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3-DIMENSIONAL NON-COMPACT INFRA-NILMANIFOLDS

HEUNG KI KIM AND SUNG MO IM

ABSTRACT. Let G be the 3-dimensional Heisenberg group. A discrete subgroup of Isom(G), acting freely on G with non-compact quotient, must be isomorphic to either $\mathbf{1}$, $\mathbb{Z}$, $\mathbb{Z}^2$ or the fundamental group of the Klein bottle. We classify all discrete representations of such groups into Isom(G) up to affine conjugacy. This yields an affine classification of 3-dimensional non-compact infra-nilmanifolds.

A Euclidean space-form is the orbit space of the Euclidean space by an action of a group of isometries. They are the spaces of constant curvature 0. Due to the three theorems by Bieberbach, the structure, finiteness and rigidity of such groups are known. In dimension 3, $\mathbb{R}^3$ is one of the so-called 8 geometries, and there are 10 compact Euclidean space-forms (up to affine equivalence), see for example, [7] Chapter 3.

A connected, simply connected nilpotent Lie group with a left invariant metric yields a geometry. An infra-nilmanifold is the orbit space of the nilpotent Lie group by an action of a discrete group of isometries. It is well known that compact infra-nilmanifolds are exactly the almost flat manifolds of Gromov. All of the Bieberbach theorems generalize to nilpotent groups, due to Auslander [1], [2] and Lee-Raymond [5]. Thus, there are structure, finiteness and rigidity theorems for infra-nilmanifolds. In dimension 3, the Heisenberg group yields the Nil-geometry, also one the 8 geometries. All compact infra-nilmanifolds with Nil-geometry are classified up to affine equivalence in [4].

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In this paper, we are concerned with non-compact spaces with the Nil-geometry. Especially, we shall classify all the 3-dimensional non-compact infra-nilmanifolds up to affine equivalence. The main results are stated in Theorem 1.1, Theorem 2.3 and Theorem 3.1.

0. Generalities

Let G be the 3-dimensional Heisenberg group; i.e., G consists of all 3×3 real upper triangular matrices with diagonal entries 1. Thus G is a simply connected, nilpotent Lie group, and there is an exact sequence

$$1 \to \mathcal{Z}(G) \to G \to \mathbb{R}^2 \to 1,$$

where

$$\mathcal{Z}(G) = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : z \in \mathbb{R} \right\}$$

is the center of G, and is isomorphic to $\mathbb{R}$. *From now on, G refers to the Heisenberg group.*

Automorphisms of the Heisenberg group

As is well known, the group of automorphisms of G is $\mathrm{Aut}(G) = \mathbb{R}^2 \rtimes \mathrm{GL}(2,\mathbb{R})$. See [6]. In fact, the $\mathbb{R}^2$-factor is the group of inner automorphisms of G, and the $\mathrm{GL}(2,\mathbb{R})$-factor induces the outer automorphism group. An element

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right) \in \mathbb{R}^2 \rtimes \mathrm{GL}(2,\mathbb{R}) = \mathrm{Aut}(G)$$

sends

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$\begin{aligned} x' &= \alpha x + \beta y \\ y' &= \gamma x + \delta y \\ z' &= (\alpha\delta - \beta\gamma)z + \beta\gamma xy + \frac{\alpha\gamma}{2}x^2 + \frac{\beta\delta}{2}y^2 + (uy - vx). \end{aligned}$$

NOTATION. Throughout this paper, we shall use the following notation.

$$R_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \quad \tau = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{O}(2) \subset \mathrm{Aut}(G)$$

$$\mathbf{z}_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in G.$$

On the quotient space $\mathbb{R}^2 = G/\mathcal{Z}(G)$, the elements $\mathbf{z}_0$, $\mathbf{x}_0$, $\mathbf{y}_0$ project down to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. We denote the images in $\mathbb{R}^2$ by $\overline{\mathbf{z}}_0$, $\overline{\mathbf{x}}_0$, $\overline{\mathbf{y}}_0$, etc.

Note that an automorphism $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GL}(2,\mathbb{R}) \subset \mathrm{Aut}(G)$ induces an automorphism on $\mathbb{R}^2 = G/\mathcal{Z}(G)$ by matrix multiplication (which is a linear transformation of a vector space).

LEMMA 0.1. *Let B be an element of $\mathrm{O}(2)-\mathrm{SO}(2)$. Then $B = R_{2\varphi}\tau$ for some φ, and $R_\varphi^{-1} B R_\varphi = \tau$. Therefore, every element of $\mathrm{O}(2)-\mathrm{SO}(2)$ is conjugate to τ by an element of $\mathrm{SO}(2)$.*

The isometry group of the Heisenberg group

We shall give a left-invariant Riemannian metric on the Heisenberg group G by choosing an *orthonormal basis* for its Lie algebra

$$\mathfrak{G} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

as follows:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

With this Riemannian metric, the group of isometries $\mathrm{Isom}(G)$ lies in $G \rtimes \mathrm{Aut}(G)$:

$$\mathrm{Isom}(G) = G \rtimes \mathrm{O}(2) \subset G \rtimes \mathrm{Aut}(G).$$

Further, it is well known that $G \rtimes \mathrm{Aut}(G)$ is the group of affine diffeomorphisms so that $\mathrm{Aff}(G) = G \rtimes (\mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R}))$.

Generalized Bieberbach theorem

The following is a generalization of a theorem by Bieberbach to the nilpotent case. It will be used to study representations of the group π into $\mathrm{Isom}(G)$.

LEMMA 0.2. [2; Theorem 3] *Let N be a connected, simply connected nilpotent Lie group, C a compact group of automorphisms of N. Suppose $\Gamma \subset N \rtimes C$ is a discrete subgroup. Then there exists a connected Lie subgroup N^* of N and a finite index subgroup Γ^* of Γ such that Γ^* is a uniform subgroup of N^*.*

Fundamental groups of 3-dimensional non-compact infranil-manifolds

PROPOSITION 0.3. *Let π be a discrete subgroup of $\mathrm{Isom}(G)$ acting freely on G. If π is not cocompact, then it is isomorphic to either 1, $\mathbb{Z}$, $\mathbb{Z}^2$ or the fundamental group of the Klein bottle.*

Proof. For a discrete group π to act freely, it is necessary and sufficient that π be torsion free. Furthermore, according to Lemma 0.2, π contains a normal subgroup Γ of finite index, which is isomorphic to a discrete subgroup of G. Since a lattice of G has (Hirsh) rank 3, Γ can be nilpotent of rank 0, 1 or 2. But Γ being nilpotent of rank ≤ 2 implies it is abelian. A torsion free group which contains $\mathbb{Z}$ of finite index is $\mathbb{Z}$ itself. A torsion free group which contains $\mathbb{Z}^2$ of finite index is either $\mathbb{Z}^2$ or the fundamental group of Klein bottle. □

We denote the fundamental group of Klein bottle by K, so it has a presentation

$$K = \langle t_1, t_2, \alpha \mid [t_1, t_2] = 1, \alpha^2 = t_1, \alpha t_2 \alpha^{-1} = t_2^{-1} \rangle.$$

Affine conjugacy

Let $\pi = 1$, $\mathbb{Z}$, $\mathbb{Z}^2$ or K. Let

$$\theta_1, \theta_2 : \pi \longrightarrow \mathrm{Isom}(G)$$

be two discrete embeddings (one-to-one homomorphisms). We say θ_2 is *conjugate* to θ_1 in $\mathrm{Aff}(G)$ if there exists an element $\alpha \in \mathrm{Aff}(G)$ such that

$$\theta_2(t) = \alpha \cdot \theta_1(t) \cdot \alpha^{-1}$$

for all $t \in \pi$. If this is the case, the resulting manifolds $G/\theta_1(\pi)$ and $G/\theta_2(\pi)$ are affinely diffeomorphic. Therefore, we try to classify all such discrete embeddings up to affine conjugation.

A 1-dimensional subgroup of G is of the form

$$\{\mathbf{x}^t : t \in \mathbb{R}\},$$

for a fixed element $\mathbf{x}(\neq e) \in G$. Let L be a 2-dimensional subgroup of G. Of course, L is then abelian. It is obvious that any 2-dimensional abelian subgroup of G must contain the center $\mathcal{Z}(G)$. Thus $\mathcal{Z}(G) \subset L$.

When π is the trivial group, there is nothing to do, so we start with the case where $\pi = \mathbb{Z}$.

1. The case: $\pi = \mathbb{Z}$

THEOREM 1.1. *An injective discrete representation of $\mathbb{Z} = \langle \zeta \rangle$ into $\mathrm{Isom}(G)$ is affinely conjugate to θ_1, θ_2 or θ_3 where*

$$\begin{aligned} \theta_1(\zeta) &= (\mathbf{x}_0, I), \\ \theta_2(\zeta) &= (\mathbf{z}_0, A) \ \textit{with} \ A \in \mathrm{SO}(2), \\ \theta_3(\zeta) &= (\mathbf{x}_0, \tau). \end{aligned}$$

Proof. Clearly each of these representations is injective and discrete. Moreover, they are affinely distinct from each other.

Suppose $\theta(\zeta) = (\mathbf{x}, I)$ where

$$\mathbf{x} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

We want to find

$$\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right) \in \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R}) = \mathrm{Aut}(G)$$

which conjugates $\mathbf{x}$ to $\mathbf{z}_0$ or $\mathbf{x}_0$.

Let us denote the image of $\mathbf{x} \in G$ on $G/\mathcal{Z}(G) = \mathbb{R}^2$ by $\overline{\mathbf{x}}$, etc. If $\overline{\mathbf{x}} = 0$ (i.e., $x = y = 0$), then $\mathbf{x} \in \mathcal{Z}(G)$. Any element of $\mathrm{GL}(2, \mathbb{R})$ of determinant $\frac{1}{z}$ maps $\mathbf{x}$ to $\mathbf{z}_0$. Thus, conjugation by this automorphism sends $(\mathbf{x}, I)$ to $(\mathbf{z}_0, I)$. This is of the form $\theta_2(\zeta) = (\mathbf{z}_0, A)$ with $A = I$.

If $\overline{\mathbf{x}} \neq 0$ then $x^2 + y^2 \neq 0$. One can find a matrix which maps $\overline{\mathbf{x}}$ to $\overline{\mathbf{x}}_0$ on the quotient $G/\mathcal{Z}(G)$. On G, this automorphism maps $\mathbf{x}$ to $\mathbf{x}_0 {\mathbf{z}_0}^s$. But then an inner automorphism can map $\mathbf{x}_0 {\mathbf{z}_0}^s$ to $\mathbf{x}_0$. Thus, we have found an automorphism which conjugates $(\mathbf{x}, I)$ to $(\mathbf{x}_0, I)$, Thus, in this case, we have $\theta_1(\zeta) = (\mathbf{x}_0, I)$.

Suppose $\theta(\zeta) = (\mathbf{x}, A)$ for some $A \in \mathrm{SO}(2)$. We want to find $\mathbf{a} \in G$ such that

$$(\mathbf{a}, I)(\mathbf{x}, A)(\mathbf{a}, I)^{-1} = (\mathbf{z}_0, A),$$

which is equivalent to $\mathbf{a} \cdot \mathbf{x} \cdot A(\mathbf{a})^{-1} = \mathbf{z}_0$. On the quotient $G/\mathcal{Z}(G)$, we have

$$\overline{\mathbf{a}} + \overline{\mathbf{x}} - A(\overline{\mathbf{a}}) = \overline{\mathbf{z}}_0.$$

Since $\overline{\mathbf{z}}_0 = \overline{0}$, this is $(A - I)\overline{\mathbf{a}} = \overline{\mathbf{x}}$. If $A \in \mathrm{SO}(2)$ and $A \neq I$, then $A - I$ is non-singular. Therefore, there always exists $\mathbf{a} \in G$ satisfying $(A - I)\overline{\mathbf{a}} = \overline{\mathbf{x}}$. With such an $\mathbf{a} \in G$, we have

$$\mathbf{a} \cdot \mathbf{x} \cdot A(\mathbf{a})^{-1} = {\mathbf{z}_0}^{\pm t^2}$$

for some $t \in \mathbb{R}$. Now $A_t = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R})$ maps

$$\mathbf{z}_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{to} \quad {\mathbf{z}_0}^{t^2} = \begin{bmatrix} 1 & 0 & t^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$\begin{aligned}(e,A_t)^{-1}[(\mathbf{a},I)(\mathbf{x},A)(\mathbf{a},I)^{-1}](e,A_t) &= (e,A_t)^{-1}({\mathbf{z}_0}^{\pm t^2},A)(e,A_t)\\ &= ({\mathbf{z}_0}^{\pm 1},A).\end{aligned}$$

Notice that we used the fact that $AA_t = A_tA$. For $({\mathbf{z}_0}^{-1}, A)$, we conjugate once again by (e,τ) to get $(\mathbf{z}_0, A^{-1})$. Note that $A^{-1} \in \mathrm{SO}(2)$ also. Consequently, we have obtained $\theta_2(\zeta) = (\mathbf{z}_0, A)$ with $A \in \mathrm{SO}(2)$ and $A \neq I$.

Finally, suppose $\theta(\zeta) = (\mathbf{x}, B)$ with $B \in \mathrm{O}(2) - \mathrm{SO}(2)$. Let $B = R_{2\varphi}\tau$. Then

$$(e,R_\varphi)^{-1}(\mathbf{x},B)(e,R_\varphi) = (\mathbf{x}',\tau)$$

where $\mathbf{x}' = R_\varphi^{-1}(\mathbf{x}) \in G$ by Lemma 0.1. Let

$$\mathbf{x}' = \begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}.$$

If $x = 0$, $(\mathbf{x}',\tau)^2 = (e, I)$. Thus, for the representation $\zeta \mapsto (\mathbf{x}',\tau)$ to be injective, we must have $x \neq 0$. A simple calculation shows that

$$(\mathbf{u},I)(\mathbf{x}',\tau)(\mathbf{u},I)^{-1} = ((\mathbf{x}_0)^x,\tau), \quad \text{where } \mathbf{u} = \begin{bmatrix}1 & 0 & (xy-2z)/4\\ 0 & 1 & -y/2\\ 0 & 0 & 1\end{bmatrix}.$$

Now conjugation by $\begin{bmatrix}1/x & 0\\ 0 & x\end{bmatrix} \in \mathrm{GL}(2,\mathbb{R}) \subset \mathrm{Aut}(G)$ sends $((\mathbf{x}_0)^x,\tau)$ to $(\mathbf{x}_0,\tau)$ as we desired. (Again, note that this matrix commutes with τ). This is of the type $\theta_3(\zeta) = (\mathbf{x}_0,\tau)$. □

2. The case: $\pi = \mathbb{Z}^2$

PROPOSITION 2.1. *A representation*

$$\theta : \mathbb{Z}^2 = \langle \zeta_1, \zeta_2\rangle \longrightarrow \mathrm{Isom}(G) = G \rtimes O(2)$$

is injective and discrete if and only if

$$\theta(\zeta_1) = (\mathbf{x}, I) \qquad \textit{and} \qquad \theta(\zeta_2) = (\mathbf{y}, I)$$

where

$$\mathbf{x} = \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 & y_1 & y_3 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix}$$

satisfy

$$x_1{}^2 + x_2{}^2 \neq 0, \ (y_1, y_2) = \lambda(x_1, x_2) \textit{ for some } \lambda,$$

$$\textit{and} \ \ y_3 \neq \lambda x_3 + \frac{\lambda^2 - \lambda}{2} x_1 x_2.$$

In this case, the smallest connected Lie subgroup containing $\mathbf{x}$ *and* $\mathbf{y}$ *is 2-dimensional, and contains the center* $\mathcal{Z}(G)$.

Proof. Suppose θ is of the form described in the statement. We shall prove θ is a discrete embedding. Since $(y_1, y_2) = \lambda(x_1, x_2)$ for some λ, the image of $\theta(\mathbb{Z}^2)$ in $\mathbb{R}^2 = G/\mathcal{Z}(G)$ lies in a 1-dimensional subgroup. Therefore, the smallest connected Lie subgroup containing $\mathbf{x}$ and $\mathbf{y}$ is at most 2-dimensional. Now the last condition ensures that the image is not contained in a 1-dimensional subgroup and hence, $\theta(\mathbb{Z}^2)$ is discrete. In fact, the two conditions

$$(y_1, y_2) = \lambda(x_1, x_2) \text{ for some } \lambda, \text{ and } y_3 \neq \lambda x_3 + \frac{\lambda^2 - \lambda}{2} x_1 x_2$$

is equivalent to

$$\mathbf{y} \neq \exp(\lambda \log \mathbf{x}) \quad (\text{i.e., } \mathbf{y} \neq \mathbf{x}^{\lambda}).$$

Conversely, suppose $\theta : \mathbb{Z}^2 \to \mathrm{Isom}(G) = G \rtimes O(2)$ is an injective discrete homomorphism. According to Lemma 0.2, there is a 2-dimensional subgroup L and a subgroup ${}'\mathbb{Z}^2$ of $\mathbb{Z}^2$ of finite index so that $\theta({}'\mathbb{Z}^2)$ acts on L as pure translations. Clearly, such a subgroup $L \cong \mathbb{R}^2$ must contain the center $\mathcal{Z}(G)$. This implies that an element of $\theta({}'\mathbb{Z}^2)$ either reverses the orientation of the orthogonal complement of L or leaves it fixed. Since there is no element of $\mathrm{Isom}(G)$ which reverses the orientation, $\theta('\mathbb{Z}^2)$ lies in the group of pure translations $G \subset \mathrm{Isom}(G)$.

We claim further that the whole group $\mathbb{Z}^2$ (not just the subgroup $'\mathbb{Z}^2$) is mapped into G. Suppose $\theta(\zeta) = (\mathbf{u}, A) \in G \rtimes O(2)$ for some $\zeta \in \mathbb{Z}^2$. We shall show $A = I$. Since $\mathbb{Z}^2$ is abelian,

$$(\mathbf{u}, A)(\mathbf{x}, I)(\mathbf{u}, A)^{-1} = (\mathbf{x}, I)$$

for every $\mathbf{x} \in' \mathbb{Z}^2 \subset L$ implies that $A(\mathbf{x}) = \mathbf{u}^{-1}\mathbf{x}\mathbf{u}$ for every $\mathbf{x} \in G$. However, $A \in \mathrm{O}(2)$ can never be equal to an inner automorphism, unless $A = I$. (That is, $\mathrm{O}(2) \subset \mathrm{Aut}(G) \to \mathrm{Out}(G)$ is injective). Therefore, A must be the identity. We have shown that $\theta(\mathbb{Z}^2) \subset G$. Let

$$\theta(\zeta_1) = (\mathbf{x}, I) \qquad \text{and} \qquad \theta(\zeta_2) = (\mathbf{y}, I).$$

Since $L \subset G$ contains $\mathcal{Z}(G)$, the image $\theta(\mathbb{Z}^2)$ lies in a line when we project down G to $\mathbb{R}^2 = G/\mathcal{Z}(G)$. Thus, there exists $\lambda \in \mathbb{R}$ such that $(y_1, y_2) = \lambda(x_1, x_2)$. The only condition for $\langle \mathbf{x}, \mathbf{y} \rangle$ to be discrete is that they do not lie in a 1-dimensional subgroup of L. Consider the images in the Lie algebra: $\log \mathbf{y} = \lambda \log \mathbf{x}$ in $\mathfrak{G}$ if and only if

$$(y_1, y_2) = \lambda(x_1, x_2) \quad \text{and} \quad y_3 = \lambda x_3 + \frac{\lambda^2 - \lambda}{2} x_1 x_2$$

Thus, $\langle \mathbf{x}, \mathbf{y} \rangle$ is discrete if and only if $y_3 \neq \lambda x_3 + \frac{\lambda^2 - \lambda}{2} x_1 x_2$. □

The following lemma will be used in Theorem 2.3 and Theorem 3.1.

LEMMA 2.2. *Let $\theta : \mathbb{Z}^2 \to \mathrm{Isom}(G)$ be an injective discrete representation. For any set of generators $\{\zeta_1, \zeta_2\}$ of $\mathbb{Z}^2$, there exists an element f of $\mathbb{R}^2 \rtimes (\mathrm{O}(2) \times \mathbb{R}^*) \subset \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$ such that, if we set $\theta'(\zeta) = f \cdot \theta(\zeta) \cdot f^{-1}$, then θ' is of the form*

$$\theta'(\zeta_1) = (\mathbf{x}_0, I), \quad \theta'(\zeta_2) = (\mathbf{x}_0{}^s \mathbf{z}_0{}^t, I),$$

or

$$\theta'(\zeta_2) = (\mathbf{x}_0, I), \quad \theta'(\zeta_1) = (\mathbf{x}_0{}^s \mathbf{z}_0{}^t, I),$$

with some s and non-zero t.

Proof. According to Proposition 2.1, every injective discrete representation of $\mathbb{Z}^2$ into $\mathrm{Isom}(G)$ has its image in the pure translation group G. Clearly, $\theta(\mathbb{Z}^2) \not\subset \mathcal{Z}(G) = \mathbb{R}$.

One of $\theta(\zeta_1)$ and $\theta(\zeta_2)$ is not in $\mathcal{Z}(G)$. Assume $\theta(\zeta_1) \notin \mathcal{Z}(G)$. This will yield the first case. There is an element of $\mathrm{O}(2) \times \mathbb{R}^* \subset \mathrm{GL}(2, \mathbb{R}) \subset \mathrm{Aut}(G)$, where $\mathbb{R}^*$ is the group of scalar matrices, which maps $\theta(\zeta_1)$ to $(\mathbf{x}_0\mathbf{w}, I)$ for some $\mathbf{w} \in \mathcal{Z}(G)$. Now using an inner automorphism in $\mathrm{Inn}(G) \cong \mathbb{R}^2$, we can map $\mathbf{x}_0\mathbf{w}$ to just $\mathbf{x}_0$. Consequently, we conjugated $\theta(\zeta_1)$ to $(\mathbf{x}_0, I)$. In the course of this conjugation, $\theta(\zeta_2)$ becomes of the form $\theta(\zeta_2) = ({\mathbf{x}_0}^s{\mathbf{z}_0}^t, I)$, for some $s, t \in \mathbb{R}$. Thus, by conjugation by an element of $\mathbb{R}^2 \rtimes (\mathrm{O}(2) \times \mathbb{R}^*)$, we have made

$$\theta(\zeta_1) = (\mathbf{x}_0, I), \quad \theta(\zeta_2) = ({\mathbf{x}_0}^s{\mathbf{z}_0}^t, I).$$

If $\theta(\zeta_2) \notin \mathcal{Z}(G)$, we get the second case by the same argument. Clearly, t cannot be zero, for otherwise the representation will not be discrete.□

THEOREM 2.3. *An injective discrete representation of* $\mathbb{Z}^2 = \langle \zeta_1, \zeta_2 \rangle$ *into* $\mathrm{Isom}(G)$ *is affinely conjugate to* θ_1 *or* θ_2 *where*

$$\theta_1(\zeta_1) = (\mathbf{x}_0, I), \quad \theta_1(\zeta_2) = ({\mathbf{x}_0}^s\mathbf{z}_0, I),$$
$$\theta_2(\zeta_1) = ({\mathbf{x}_0}^s\mathbf{z}_0, I), \quad \theta_2(\zeta_2) = (\mathbf{x}_0, I),$$

where $s \in \mathbb{R}$. *For distinct* s, *the representations are affinely distinct. However, the resulting manifolds* $\theta(\mathbb{Z}^2)\backslash G$ *are all diffeomorphic.*

Proof. By Lemma 2.2, we may assume that, after conjugating by an element of $\mathbb{R}^2 \rtimes (\mathrm{O}(2) \times \mathbb{R}^*)$,

$$\theta(\zeta_1) = (\mathbf{x}_0, I), \quad \theta(\zeta_2) = ({\mathbf{x}_0}^s{\mathbf{z}_0}^t, I)$$

with $t \neq 0$, or the generators ζ_1 and ζ_2 interchanged. The automorphism $C = \begin{bmatrix} 1 & 0 \\ 0 & 1/t \end{bmatrix}$ maps ${\mathbf{x}_0}^s{\mathbf{z}_0}^t$ to ${\mathbf{x}_0}^s\mathbf{z}_0$. Observe that this conjugation does not change $\theta(\zeta_1) = (\mathbf{x}_0, I)$. Therefore, conjugation by (e, C) changes our representation to the desired form. This yields the representation θ_1. If the generators ζ_1 and ζ_2 are interchanged, we get the representation θ_2. It is clear that, for distinct s, the representations are affinely distinct.

For all s, the resulting spaces $\theta(\mathbb{Z}^2)\backslash G$ are torus bundles over $\mathbb{R}^1$, therefore they are equivalent as bundles smoothly. So they are diffeomorphic to each other. □

3. The case: $\pi = K$ (Klein bottle group)

THEOREM 3.1. *An injective discrete representation of*

$$K = \langle t_1, t_2, \alpha \mid [t_1, t_2] = 1, \alpha^2 = t_1, \alpha t_2 \alpha^{-1} = t_2^{-1} \rangle$$

into $\mathrm{Isom}(G)$ *is affinely conjugate to* θ_1 *or* θ_2 *where*

$$\theta_1(t_1) = (\mathbf{x}_0, I), \quad \theta_1(t_2) = (\mathbf{z}_0, I), \quad \theta_1(\alpha) = (\mathbf{x}_0{}^{\frac{1}{2}}, \tau)$$
$$\theta_2(t_1) = (\mathbf{z}_0, I), \quad \theta_2(t_2) = (\mathbf{x}_0, I), \quad \theta_2(\alpha) = (\mathbf{z}_0{}^{\frac{1}{2}}, -I).$$

Proof. Let $\theta : K \hookrightarrow \mathrm{Isom}(G)$ be a discrete injective representation. From Proposition 2.1, we know $\theta(\mathbb{Z}^2)$ lies in $L \subset G \subset \mathrm{Isom}(G)$ which contains the center $\mathcal{Z}(G)$. Furthermore, Lemma 2.2 says that, after conjugation by an element of $\mathbb{R}^2 \rtimes (\mathrm{O}(2) \times \mathbb{R}^*)$, we may assume that

$$\theta(t_1) = (\mathbf{x}_0, I), \quad \theta(t_2) = (\mathbf{x}_0{}^s \mathbf{z}_0{}^t, I), \ t \neq 0 \tag{1}$$

or

$$\theta(t_2) = (\mathbf{x}_0, I), \quad \theta(t_1) = (\mathbf{x}_0{}^s \mathbf{z}_0{}^t, I), \ t \neq 0. \tag{2}$$

Since $\mathrm{O}(2) \times \mathbb{R}^*$ normalizes $\mathrm{O}(2)$, we have

$$\theta(\alpha) = (c, C), \quad C \in \mathrm{O}(2).$$

The point here is that the matrix C still lies in $\mathrm{O}(2)$ after conjugation. The automorphism C has order 2. The center is a characteristic subgroup so that $C(\mathbf{z}_0) = \mathbf{z}_0$ or $\mathbf{z}_0^{-1}$.

<u>Case 1: $C(\mathbf{z}_0) = \mathbf{z}_0{}^{-1}$</u>. In this case, $\det(C) = -1$ because $C(\mathbf{z}_0) = \det(C)\ \mathbf{z}_0$. From the presentation of K, we must have

$$(c, C)^2 = \theta(\zeta_1), \quad (c, C)\theta(\zeta_2)(c, C)^{-1} = \theta(\zeta_2)^{-1}$$

since $\theta(\alpha) = (c, C)$.

Suppose we are in case (2). Interpreting the second equality on the quotient $\mathbb{R}^2 = G/\mathcal{Z}(G)$, we have

$$C \in \mathrm{O}(2), \quad C \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

(Note that the image of $\mathbf{x}_0$ in $\mathbb{R}^2$ is $\begin{bmatrix}1\\0\end{bmatrix}$). This makes $C = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$. We try to find $c \in G$ for which

$$(c, C)^2 = ({\mathbf{x}_0}^s {\mathbf{z}_0}^t, I),\ t \neq 0$$

is satisfied. Let

$$c = \begin{bmatrix}1 & x_1 & z_1\\ 0 & 1 & y_1\\ 0 & 0 & 1\end{bmatrix}.$$

On the quotient $\mathbb{R}^2$, the above equation is

$$\bar{c} + C\bar{c} = s\overline{\mathbf{x}_0}, \quad \text{that is,}\ \begin{bmatrix}x_1\\ y_1\end{bmatrix} + \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ y_1\end{bmatrix} = \begin{bmatrix}s\\ 0\end{bmatrix}$$

which implies $y_1 = 0$ and $s = 0$. Now one can see easily that

$$Cc = c^{-1}.$$

This implies that the equality $(c, C)^2 = ({\mathbf{x}_0}^s {\mathbf{z}_0}^t, I),\ t \neq 0$ is never possible.

Consequently, we must be in case (1).

$$(c, C)^2 = (\mathbf{x}_0, I), \quad (c, C)({\mathbf{x}_0}^s {\mathbf{z}_0}^t, I)(c, C)^{-1} = ({\mathbf{x}_0}^s {\mathbf{z}_0}^t, I)^{-1}$$

The first equality implies that (c, C) and $(\mathbf{x}_0, I)$ commute with each other. In particular, $C(\overline{\mathbf{x}_0}) = \overline{\mathbf{x}_0}$ on the quotient $\mathbb{R}^2$. Any $C \in O(2)$ which fixes $\overline{\mathbf{x}_0} = \mathbf{e}_1$ and has determinant -1 is $C = \tau$. Then it is easy to see that s must be 0, and $c = {\mathbf{x}_0}^{\frac{1}{2}}$. Thus

$$\theta(t_1) = (\mathbf{x}_0, I), \quad \theta(t_2) = ({\mathbf{z}_0}^t, I), \quad \theta(\alpha) = ({\mathbf{x}_0}^{\frac{1}{2}}, \tau).$$

Now conjugation by the automorphism $\begin{bmatrix}1 & 0\\ 0 & \frac{1}{t}\end{bmatrix}$ maps ${\mathbf{z}_0}^t$ to $\mathbf{z}_0$ without changing any other generators, because this matrix commutes with τ. Consequently, we have obtained the representation θ_1.

Case 2: $C(\mathbf{z}_0) = \mathbf{z}_0$. If $C(\mathbf{z}_0) = \mathbf{z}_0$, then $C \in \mathrm{SO}(2)$. The only element of SO(2) of order 2 is $-I$. Then, necessarily, it is the case of (2). The presentation of group K yields $s = 0$. Thus

$$\theta(t_2) = (\mathbf{x}_0, I), \quad \theta(t_1) = ({\mathbf{z}_0}^t, I), \quad \theta(\alpha) = (c, -I).$$

Moreover, $\theta(\alpha)^2 = \theta(t_1)$ yields $c = {\mathbf{z}_0}^{\frac{1}{2}t}$. Now conjugation by the automorphism $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{t} \end{bmatrix}$ maps ${\mathbf{z}_0}^t$ to $\mathbf{z}_0$ without changing any other generators, because this matrix commutes with $-I$. Consequently, we have obtained the representation θ_2. □

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Heung Ki Kim
Dankook University
Seoul 140–714, Korea
E-mail: hkkim@ns.dankook.ac.kr

Sung Mo Im
Chungbuk National University
Cheongju 361–763, Korea
E-mail: sungim@cbucc.chungbuk.ac.kr