J. Korean Math. Soc. **36** (1999), No. 1, pp. 51–71

FLYPES OF CLOSED 3-BRAIDS IN THE STANDARD CONTACT SPACE

KI HYOUNG KO AND SANG JIN LEE

ABSTRACT. We classify all conjugacy classes of 3-braids that are related by flypes on representatives. Among them we determine which classes have representatives that admit both (+) and (−)-flypes as an effort to search for a potential example of a pair of transversal knots that are topologically isotopic and have the same Bennequin number but are not transversally isotopic.

1. Introduction

The *standard contact structure* Δ on $\mathbf{R}^3$ is the field of two-dimensional tangent planes determined by the differential 1-form $dz + r^2 d\theta$, where (r, θ, z) is the cylindrical coordinate for the 3-space. Then we can consider two special types of knots: *Legendrian knots* which are everywhere tangent to the contact planes, and *transversal knots* which are nowhere tangent to them. It is natural to classify Legendrian and transversal knots up to Legendrian and transversal isotopies and compare those isotopies with the usual topological isotopy.

For Legendrian knots one introduces two integer-valued Legendrian isotopy invariants. The first measures the rotation of a (oriented) knot with respect to the contact structure — we call it the *Maslov number.* The second one, which we call the *Bennequin number*, is defined as the contact self-linking number of the knot. Transversal knots have no Maslov numbers, but also have Bennequin numbers.

These kinds of knots have become very popular in contact geometry since the seminal work of Bennequin[1], published in 1983. The aim of that paper was to prove the existence of contact structures not isomorphic to the standard contact structure. It is based on two inequalities

Received September 7, 1997.
1991 Mathematics Subject Classification: Primary 57M25; Secondary 57M05.
Key words and phrases: flype, 3-braid, contact structure, transversal knot.

between the above two invariants and the Euler characteristic of incompressible spanning surfaces. To prove the main inequalities he reduced them to problems of knots and braids: He proved that any transversal knot can be deformed through transversal isotopies to a closed braid representative relative to the z-axis. He then proceeded to study certain incompressible spanning surfaces which are bounded by these closed braids, using the foliation on these surface which are induced by the Reeb foliation.

Since the work of Bennequin, many mathematicians studied the Legendrian and transversal knots in standard or general contact manifolds. One of the most exciting problems in the theory of transversal (oriented Legendrian) knots is the following.

QUESTION. Are topologically isotopic transversal (oriented Legendrian, respectively) knots with equal Bennequin numbers (Bennequin and Maslov numbers) transversally (oriented Legendrian) isotopic?

There are many evidences that this question may have affirmative answer or it is hard to disprove. For examples, see the followings:

THEOREM 1 (Eliashberg [6]). *Transversal (oriented Legendrian, respectively) knots with equal Bennequin numbers (Bennequin and Maslov numbers), which are topologically isotopic to the trivial knot, are transversally (oriented Legendrian) isotopic.*

THEOREM 2 (Fuchs-Tabachnikov [7]). *Topologically isotopic Legendrian knots with equal Bennequin and Maslov numbers can not be distinguished by finite-order Legendrian knot invariants.*

THEOREM 3 (Swiakowski [12]). *Two Legendrian knots with generic nonintersecting fronts are Legendrian isotopic if and only if they have same Bennequin and Maslov numbers.*

The closed braids are closely related to the transversal and Legendrian knots. And the links with braid index ≤ 3 are classified completely via closed 3-braid representatives by Birman-Menasco [3] (see Theorem 4) and the conjugacy problem in the braid groups has algorithmic solutions [4, 5, 8, 9, 10, 13]. So it is worthwhile to check whether the question is affirmative or not for the closed 3-braids.

The classical presentation of the 3-braid group B_3 is the Artin's presentation, $\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. In this article we use the *band generator presentation* of Xu [13], Kang-Ko-Lee [10] and Birman-Ko-Lee

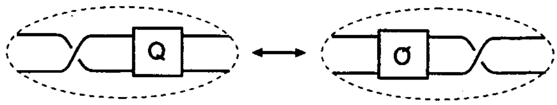

FIGURE 1. A flype

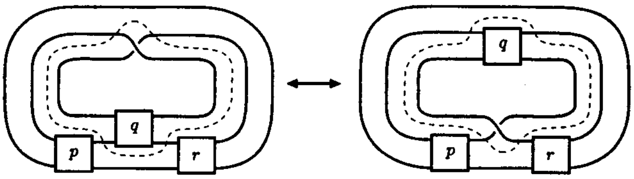

FIGURE 2. A braid-preserving flype

[4]. So B_3 is generated by three cyclically symmetric elementary braids $a_1 = \sigma_1$, $a_2 = \sigma_2$, $a_3 = \sigma_1^{-1}\sigma_2\sigma_1$ with the relations $a_1a_3 = a_2a_1 = a_3a_2$, that is,

$$B_3 = \langle a_1, a_2, a_3 \mid a_3a_2 = a_2a_1 = a_1a_3 \rangle.$$

DEFINITION. A 3-braid is called *flype admissible* if it is conjugate to a word of *flype admissible form* $a_1^p a_2^q a_1^r a_2^\epsilon$, where p, q, r, ϵ are integers and $\epsilon = \pm 1$. It is also called *ϵ-flype admissible* according to $\epsilon = \pm 1$. For this braid, the *ϵ-flype* is defined by

$$a_1^p a_2^q a_1^r a_2^\epsilon \to a_1^p a_2^\epsilon a_1^r a_2^q.$$

A flype is called *effective* if it changes the conjugacy class. The *ϵ-flype equivalence* is the equivalence relation generated by ϵ-flypes and conjugations.

Geometrically, Figure 1 shows the flype operation. For closed 3-braids in Figure 2 we can perform the flype operation on the left one to get the right one. This move may or may not change the conjugacy class of the braids. The following theorem shows that it is the only move necessary to connect two conjugacy classes of 3-braids representing the same link.

THEOREM 4 (Birman-Menasco [3]). *Any link of braid index ≤ 3 admits a unique conjugacy class of 3-braid representative with the following exceptions:*

- *unknot: it has three conjugacy classes of 3-braid representatives,* a_1a_2, $a_1a_2^{-1}$ *and* $a_1^{-1}a_2^{-1}$.
- $(2, k)$ *torus links,* $k \neq 1$*: they have two conjugacy classes of 3-braid representatives,* $a_1^k a_2$ *and* $a_1^k a_2^{-1}$.
- *special class of links with braid index 3 which have 3-braid representatives which admit "braid-preserving flypes". These links have at most two conjugacy classes of 3-braid representatives,* $a_1^p a_2^q a_1^r a_2^\epsilon$ *and* $a_1^p a_2^\epsilon a_1^r a_2^q$*, where* p*,* q*,* r *are distinct integers having absolute value at least 2 and* $\epsilon = \pm 1$.

The conjugacy problem in braid groups is easy for 3-braids. Thus the answer to the following question will complete the Birman's classification in the level of algorithm.

QUESTION A. Classify all flype admissible forms which admit effective flype.

Closed braids can be considered naturally as transversal links in the standard contact space [1, Theorem 10] and

(i) two closed braids are transversally isotopic if and only if one can be transformed to the other by (+)-Markov moves, i.e., $W \leftrightarrow W\sigma_n$ for $W \in B_n$;
(ii) an ϵ-flype can be realized by ϵ-Markov moves and isotopies of closed braids.

For (ii) see Figure 3 for $\epsilon = +$. In the figure, (a) $\longrightarrow$ (b) (respectively, (c) $\longrightarrow$ (d)) is a (+)-Markov move which increases (respectively, decreases) the braid index and (b) $\longrightarrow$ (c) is an isotopy of the closed braid. The question on the transversal isotopy of links with braid index ≤ 3 can be restated as follows:

QUESTION B. Is it possible that for two ϵ-flype equivalent 3-braids, one can be transformed to the other by $(-\epsilon)$-Markov moves and isotopies?

We reduce the above question to a weak problem as follows:

QUESTION C. Is it possible that two ϵ-flype equivalent 3-braids are $(-\epsilon)$-flype equivalent?

The aim of this article is to answer the questions A and C. Our strategy goes as follows. Consider a flype admissible form $a_1^p a_2^q a_1^r a_2^\epsilon$.

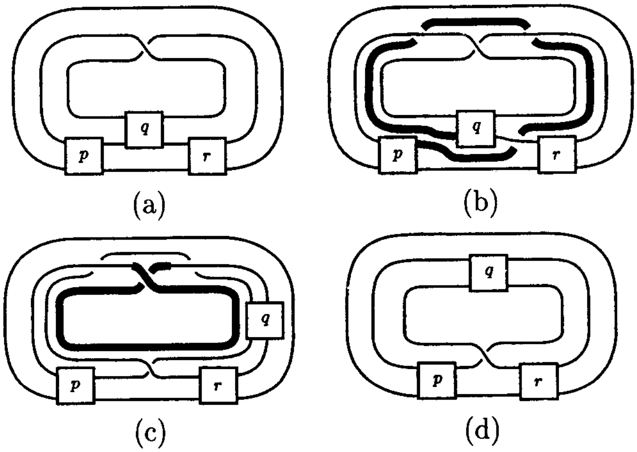

FIGURE 3.

(1) Find necessary conditions on p, q, r so that the flype on $a_1^p a_2^q a_1^r a_2^\epsilon$ is effective.
(2) Divide the flype admissible forms into some classes and compute the canonical representatives for the conjugacy classes.
(3) Prove that for all the 3-braids in the list obtained from step (2), the flype is effective if it satisfies the conditions in step (1). This step solves the question A.
(4) Find all 3-braids which admit both (+) and (−)-flypes by using the list of canonical representatives obtained from the step (2). This step solves the question C.

The main tool in this work is the algorithm on the conjugacy problem of 3-braids. The conjugacy problem for braid groups was solved first by Garside [9] in the late sixties. His algorithm was improved by Thurston and Elrifai-Morton [8]. To get an algorithm appropriate to the classification theorem of Birman-Menasco (Theorem 4), Xu made a new presentation of B_3 and an algorithm for the conjugacy problem for this group in [13], which was generalized by Kang-Ko-Lee [10] to the 4-braid group and by Birman-Ko-Lee [4, 5] to the n-braid groups for general n. We will recall necessary facts in the next section.

We say that a flype admissible form $a_1^p a_2^q a_1^r a_2^\epsilon$ is *faithful* if

1. Neither p nor r is equal to 0, ϵ, 2ϵ or $q+\epsilon$ and $p \neq r$.
2. q has absolute value ≥ 2.

Now the following theorem is our answer to the question A.

THEOREM 5. *Let $a_1^p a_2^q a_1^r a_2^\epsilon$ be a flype admissible form. Then the flype on this form is effective if and only if it is faithful. Moreover in this case, the closed 3-braid represented by $a_1^p a_2^q a_1^r a_2^\epsilon$ is a prime, non-split link with braid index 3.*

The following theorem is our answer to the question C. For a precise version see Theorem 12.

THEOREM 6. *Let $W \in B_3$ be represented by an ϵ-flype admissible form $a_1^p a_2^q a_1^r a_2^\epsilon$. Then it also admits a $(-\epsilon)$-flype admissible form if and only if $p = -\epsilon$ or $r = -\epsilon$ or $q = -2\epsilon$.*

We think that the closed 3-braids which do not admit both $(+)$ and $(-)$-flypes are potential counterexamples (if there are any) for the question about transversal knots that was mentioned at the beginning.

2. Calculations

Before starting discussions, we recall some results on the conjugacy problem of 3-braids and fix some notational conventions. The conjugacy problem for 3-braid group is very easy because the number of strings is too small and so any complicated phenomena can not occur. We recall the result of Xu [13], which is specific to the 3-braid group. For the solution of the conjugacy problem for the general braid group see [4, 5, 10].

We use the indices of a_i in mod 3. So $a_{-3} = a_0 = a_3 = \cdots$, $a_{-2} = a_1 = a_4 = \cdots$ and so on. Let α be the *fundamental element* of B_3 defined by $\alpha = a_{i+1}a_i$. It has the property $a_i^r\alpha = \alpha a_{i+1}^r$ for any i and r.

A positive word $P = a_{\mu_1}\cdots a_{\mu_k}$ in B_3 is said to be in *nondecreasing order* (ND-order) if the array of its subscripts $(\mu_1, \ldots, \mu_k)$ satisfies $\mu_{j+1} = \mu_j$ or $\mu_{j+1} = \mu_j + 1$.

THEOREM 7 (Xu). *In B_3, every element W can be expressed uniquely in the form $\alpha^m P$, where P is a positive word in ND-order.*

The form $\alpha^m P$ in the above theorem is called a *canonical form* of W. If a canonical form $W \equiv \alpha^m P \equiv \alpha^m a_{\mu_1}\cdots a_{\mu_k}$ is rewritten in the form

$$W = \alpha^m a_{\lambda_1}^{l_1} a_{\lambda_2}^{l_2} \cdots a_{\lambda_t}^{l_t},$$

where $l_1, \dots, l_t$ are positive integers and the indices $(\lambda_1, \dots, \lambda_t)$ are in strictly increasing order, i.e., $\lambda_{i+1} = \lambda_i + 1$, then it is called the *syllable form* of W. The *syllable length* of W is $\ell_s(W) = t$, and the *extended syllable length* of W is $\bar{\ell}_s(W) = m + t$. The *symbol* of W is $\Sigma\{W\} = [m\,; l_1, \dots, l_t]$. m is called the *power* and $(l_1, \dots, l_t)$ is called the *tail.*

Note that a symbol determines the braid up to the *meridian conjugation equivalence*, which is generated by

$$W = \alpha^m a_{\lambda_1}^{l_1} \cdots a_{\lambda_t}^{l_t} \;\to\; \alpha^{-1} W \alpha = \alpha^m a_{\lambda_1+1}^{l_1} \cdots a_{\lambda_t+1}^{l_t}.$$

The *summit set* of a 3-braid W is defined as the set of all conjugates W' with maximal power. It is clear by definition that conjugate braids have the same summit set. Any symbol of an element in the summit set of W will be called a *summit symbol* of W. We define the extended syllable length and power of a summit symbol of W by *minimal extended syllable length* (MESL) and the *summit power* of W. Clearly, the MESL and the summit power are invariants of conjugacy classes.

Xu gives an algorithm to find a representative symbol among all the symbols of the 3-braids in the summit set. We summarize her algorithm.

1. Define the *representative symbol* $\Sigma^*[W]$ of a 3-braid W as the symbol of a braid in the summit set whose syllable length is minimal. Then it is characterized by the following property:
   (a) $\bar{\ell}_s \equiv 0 \pmod 3$ or
   (b) tail is ( ) or (1) or
   (c) $\bar{\ell}_s \equiv 1 \pmod 3$ and $\ell_s = 1$ (i.e., $\Sigma^*[W] = [3k\,; \ell]$).
2. The representative symbol is unique up to cyclic moves on the tails of a given representative symbol. (In fact Xu defined the representative symbol as the summit symbol whose tail is minimal in the lexicographic order.)
3. The representative symbol can be obtained from any symbol

   $$[m\,; l_1, \dots, l_t]$$

   by the following two moves.
   (I) if $\bar{\ell}_s \equiv 1 \pmod 3$ and $\ell_s > 1$, then

   $$[m\,; l_1, \dots, l_t] \xrightarrow{I} [m\,; l_2, \dots, l_t + l_1].$$

   In this case, the result is the representative symbol of the conjugacy class of W. Thus the MESL of W is $\bar{\ell}_s - 1 = m + t - 1$.

(II) if $\bar{\ell}_s \equiv 2 \pmod 3$ and the tail is not (1), then

$$[m\,;\,l_1,\dots,l_t] \xrightarrow{II} [m+1\,;\,l_1-1,\dots,l_{t-1},l_t-1].$$

$$(\text{if } t=1 \text{ and } l_1 \geq 2, \text{ then } [m\,;\,l_1] \to [m+1\,;\,l_1-2].$$

Here we note that the conjugacy problem for B_n, $n \geq 4$ is not so easy as for B_3.

1. The word problem is solved by obtaining the canonical form for a given word. For 3-braids, it is just the positive word in ND-order. But for B_n, $n \geq 4$, the situation is somewhat complicated and so one considers the canonical factors which are positive subwords of the so-called *fundamental word.* See [4, 5, 8, 10]. The canonical form in this case is the *left canonical form*, a product of canonical factors satisfying the left-weightedness condition.
2. The conjugacy problem for B_n is solved by computing the summit set which consists of all the canonical forms of the conjugate braids with maximal power. For B_3 this set is represented by the *representative symbol* which can be computed easily. But for general braid groups, the conjugacy class can not be represented by a single symbol, even though one reduces the summit set to the so-called *super summit set*, which requires another condition that the number of factors is minimized.
3. The notions such as syllable length, minimal extended syllable length and representative symbol can not be generalized to the general braid groups.

We prove the next lemma since it can be used to simplify the proof of theorem 5.

LEMMA 8. *Let $W = \alpha^m a_1^{l_1} \cdots a_t^{l_t}$ be a canonical form with $t > 1$ and $l_i > 1$ for some i. Then W^{-1} has MESL $-m-t$.*

Proof. Note that

- $a_{i+1}^{-1} a_i^{-1} = \alpha^{-1} a_{i+2} \alpha^{-1} a_{i+1} = a_{i+3}^2 \alpha^{-2}$.
- $a_i^{-k} = \underbrace{(\alpha^{-1} a_{i+1}) \cdots (\alpha^{-1} a_{i+1})}_{k} = \underbrace{a_{i+2} \cdots a_{i+k+1}}_{k} \alpha^{-k}$
- the syllable length is just $b-a+1$, where a (respectively, b) is the first (respectively, last) index of the canonical form.

So $a_t^{-l_t}a_{t-1}^{-l_{t-1}}\cdots a_1^{-l_1}$ has syllable length $l_1+\cdots+l_t-(t-1)$ by the above observations. Clearly the power is $-(l_1+\cdots+l_t)$. Thus

$$W^{-1}=a_t^{-l_t}a_{t-1}^{-l_{t-1}}\cdots a_1^{-l_1}\alpha^{-m}$$

has syllable length $\ell_s=l_1+\cdots l_t-(t-1)$ and power $-(m+l_1+\cdots l_t)$. Thus W^{-1} has extended syllable length $\bar{\ell}_s=-m-t+1$. Since W has syllable length $t>1$, $m+t\equiv 0 \pmod 3$ and so $\bar{\ell}_s\equiv 1 \pmod 3$. Since $l_i>1$ for some i, $\ell_s>1$. By 3.(I) of the results of Xu, W^{-1} has MESL $-m-t$. □

In characterizing effective flype-admissible forms, we abbreviate k successive appearances of 1 in the symbols by 1^k. For example, we express $[0\,;p,1,1,1,1,q]$ as $[0\,;p,1^4,q]$. We denote $W_1\sim W_2$ if W_1 and W_2 are conjugate.

The following proposition proves the 'only if' part of Theorem 5.

PROPOSITION 9. *If a flype admissible form $a_1^pa_2^qa_1^ra_2^\epsilon$ is not faithful, the flype on this form is not effective.*

Proof. Before starting the proof we note that:

- $a_ia_{i-1}^\epsilon=a_{i+1}^\epsilon a_i$ for $\epsilon=\pm1$.
- For any word W on a_1,a_2, $W(a_1,a_2)\sim W(a_2,a_1)$ since $\Delta a_1\Delta^{-1}=a_2$ and $\Delta a_2\Delta^{-1}=a_1$, where $\Delta=a_1a_2a_1$.
- $a_1^\epsilon a_2^\epsilon a_1^p=a_2^pa_1^\epsilon a_2^\epsilon$ for $\epsilon=\pm1$ and p any integer as we can observe easily from braid diagrams.

We prove the proposition case by case.

(1) $p=r$: (obvious)

(2) $p=0$ or $r=0$: (obvious)

(3) $p=\epsilon$ or $r=\epsilon$: Assume $r=\epsilon$. The case $p=\epsilon$ is similar to this case. Let $W=a_1^pa_2^qa_1^\epsilon a_2^\epsilon$ and W' be the result of flype. Then $W=a_1^pa_2^qa_1^\epsilon a_2^\epsilon=a_1^pa_1^\epsilon a_2^\epsilon a_1^q\sim a_1^{p+q+\epsilon}a_2^\epsilon$ and $W'=a_1^pa_2^\epsilon a_1^\epsilon a_2^q=a_1^pa_1^qa_2^\epsilon a_1^\epsilon\sim a_1^{p+q+\epsilon}a_2^\epsilon$. Thus W and W' are conjugate. We also note that in this case the braid is reducible.

(4) $p=2\epsilon$ or $r=2\epsilon$: Assume $p=2\epsilon$ and $\epsilon=1$. The other cases are similar. Let $W=a_1^2a_2^qa_1^ra_2$ and W' be the result of flype.

$$\begin{aligned}
W &= a_1^2a_2^qa_1^ra_2\sim a_2^qa_1^r(a_2a_1)a_1=a_2^qa_1^ra_1a_3a_1=a_2^qa_1^{r+1}a_3a_1.\\
W' &= a_1^ra_2^qa_1^2a_2\sim a_2^qa_1^2a_2a_1^r=a_2^qa_1^2a_3^ra_2\\
&= a_2^qa_1a_2^ra_1a_2=a_2^{q-1}(a_2a_1)a_2^ra_1a_2=a_2^{q-1}a_3a_2a_2^ra_1a_2\\
&= a_2^{q-1}a_3a_2^{r+1}a_1a_2\sim a_2^qa_3a_2^{r+1}a_1=a_2^qa_1^{r+1}a_3a_1
\end{aligned}$$

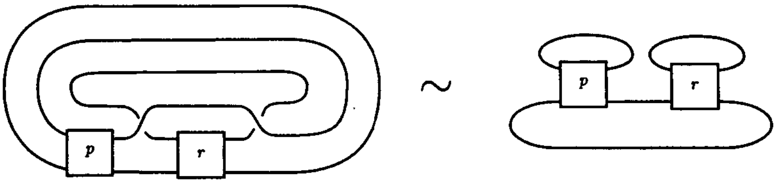

FIGURE 4. The closure of $a_1^p a_2^{-1} a_1^r a_2$

Thus W and W' are conjugate.

(5) $p = q + \epsilon$ or $r = q + \epsilon$: Let $p = q + \epsilon$. Let $W = a_1^p a_2^q a_1^r a_2^\epsilon$ and $W' = a_1^r a_2^q a_1^p a_2^\epsilon$ be the result of flype. Then

$$\begin{aligned} W &= a_1^{q+\epsilon} a_2^q a_1^r a_2^\epsilon \sim a_1^q a_2^q (a_1^r a_2^\epsilon a_1^\epsilon) \\ W' &= a_1^r a_2^q a_1^q a_1^\epsilon a_2^\epsilon \sim a_2^q a_1^q (a_1^\epsilon a_2^\epsilon a_1^r) \sim a_1^q a_2^q (a_2^\epsilon a_1^\epsilon a_2^r) \end{aligned}$$

Since $a_1^r a_2^\epsilon a_1^\epsilon = a_2^\epsilon a_1^\epsilon a_2^r$, W and W' are conjugate. Thus the flype does not change the conjugacy class. Similar for $r = q + \epsilon$.

(6) $q = 0$: (obvious)

(7) $q = \epsilon$: (obvious)

(8) $q = -\epsilon$: $a_1^p a_2^q a_1^r a_2^\epsilon$ represents a composite link (see Figure 4.) and so the flype is not effective since any composite link admit an unique closed 3-braid representative by Morton [11]. □

We divide flype admissible forms into sixteen (possibly not disjoint) types according to the positive-or-negativeness of the four integers p, q, r and ϵ in the flype admissible form. We divide again these sixteen types into five types, say type I, II, III, IV and V, according to the number of negative integers in p, q, r and ϵ in the flype-admissible form, that is, *type* k means that it has a flype-admissible form in which $(k-1)$ of p, q, r and ϵ are negative.

From now on p, q, r denote positive integers. Followings are all flype admissible forms.

type I: $a_1^p a_2^q a_1^r a_2$
type II: $a_1^p a_2^q a_1^r a_2^{-1}$, $a_1^p a_2^{-q} a_1^r a_2$, $a_1^p a_2^q a_1^{-r} a_2$, $a_1^{-p} a_2^q a_1^r a_2^1$
type III: $a_1^p a_2^{-q} a_1^r a_2^{-1}$, $a_1^{-p} a_2^q a_1^{-r} a_2$, $a_1^p a_2^q a_1^{-r} a_2^{-1}$, $a_1^{-p} a_2^q a_1^r a_2^{-1}$, $a_1^p a_2^{-q} a_1^{-r} a_2$, $a_1^{-p} a_2^{-q} a_1^r a_2$
type IV: $a_1^p a_2^{-q} a_1^{-r} a_2^{-1}$, $a_1^{-p} a_2^{-q} a_1^r a_2^{-1}$, $a_1^{-p} a_2^q a_1^{-r} a_2^{-1}$, $a_1^{-p} a_2^{-q} a_1^{-r} a_2$
type V: $a_1^{-p} a_2^{-q} a_1^{-r} a_2^{-1}$

We also assume that they are all faithful.

PROPOSITION 10. (1) *The representative symbols and the results of flypes of the faithful flype admissible forms of type I, II, III are as in the table 1.*

(2) *The MESL of flype-admissible 3-braids of type I, II, III, IV and V are 6, 3, 0, −3 and −6, respectively. Thus the flype-admissible 3-braids of different types can not be conjugate.*

word	representative symbol	result of flype
	type I	
$a_1^p a_2^q a_1^r a_2^1$	$[3\,;\, p-2, q-1, r-2]$	$[3\,;\, r-2, q-1, p-2]$
	type II	
$a_1^p a_2^q a_1^r a_2^{-1}$	$[0\,;\, p, q-1, r]$	$[0\,;\, r, q-1, p]$
$a_1^p a_2^{-q} a_1^r a_2$	$[2-q\,;\, p-1, 1^{q-1}, r-1]$	$[2-q\,;\, r-1, 1^{q-1}, p-1]$
$a_1^p a_2^q a_1^{-r} a_2$	$[1-r\,;\, p-1, q, 1^r]$	$[1-r\,;\, q, p-1, 1^r]$
$a_1^{-p} a_2^q a_1^r a_2$	$[1-p\,;\, q, r-1, 1^p]$	$[1-p\,;\, r-1, q, 1^p]$
	type III	
$a_1^{-p} a_2^q a_1^{-r} a_2$	$[-(p+r); 1^{p-1}, q+1, 1^{r-1}, 2]$	$[-(p+r); 1^{r-1}, q+1, 1^{p-1}, 2]$
$a_1^p a_2^{-q} a_1^{-r} a_2$	$[1-(q+r)\,;\, p, 1^{q-2}, 2, 1^{r-1}]$	$[1-(q+r)\,;\, p, 1^{r-1}, 2, 1^{q-2}]$
$a_1^{-p} a_2^{-q} a_1^r a_2$	$[1-(p+q)\,;\, r, 1^{p-1}, 2, 1^{q-2}]$	$[1-(p+q)\,;\, r, 1^{q-2}, 2, a^{p-1}]$
$a_1^p a_2^{-q} a_1^r a_2^{-1}$	$[-(q+1)\,;\, r+1, p+1, 1^{q-1}]$	$[-(q+1)\,;\, p+1, r+1, 1^{q-1}]$
$a_1^p a_2^q a_1^{-r} a_2^{-1}$	$[-r\,;\, p+1, q, 1^{r-2}]$	$[-r\,;\, q, p+1, 1^{r-2}]$
$a_1^{-p} a_2^q a_1^r a_2^{-1}$	$[-p\,;\, q, r+1, 1^{p-2}]$	$[-p\,;\, r+1, q, 1^{p-2}]$

TABLE 1.

Proof. We denote k successively increasing generators as $a_l a_{l+1} \cdots a_{l+k-1}$ and k occurrence of same word as $(\alpha_2^{-1}) \cdots (\alpha_2^{-1})$ by

$$\underbrace{a_l \cdots a_{l+k-1}}_{k} \quad \text{and} \quad \underbrace{(\alpha_2^{-1}) \cdots (\alpha_2^{-1})}_{k}.$$

We will prove (1) by computing the representative symbols of all the eleven types of flype admissible forms. It will be helpful to keep in mind that $a_j \alpha^m = \alpha^m a_{j+m} = \alpha^m a_{j+m-3}$ and $a_i^{-1} = \alpha^{-1} a_{i+1}$. Thus

$$a_i^{-k} = \underbrace{(\alpha^{-1} a_{i+1}) \cdots (\alpha^{-1} a_{i+1})}_{k} = \underbrace{a_{i-1} \cdots a_{k+i-2}}_{k} \alpha^{-k}.$$

Case 1: $a_1^p a_2^q a_1^r a_2$.

$$\begin{aligned} a_1^p a_2^q a_1^r a_2 &\sim a_2 a_1^p a_2^q a_1^r = (a_2 a_1) a_1^{p-1} a_2^{q-1} (a_2 a_1) a_1^{r-1} = \alpha^2 a_2^{p-1} a_3^{q-1} a_1^{r-1} \\ &= [2\,;\, p-1, q-1, r-1] \xrightarrow{II} [3\,;\, p-2, q-1, r-2] \end{aligned}$$

Since $a_1^p a_2^q a_1^r a_2$ is faithful, $p, r \geq 3$ and $q \geq 2$. So its representative symbol is $[3\,;\, p-2, q-1, r-2]$ and the MESL is 6.

Case 2: $a_1^p a_2^q a_1^r a_2^{-1}$.

$$\begin{aligned} a_1^p a_2^q a_1^r a_2^{-1} &= a_1^p a_2^{q-1} (a_2 a_1) a_1^{r-1} (\alpha^{-1} a_3) = a_1^p a_2^{q-1} \alpha a_1^{r-1} \alpha^{-1} a_3 \\ &= a_1^p a_2^{q-1} a_3^r = [0\,;\, p, q-1, r] \end{aligned}$$

So its representative symbol is $[0\,;\, p, q-1, r]$ and the MESL is 3.

Case 3: $a_1^p a_2^{-q} a_1^r a_2$.

$$\begin{aligned} a_1^p a_2^{-q} a_1^r a_2 &= a_1^p \underbrace{a_1 a_2 \cdots a_q}_{q} \alpha^{-q} a_1^r a_2 = a_1^p \underbrace{a_1 a_2 \cdots a_q}_{q} a_{q+1}^r a_{q+2} \alpha^{-q} \\ &= [-q\,;\, p+1, 1^{q-1}, r, 1] \xrightarrow{II} [1-q\,;\, p, 1^{q-1}, r] \\ &\xrightarrow{II} [2-q\,;\, p-1, 1^{q-1}, r-1] \end{aligned}$$

So its representative symbol is $[2-q\,;\, p-1, 1^{q-1}, r-1]$ and the MESL is 3.

Case 4: $a_1^p a_2^q a_1^{-r} a_2$.

$$\begin{aligned} a_1^p a_2^q a_1^{-r} a_2 &= a_1^p a_2^q \underbrace{a_3 \cdots a_{r+2}}_{r} \alpha^{-r} a_2 = a_1^p a_2^q \underbrace{a_3 \cdots a_{r+2}}_{r} a_{r+2} \alpha^{-r} \\ &= [-r\,;\, p, q, 1^{r-1}, 2] \xrightarrow{II} [1-r\,;\, p-1, q, 1^r] \end{aligned}$$

So its representative symbol is $[1-r\,;\, p-1, q, 1^r]$ and the MESL is 3.

Case 5: $a_1^{-p}a_2^q a_1^r a_2$.

$$\begin{aligned}
a_1^{-p}a_2^q a_1^r a_2 &= \underbrace{a_3a_4\cdots a_{p+2}}_{p}\,\alpha^{-p}a_2^{q-1}(a_2a_1)a_1^{r-1}a_2\\
&= \underbrace{a_3a_4\cdots a_{p+2}}_{p}\,a_{p+2}^{q-1}a_p^{r-1}a_{p+1}\alpha^{1-p}\\
&= \underbrace{a_3a_4\cdots a_{p+1}}_{p-1}\,a_{p+2}^{q}a_{p+3}^{r-1}a_{p+4}\alpha^{1-p}\\
&= [1-p\,;\,1^{p-1},q,r-1,1] \xrightarrow{I} [1-p\,;q,r-1,1^p]
\end{aligned}$$

So its representative symbol is $[1-p\,;\,q,r-1,1^p]$ and the MESL is 3.

Case 6: $a_1^{-p}a_2^q a_1^{-r} a_2$.

$$\begin{aligned}
a_1^{-p}a_2^q a_1^{-r} a_2 &= \underbrace{a_3a_4\cdots a_{p+2}}_{p}\,\alpha^{-p}a_2^q\,\underbrace{a_3a_4\cdots a_{r+2}}_{r}\,\alpha^{-r}a_2\\
&= \underbrace{a_3a_4\cdots a_{p+2}}_{p}\,a_{p+2}^q\,\underbrace{a_{p+3}\cdots a_{p+r+2}}_{r}\,a_{p+r+2}\alpha^{-(p+r)}\\
&= \underbrace{a_3a_4\cdots a_{p+1}}_{p-1}\,a_{p+2}^{q+1}\,\underbrace{a_{p+3}\cdots a_{p+r+1}}_{r-1}\,a_{p+r+2}^2\alpha^{-(p+r)}\\
&= [-(p+r)\,;\,1^{p-1},q+1,1^{r-1},2]
\end{aligned}$$

So its representative symbol is $[-(p+r)\,;\,1^{p-1},q+1,1^{r-1},2]$ and the MESL is 0.

Case 7: $a_1^{p}a_2^{-q} a_1^{-r} a_2$.

$$\begin{aligned}
a_1^{p}a_2^{-q} a_1^{-r} a_2 &= a_1^p\,\underbrace{a_1a_2\cdots a_q}_{q}\,\alpha^{-q}\,\underbrace{a_0a_1a_2\cdots a_{r-1}}_{r}\,\alpha^{-r}a_2\\
&= a_1^p\,\underbrace{a_1a_2\cdots a_q}_{q}\,\underbrace{a_qa_{q+1}\cdots a_{q+r-1}}_{r}\,a_{q+r+2}\alpha^{-(q+r)}\\
&= a_1^{p+1}\,\underbrace{a_2a_3\cdots a_{q-1}}_{q-2}\,a_q^2\,\underbrace{a_{q+1}a_{q+2}\cdots a_{q+r-2}}_{r-2}\,a_{q+r-1}^2\alpha^{-(q+r)}\\
&= [-(q+r)\,;\,p+1,1^{q-2},2,1^{r-2},2]\\
&\xrightarrow{II} [1-(q+r)\,;\,p,1^{q-2},2,1^{r-1}]
\end{aligned}$$

So its representative symbol is $[1-(q+r)\,;\,p,1^{q-2},2,1^{r-1}]$ and the MESL is 0.

Case 8: $a_1^{-p}a_2^{-q}a_1^r a_2$.

$$\begin{aligned}
a_1^{-p}a_2^{-q}a_1^r a_2 &= \underbrace{a_3a_4\cdots a_{p+2}}_{p}\,\alpha^{-p}\,\underbrace{a_1a_2\cdots a_q}_{q}\,\alpha^{-q}a_1^r a_2\\
&= \underbrace{a_3a_4\cdots a_{p+2}}_{p}\,\underbrace{a_{p+1}a_{p+2}\cdots a_{p+q}}_{q}\,a_{p+q+1}^r a_{p+q+2}^{-(p+q)}\\
&= [-(p+q)\,;\,1^p,1^q,r,1] \xrightarrow{II} [1-(p+q)\,;\,r,1^{p-1},2,1^{q-2}]
\end{aligned}$$

So its representative symbol is $[1-(p+q)\,;\,r,1^{p-1},2,1^{q-2}]$ and the MESL is 0.

Case 9: $a_1^p a_2^{-q}a_1^r a_2^{-1}$.

$$\begin{aligned}
a_1^p a_2^{-q}a_1^r a_2^{-1} &= a_1^p\,\underbrace{a_1a_2\cdots a_q}_{q}\,\alpha^{-q}a_1^r(\alpha^{-1}a_3)\\
&= a_1^{p+1}\,\underbrace{a_2\cdots a_q}_{q-1}\,a_{q+1}^{r+1}\alpha^{-(q+1)}\\
&= [-(q+1)\,;\,p+1,1^{q-1},r+1]
\end{aligned}$$

So its representative symbol is $[-(q+1)\,;\,p+1,1^{q-1},r+1]$ and the MESL is 0.

Case 10: $a_1^p a_2^q a_1^{-r}a_2^{-1}$.

$$\begin{aligned}
a_1^p a_2^q a_1^{-r}a_2^{-1} &= a_1^p a_2^q a_1^{1-r}(a_2a_1)^{-1} = a_1^p a_2^q\,\underbrace{a_3a_4\cdots a_{r+1}}_{r-1}\,\alpha^{1-r}\alpha^{-1}\\
&= [-r\,;\,p,q,1^{r-1}] \xrightarrow{I} [-r\,;\,p+1,q,1^{r-2}]
\end{aligned}$$

So its representative symbol is $[-r\,;\,p+1,q,1^{r-2}]$ and the MESL is 0.

Case 11: $a_1^{-p}a_2^q a_1^r a_2^{-1}$.

$$\begin{aligned}
a_1^{-p}a_2^q a_1^r a_2^{-1} &= \underbrace{a_3a_4\cdots a_{p+2}}_{p}\,\alpha^{-p}a_2^{q-1}(a_2a_1)a_1^{r-1}\alpha^{-1}a_3^r\\
&= \underbrace{a_3a_4\cdots a_{p+2}}_{p}\,a_{p+2}^{q-1}a_{p+3}^r\alpha^{-p}\\
&= [-p;\,1^{p-1},q,r]\\
&\xrightarrow{I} [-p;\,1^{p-2},q,r+1] = [-p;\,q,r+1,1^{p-2}]
\end{aligned}$$

Thus it has the summit symbol $[-p;\,q,r+1,1^{p-2}]$ and MESL is 0.

Now the above eleven cases prove (1). It is easy to check that the result of flypes are as in the table 1.

By (1) the MESL of type I, II and III are 6, 3 and 0 respectively. Note that if W is of type IV (respectively, V), then W^{-1} is of type II (respectively I) and so has the MESL 3 (respectively 6). By Lemma 8, W has MESL -3 (respectively -6). Since the MESL is an invariant of conjugacy relation, any two word of different types can not be conjugate. $\square$

The following proposition is the 'if' part of Theorem 5.

PROPOSITION 11. *The flype on a faithful flype admissible form is effective.*

Proof. Note that the flype reverses the cyclic order of the tail. The faithfulness of the flype admissible form of type I, II, III guarantees that the cyclic order of the tail is different from its reverse. (It is easy to check.) For the forms of type IV and V note that the ϵ-flype on $W = a_1^p a_2^q a_1^r a_2^\epsilon$ is just the inverse of the $(-\epsilon)$-flype on $W^{-1} \sim a_1^{-r} a_2^{-q} a_1^{-p} a_2^{-\epsilon}$. Thus all flypes are effective. $\square$

Proof of Theorem 5. Proposition 9 and Proposition 11 proves the 'if and only if'. It remains to show that if a 3-braid is represented by a faithful flype admissible form, it is neither composite, split nor reducible. If it is a composite (respectively split) link, it has unique closed 3-braid representative by Morton [11] (respectively, Birman-Menasco [2]) so that any flype cannot be effective.

Assume that $W \in B_3$ represents a reducible braid, i.e., a $(2,k)$ torus link $k \neq 1$. By the classification theorem of Birman-Menasco (Theorem 4), W is conjugate to $a_1^{\pm k} a_2^{\pm 1}$ for some integer $k \geq 2$. Note that

$$\begin{aligned}
a_1^k a_2 &= [0\,;\,k,1] \xrightarrow{II} [1\,;\,k-1] \xrightarrow{II} [2\,;\,k-3],\\
a_1^k a_2^{-1} &= a_1^k a_1 \alpha^{-1} = [-1\,;\,k+1],\\
a_1^{-k} a_2 &\sim a_2 a_1^{-k} = a_2 \underbrace{a_3 a_4 \cdots a_{k+2}}_{k} \alpha^{-k} = [-k\,;\,1^{k+1}]\\
a_1^{-k} a_2^{-1} &= a_1^{1-k}(a_2 a_1)^{-1} = \underbrace{a_3 \cdots a_{k+1}}_{k-1} \alpha^{-k} = [-k\,;\,1^{k-1}] \xrightarrow{II} [1-k\,;\,1^{k-3}]
\end{aligned}$$

In each case the tail of the representative symbol of W has tail with syllable length 1 or is just a sequence of 1's so that it cannot be a representative symbol of a faithful flype admissible form. $\square$

word / result of flype	representative symbols
$a_1^x a_2^y a_3 (x \neq y \geq 2)$ $\Longrightarrow a_1^y a_2^x a_3$	$[0\,;\, x, y, 1]$ $\Longrightarrow [0\,;\, x, y, 1]$
$a_1^{-x} a_2^y a_3 (x \geq 3,\ y \geq 2)$ $\Longrightarrow a_1^y a_2^{-x} a_3$	$[-x\,;\, y+1, 2, 1^{x-2}]$ $\Longrightarrow [-x\,;\, 2, y+1, 1^{x-2}]$
$a_1^y a_2^{-x} a_3 (x \geq 3,\ y \geq 2)$ $\Longrightarrow a_1^{-x} a_2^y a_3$	$[-x\,;\, 2, y+1, 1^{x-2}]$ $\Longrightarrow [-x\,;\, y+1, 2, 1^{x-2}]$
$a_2^{-x} a_1^{-y} a_3^{-1} (x \neq y \geq 2)$ $\Longrightarrow a_2^{-y} a_1^{-x} a_3^{-1}$	$[-(x+y+1)\,;\, 3, 1^{x-2}, 2, 1^{y-2}]$ $\Longrightarrow [-(x+y+1)\,;\, 3, 1^{y-2}, 2, 1^{x-2}]$

TABLE 2.

The following theorem, which is a refined version of Theorem 6, is our answer to the question C.

THEOREM 12. *If $W \in B_3$ is both $(+)$ and $(-)$-flype admissible, then W has MESL 3, 0 or -3 and in each case it has the property described below (see the table 2):*

1. *If W has MESL 3, then it is conjugate to $a_1^x a_2^y a_3$ $(x \neq y \geq 2)$ with representative symbol $[0\,;\, x, y, 1]$. W can be written as any flype admissible form with MESL 3 in the table 1 and the result of any flype is conjugate to $a_1^y a_2^x a_3$ with representative symbol $[0\,;\, y, x, 1]$;*
2. *If W has MESL 0, then it is conjugate to $W_1 = a_1^{-x} a_2^y a_3$ or $W_2 = a_1^y a_2^{-x} a_3$ $(x \geq 3, y \geq 2)$. Their representative symbols are $[-x\,;\, y+1, 2, 1^{x-2}]$ and $[-x\,;\, 2, y+1, 1^{x-2}]$ respectively. W can be written as any flype admissible form with MESL 0 in the table 1 and the result of any flype on W_1 (respectively W_2) is conjugate to W_2 (respectively W_1).*
3. *If W has MESL -3, then it is conjugate to $a_2^{-x} a_1^{-y} a_3^{-1}$ $(x \neq y \geq 2)$ with representative symbol $[-(x+y+1)\,;\, 3, 1^{x-2}, 2, 1^{y-2}]$. W can be written as any flype admissible form with MESL -3 which is the inverse of a flype admissible form in the table 1 with MESL 3 and the result of any flype is $a_2^{-y} a_1^{-x} a_3^{-1}$ with representative symbol $[-(x+y+1)\,;\, 3, 1^{y-2}, 2, 1^{x-2}]$.*

Moreover, if $W \in B_3$ is represented by an ϵ-flype admissible form $a_1^p a_2^q a_1^r a_2^\epsilon$, then it also admits an $(-\epsilon)$-flype admissible form if and only if $p = -\epsilon$, $r = -\epsilon$ or $q = -2\epsilon$.

Proof. If W is both (+) and (−)-flype admissible, its MESL cannot be 6 or −6 by Proposition 10. Thus W has MESL 3, 0 or −3

Assume that W has MESL 3. Since it is (−)-flype admissible, it has summit power 0. Since W is also (+)-flype admissible, it is conjugate to one of $a_1^p a_2^{-q} a_1^r a_2$, $a_1^p a_2^q a_1^{-r} a_2$ or $a_1^{-p} a_2^q a_1^r a_2$.

1. If W is conjugate to $a_1^p a_2^{-q} a_1^r a_2$ which has representative symbol $[2-q\,;\, r-1, p-1, 1^{q-1}]$, then $q = 2$ and W is conjugate to $a_1^p a_2^{-2} a_1^r a_2$ with representative symbol $[0\,;\, r-1, p-1, 1]$. Since the flype is faithful $p \neq r \geq 3$. Thus it is conjugate to the word $a_1^x a_2^y a_3$ with $x = r - 1$ and $y = p - 1$.
2. If W is conjugate to $a_1^p a_2^q a_1^{-r} a_2$ which has representative symbol $[1 - r\,;\, p - 1, q, 1]$, then $r = 1$ and W is conjugate to $a_1^p a_2^q a_1^{-1} a_2$ with representative symbol $[0\,;\, p-1, q, 1]$. Since the flype is faithful, $p \neq q + 1$ and $p \geq 3$. Thus it is conjugate to the word $a_1^x a_2^y a_3$ with $x = p - 1$ and $y = q$.
3. If W is conjugate to $a_1^{-p} a_2^q a_1^r a_2$ which has representative symbol $[1 - p\,;\, q, r - 1, 1^p]$, then $p = 1$ and W is conjugate to $a_1^{-1} a_2^q a_1^r a_2$ with representative symbol $[0\,;\, q, r-1, 1]$. Since the flype is faithful, $r \neq p + 1$ and $r \geq 3$. Thus it is conjugate to $a_1^x a_2^y a_3$ with $x = q$ and $y = r - 1$.

From the above observations, W is conjugate to $a_1^x a_2^y a_3$ as in the statement. It is easy to see that W has the following flype admissible forms

$$\begin{aligned} a_1^x a_2^y a_3 \;&\sim\; a_1^y a_2^2 a_1^x a_2^{-1} \sim a_1^x a_2^{y+1} a_1 a_2^{-1} \sim a_1 a_2^{x+1} a_1^y a_2^{-1} \\ &\sim\; a_1^{x+1} a_2^{-2} a_1^{y+1} a_2 \sim a_1^{y+1} a_2^x a_1^{-1} a_2 \sim a_1^{-1} a_2^y a_1^{x+1} a_2 \end{aligned}$$

and any flype on one of those forms transforms W to $a_1^y a_2^x a_3$.

Note that both (+) and (−)-flype admissible forms with MESL 3 have the property that (i) the summit power is 0 and (ii) there is exactly one '1' in the tail. Thus if an ϵ-flype admissible form with MESL 3 admits also a $(-\epsilon)$-flype admissible form, it should satisfy the conditions on the exponents as in the table 3, which proves the 'moreover' part.

Now assume that W has MESL −3. Since W^{-1} has MESL 3 and both (+) and (−)-flype are admissible, W^{-1} is conjugate to $a_1^y a_2^x a_3$ ($x \neq y \geq 2$). So W is conjugate to $a_3^{-1} a_2^{-x} a_1^{-y} \sim a_2^{-x} a_1^{-y} a_3^{-1}$ and any flype on W results in $a_2^{-y} a_1^{-x} a_3^{-1}$. Then W has representative symbol $[-(x + y + 1)\,;\, 3, 1^{x-2}, 2, 1^{y-2}]$ and the result of any flype on W is $[-(x + y +$

form	representative symbol	condition on exponents
$a_1^p a_2^q a_1^r a_2^{-1}$	$[0\,;\,p, q-1, r]$	$p=1$, $q=2$ or $r=1$
$a_1^p a_2^{-q} a_1^r a_2$	$[2-q\,;\,p-1, 1^{q-1}, r-1]$	$q=2$
$a_1^p a_2^q a_1^{-r} a_2$	$[1-r\,;\,p-1, q, 1^r]$	$r=1$
$a_1^{-p} a_2^q a_1^r a_2$	$[1-p\,;\,q, r-1, 1^p]$	$p=1$

TABLE 3.

$1)\,;\,3, 1^{y-2}, 2, 1^{x-2}]$ since

$$\begin{aligned}
a_2^{-x}a_1^{-y}a_3^{-1} &\sim \underbrace{a_1a_2\cdots a_x}_{x}\alpha^{-x}\underbrace{a_0a_1\cdots a_{y-1}}_{y}\alpha^{-y}a_2\alpha^{-1}\\
&= \underbrace{a_1a_2\cdots a_x}_{x}\underbrace{a_xa_{x+1}\cdots a_{x+y-1}}_{y}a_{x+y+2}\alpha^{-(x+y+1)}\\
&= [-(x+y+1)\,;\,1^{x-1}, 2, 1^{y-1}, 2]\\
&\xrightarrow{I} [-(x+y+1)\,;\,3, 1^{x-2}, 2, 1^{y-1}]
\end{aligned}$$

Note that the 'moveover' part of the theorem follows from the condition on the exponents of the flype admissible form case of MESL 3.

Assume that W has MESL 0. First we show that $W_1 = a_1^{-x}a_2^ya_3$ (respectively, $W_2 = a_1^ya_2^{-x}a_3$) has representative symbol $[-x\,;\,y+1, 2, 1^{x-2}]$ (respectively, $[-x\,;\,2, y+1, 1^{x-2}]$).

$$\begin{aligned}
W_1 &= a_1^{-x}a_2^ya_3 = \underbrace{a_3a_4\cdots a_{x+2}}_{x}\alpha^{-x}a_2^ya_3\\
&= \underbrace{a_3a_4\cdots a_{x+2}}_{x}a_{x+2}^ya_{x+3}\alpha^{-x}\\
&= [-x\,;\,1^{x-1}, y+1, 1] \xrightarrow{I} [-x\,;\,1^{x-2}, y+1, 2]\\
&= [-x\,;\,y+1, 2, 1^{x-2}]\\
W_2 &= a_1^y\underbrace{a_1a_2\cdots a_x}_{x}\alpha^{-x}a_3 = a_1^y\underbrace{a_1a_2\cdots a_x}_{x}a_x\alpha^{-x}\\
&= [-x\,;\,y+1, 1^{x-2}, 2] = [-x\,;\,2, y+1, 1^{x-2}]
\end{aligned}$$

Note that the tail of any $(-)$-flype admissible forms with MESL 0 has exactly one series of consecutive 1's. Since W is also $(+)$-flype admissible, it is conjugate to one of $a_1^{-p}a_2^qa_1^{-r}a_2$, $a_1^pa_2^{-q}a_1^{-r}a_2$ and $a_1^{-p}a_2^{-q}a_1^ra_2$.

1. If W is conjugate to $a_1^{-p}a_2^q a_1^{-r}a_2$ which has representative symbol $[-(p+r)\,;\,1^{p-1}, q+1, 1^{r-1}, 2]$, then $r = 1$ or $p = 1$.

   If $r = 1$, then W is conjugate to $a_1^{-p}a_2^q a_1^{-1}a_2$ with representative symbol $[-(p+1)\,;\,q+1, 2, 1^{p-1}]$ $(p, q \geq 2)$ which is just $W_1 = a_1^{-x}a_2^y a_3$ with $x = p+1$ and $y = q$.

   If $p = 1$, then W is conjugate to $a_1^{-1}a_2^q a_1^{-r}a_2$ with representative symbol $[-(r+1)\,;\,q+1, 1^{r-1}, 2] = [-(r+1)\,;\,2, q+1, 1^{r-1}]$ $(q, r \geq 2)$ which is just $W_2 = a_1^y a_2^{-x} a_3$ with $x = r+1$ and $y = q$.
2. If W is conjugate to $a_1^p a_2^{-q} a_1^{-r} a_2$ which has representative symbol $[1-(q+r)\,;\,p, 1^{q-2}, 2, 1^{r-1}]$, then $r = 1$ or $q = 2$.

   If $q = 2$, then W is conjugate to $a_1^p a_2^{-2} a_1^{-r} a_2$ with representative symbol $[-(r+1)\,;\,p, 2, 1^{r-1}]$ $(p \geq 3$ and $r \geq 2)$ which is just W_1 with $x = r+1$ and $y = p-1$.

   If $r = 1$, then W is conjugate to $a_1^p a_2^{-q} a_1^{-1} a_2$ with representative symbol $[-q\,;\,p, 1^{q-2}, 2] = [-q\,;\,2, p, 1^{q-2}]$ $(p, q \geq 3)$ which is just W_2 with $x = q$ and $y = p-1$.
3. If W is conjugate to $a_1^{-p} a_2^{-q} a_1^r a_2$ which has representative symbol $[1-(p+q)\,;\,r, 1^{p-1}, 2, 1^{q-2}]$, then $p = 1$ or $q = 2$.

   If $p = 1$, then W is conjugate to $a_1^{-1} a_2^{-q} a_1^r a_2$ with representative symbol $[-q\,;\,r, 2, 1^{q-2}]$ $(q, r \geq 3)$ which is just W_1 with $x = q$ and $y = r-1$.

   If $q = 2$, then W is conjugate to $a_1^{-p} a_2^{-2} a_1^r a_2$ with representative symbol $[-(p+1)\,;\,2, r, 1^{p-1}]$ $(p \geq 2$ and $r \geq 3)$ which is just W_2 with $x = p+1$ and $y = r-1$.

From the above observations, W is conjugate to $W_1 = a_1^{-x}a_2^y a_3$ or $W_2 = a_1^y a_2^{-x} a_3$ as in the statement. It is easy to see that W_1 and W_2 has all the flype admissible forms with MESL 0 in the table 1 and any flype on one of those forms transforms W_1 (respectively W_2) to W_2 (respectively W_1).

Note that both $(+)$ and $(-)$-flype admissible forms with MESL 0 have the property that (i) there is exactly one '2' in the tail and (ii) there is exactly one series of consecutive 1's in the tail. Thus if an ϵ-flype admissible form with MESL 0 admits also a $(-\epsilon)$-flype admissible form, it should satisfy the conditions on the exponents as in the table 4, which proves the 'moreover' part. $\square$

form	representative symbol	condition on exponents
$a_1^{-p}a_2^q a_1^{-r}a_2$	$[-(p+r)\,;\,1^{p-1},q+1,1^{r-1},2]$	$p=1$ or $r=1$
$a_1^p a_2^{-q}a_1^{-r}a_2$	$[1-(q+r)\,;\,p,1^{q-2},2,1^{r-1}]$	$q=2$ or $r=1$
$a_1^{-p}a_2^{-q}a_1^r a_2$	$[1-(p+q)\,;\,r,1^{p-1},2,1^{q-2}]$	$p=1$ or $q=2$
$a_1^p a_2^{-q}a_1^r a_2^{-1}$	$[-(q+1)\,;\,r+1,p+1,1^{q-1}]$	$p=1$ or $r=1$
$a_1^p a_2^q a_1^{-r}a_2^{-1}$	$[-r\,;\,p+1,q,1^{r-2}]$	$p=1$ or $q=2$
$a_1^{-p}a_2^q a_1^r a_2^{-1}$	$[-p;\,q,r+1,1^{p-2}]$	$q=2$ or $r=1$

TABLE 4.

References

[1] D. Bennequin, *Entrelacements et équations de Pfaff,* Astérisque **107-108** (1983), 87–161.

[2] J. S. Birman and W. W. Menasco, *Studying links via closed braids.* I*: A finiteness theorem,* Pacific J. Math. **154** (1992), 17–36.

[3] ———, *Studying links via closed braids* III*: Classifying links which are closed* 3*-braids,* Pacific J. Math. **161** (1993), 25–113.

[4] J. S. Birman, K. H. Ko and S. J. Lee, *A new approach to the word and conjugacy problem in the braid groups,* Advances in Mathematics **139** (1998), 322-353.

[5] ———, *A polynomial-time solution to the conjugacy problem in the* 4*-braid group,* manuscript in preparation.

[6] Y. Eliashberg, *Legendrian and transversal knots in tight contact* 3*-manifolds,* In *Topological methods in modern mathematics,* Publish or Perish, Berkeley, CA (1993), 171–193.

[7] D. Fuchs and S. Tabachnikov, *Invariants of legendrian and transverse knots in the standard contact space,* Topology **36** (1997), 1025–1053.

[8] E. A. Elrifai and H. R. Morton, *Algorithms for positive braids,* Quart. J. Math. Oxford **45** (1994), 479–497.

[9] F. A. Garside, *The braid group and other groups,* Quart. J. Math. Oxford **20** (1969), 235–254.

[10] E. S. Kang, K. H. Ko and S. J. Lee, *Band-generator presentation for the* 4*-braid group,* Topology and its Applications **78** (1997), 39–60.

[11] H. R. Morton, *Closed braids which are not prime knots,* Math. Proc. Cambridge Philos. Soc. **93** (1979), 421–426.

[12] J. Swiatkowski, *On the isotopy of Legendrian knots,* Ann. Glob. Anal. Geom. **10** (1992), 195–207.

[13] P. J. Xu, *The genus of closed* 3*-braids,* J. of Knot Theory and its Ramifications **1** (1992), 303–326.

Department of Mathematics
Korea Advanced Institute of Science and Technology
Taejon 305–701, Korea
E-mail: knot@knot.kaist.ac.kr

Department of Mathematics
Korea Advanced Institute of Science and Technology
Taejon 305–701, Korea
E-mail: sjlee@math.kaist.ac.kr