\xi A_\eta = A_\eta A_\xi$ for $\xi \in N_xM$ and $\eta \in T_xM^{\perp}$.*

Proof. Since the normal connection is lift flat, the equation of Ricci and (4.4) (a) implies

$$\begin{aligned} 0 =& h(A_\xi X,Y) - h(A_\xi Y,X) + g(Y,U_\xi)u(X) - g(X,U_\xi)u(Y) \\ &+ g(Y,V_\xi)v(X) - g(X,V_\xi)v(Y) + g(Y,W_\xi)w(X) - g(X,W_\xi)w(Y). \end{aligned}$$

In particular, for $\xi \in N_xM$, $U_\xi = V_\xi = W_\xi = 0$, and consequently

$$h(A_\xi X,Y) = h(A_\xi Y,X). \tag{4.5}$$

Hence, if $\xi \in N_xM$ and $\eta \in T_xM^{\perp}$, we have

$$\begin{aligned} &g((A_\eta A_\xi - A_\xi A_\eta)X,Y) \\ &= g(A_\eta A_\xi X,Y) - g(A_\xi A_\eta X,Y) \\ &= g(h(A_\xi X,Y),\eta) - g(h(A_\xi Y,X),\eta) = 0 \end{aligned}$$

because of (4.5). This completes the proof. □

THEOREM 4.5. *Assume that the normal connection of a quaternionic CR-submanifold M of $QP^{\frac{n+p}{4}}$ is lift flat and that NM is invariant under parallel translation with respect to the normal connection. Then there is a totally geodesic quaternionic projective space $QP^{\frac{n+q}{4}}$ such that M is a quaternionic CR-submanifold of the quaternionic projective space.*

Proof. By means of Theorem 4.3, it suffices to show that $NM = H_0M$. We choose orthonormal normal vector fields $\xi_1, \ldots, \xi_p$ in such a way that

$$\xi_1, \ldots, \xi_q \in F\mathcal{D}^\perp \oplus G\mathcal{D}^\perp \oplus H\mathcal{D}^\perp, \quad \xi_{q+1}, \ldots, \xi_p \in NM$$

(q must be a multiple of three) and denote by A_α the shape operator for ξ_α. Since NM is not only invariant under parallel translation with respect to the normal connection, but also Q-invariant, it follows that

$$\nabla^\perp_X \xi_\alpha = \sum_{\lambda=q+1}^{p} s_{\alpha\lambda}(X)\xi_\lambda, \quad \alpha = q+1, \ldots, p, \tag{4.6}$$

$$\begin{cases} F\xi_\alpha = P_F\xi_\alpha = \sum_{\lambda=q+1}^{p}(P_F)_{\alpha\lambda}\xi_\lambda, \\ G\xi_\alpha = P_G\xi_\alpha = \sum_{\lambda=q+1}^{p}(P_G)_{\alpha\lambda}\xi_\lambda, \quad \alpha = q+1, \ldots, p, \\ H\xi_\alpha = P_H\xi_\alpha = \sum_{\lambda=q+1}^{p}(P_H)_{\alpha\lambda}\xi_\lambda, \end{cases} \tag{4.7}$$

from which we have

$$\begin{cases} F\nabla^\perp_X\xi_\alpha = \sum_{\lambda,\mu=q+1}^{p} s_{\alpha\lambda}(X)(P_F)_{\lambda\mu}\xi_\mu, \\ G\nabla^\perp_X\xi_\alpha = \sum_{\lambda,\mu=q+1}^{p} s_{\alpha\lambda}(X)(P_G)_{\lambda\mu}\xi_\mu, \ \alpha = q+1, \ldots, p. \\ H\nabla^\perp_X\xi_\alpha = \sum_{\lambda,\mu=q+1}^{p} s_{\alpha\lambda}(X)(P_H)_{\lambda\mu}\xi_\mu, \end{cases} \tag{4.8}$$

On the other hand, comparing the tangential parts of (4.2) and (4.3), and using (4.7), we obtain

$$\begin{cases} \phi A_\lambda X = \sum_{\mu=q+1}^{p}(P_F)_{\lambda\mu}A_\mu X, \\ \psi A_\lambda X = \sum_{\mu=q+1}^{p}(P_G)_{\lambda\mu}A_\mu X, \quad \lambda \geq q+1. \\ \theta A_\lambda X = \sum_{\mu=q+1}^{p}(P_H)_{\lambda\mu}A_\mu X, \end{cases} \tag{4.9}$$

Substituting $A_\lambda X$ for X in (4.9) and summing over $\lambda = q+1, \ldots, p$, we have

$$\phi \sum_{\lambda=q+1}^{P} {A_\lambda}^2 X = \sum_{\lambda,\mu=q+1}^{p} (P_F)_{\lambda\mu}A_\lambda A_\mu X = 0$$

because $(P_F)_{\lambda\mu}$ is skew-symmetric with respect to λ and μ, but $A_\lambda A_\mu = A_\mu A_\lambda$ by Lemma 4.4. Thus we have

$$\phi^2 \sum_{\lambda=q+1}^{p} {A_\lambda}^2 X = -\sum_{\lambda=q+1}^{p} {A_\lambda}^2 X + \sum_{\alpha=q+1}^{p} U_{u({A_\alpha}^2 X)} = 0.$$

However, as already shown in the proof of Lemma 4.2, $u({A_\alpha}^2 X) = 0$ and consequently $\sum_{\lambda=q+1}^{n} {A_\lambda}^2 X = 0$, that is, $A_\lambda = 0$, $\lambda \geq q+1$. Thus $N_x M$ is a Q-invariant subspace of $N_0 M$. Since $H_0 M$ is maximal, it follows that $N_x M \subset H_0 M$. Let $\xi \in H_0 M$ and $\eta \in F\mathcal{D}^\perp \oplus G\mathcal{D}^\perp \oplus H\mathcal{D}^\perp$. Then there exist $Y_1, Y_2, Y_3 \in \mathcal{D}^\perp \subset TM$ such that

$$\eta = FY_1 + GY_2 + HY_3.$$

Therefore it is clear that

$$\langle \xi, \eta \rangle = -\langle F\xi, Y_1 \rangle - \langle G\xi, Y_2 \rangle - \langle H\xi, Y_3 \rangle = 0$$

since $H_0 M$ is Q-invariant. This means that $\xi \in NM$ and consequently $NM = H_0 M$. This completes the proof. □

REMARK. As already shown in the proof of Lemma 4.4, it suffices to assume only the condition (4.4)(a) instead of the condition "lift flatness" in order to prove Theorem 4.5.

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