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BYPATHS IN LOCAL TOURNAMENTS

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ABSTRACT. A digraph T is called a local tournament if for every vertex x of T, the set of in-neighbors as well as the set of out-neighbors of x induce tournaments. Let x and y be two vertices of a 3-connected and arc-3-cyclic local tournament T with $y \not\to x$. We investigate the structure of T such that T contains no (x, y)-path of length k for some k with $3 \leq k \leq |V(T)| - 1$. Our result generalizes those of [2] and [15] for tournaments.

1. Introduction

A digraph D is *arc-k-cyclic* if every arc of D is contained in a k-cycle. We say that D is *arc-pancyclic* if it is arc-k-cyclic for all k satisfying $3 \leq k \leq |V(D)|$.

In [1], it is proved that every regular tournament is arc-pancyclic. The structure of all arc-3-cyclic, but not arc-pancyclic tournaments has been completely determined in [18].

A path from a vertex x to another vertex y is said to be a *bypath* if $y \not\to x$. A digraph D is *arc-k-anticyclic* for some $k \geq 3$ if every arc of D has a bypath of length $k - 1$.

It is shown in [2] that a regular tournament on at least 7 vertices is arc-k-anticyclic for all $k \geq 4$ (i.e., every arc of such a tournament has a bypath of length m for all $m \geq 3$).

A digraph is *strongly arc-k-cyclic* if it is arc-k-cyclic and arc-k-anticyclic. A digraph is *strongly arc-pancyclic* if it is strongly arc-k-cyclic for all $k \geq 3$.

In [17], some sufficient conditions are given for tournaments to be strongly arc-k-cyclic for all $k \geq 4$. A characterization of strongly arc-pancyclic tournaments is found in [19], it states that a tournament is

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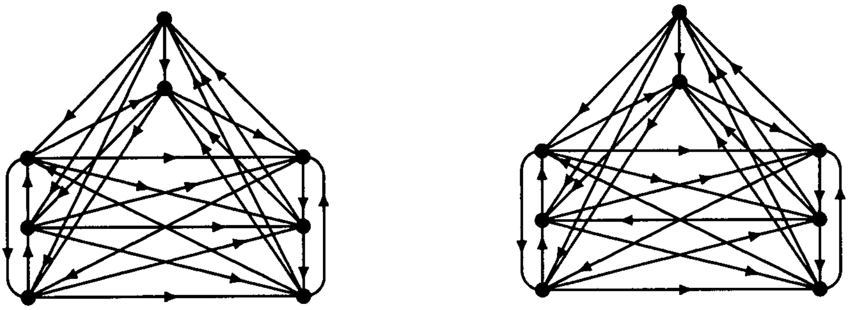
strongly arc-pancyclic if and only if it is 2-connected and strongly arc-3-cyclic.

Recently, Volkmann and the author [15] proved the following result, which generalizes the above mentioned result in [2].

THEOREM 1.1 ([15]). *Let T be a 3-connected and arc-3-cyclic tournament. Then every arc of T has a bypath of length k for all $k \geq 3$, unless T is isomorphic to T_8^4 or to T_8^5.*



T_8^4 T_8^5

Figure 1

From the before mentioned last two results, we see that the arc-3-anticyclicity condition for an arc-3-cyclic tournament to be strongly arc-pancyclic is of consequence only for those tournaments that are exactly 2-connected.

In 1990, Bang-Jensen [3] introduced a very interesting generalization of tournaments — the class of locally semicomplete digraphs. A digraph D is *locally semicomplete* if for every vertex x, the set of in-neighbors as well as the set of out-neighbors of x induce semicomplete digraphs (a digraph is *semicomplete* if for any two different vertices x and y, there is at least one arc between them).

A *local tournament* is a locally semicomplete digraph without a cycle of length two. It is obvious that the class of local tournaments is a superclass of tournaments.

Since their introduction by Bang-Jensen, locally semicomplete digraphs have been intensively studied (e.g. [3]–[14] and [16]).

The arc-pancyclicity and strongly arc-pancyclicity in local tournaments have been studied in [8] and [9], respectively.

In [12], the author considered the path-connectivity between any two vertices of a local tournament.

A digraph D is said to be *generalized arc-pancyclic* if D is arc-pancyclic and for any two nonadjacent vertices $x, y \in V(D)$, there are an (x,y)-path of length k and a (y,x)-path of length k for each $k \in \{2, 3, ..., |V(D)|-1\}$.

A digraph D is *strongly path-panconnected* if for any two vertices $x, y \in V(D)$ and any integer k with $2 \leq k \leq |V(D)| - 1$, there is an (x,y)-path of length k and a (y,x)-path of length k in D.

A characterization of generalized arc-pancyclic local tournaments is given in [12] (see Theorem 3.1 and Corollary 3.4 there). As an immediate consequence of this characterization, we note the following statement.

PROPOSITION 1.2. *Let D be a 3-connected and arc-3-cyclic local tournament. Then D is generalized arc-pancyclic, unless D is isomorphic to one of $\{T_8^1, T_8^2\}$.*

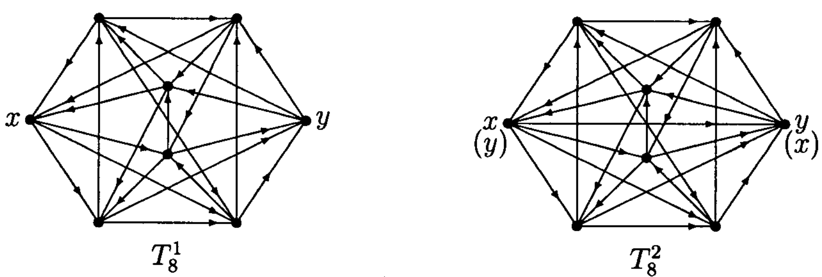


Figure 2

It is easy to see that there is no (x,y)-path of length 7 in T_8^1 or in T_8^2.

Under the condition that D is strongly arc-3-cyclic, the author [12] studied the strongly path-panconnectivity in local tournaments (see Theorem 4.2 and Corollary 4.5 there).

In this paper, we shall investigate bypaths in 3-connected and arc-3-cyclic tournaments. Our result extends Theorem 1.1 above to local tournaments.

2. Terminology and preliminaries

We only consider finite digraphs without loops and multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively.

If xy is an arc of D, then we say that x *dominates* y. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \to B$.

The *outset* of a vertex x of a digraph D is the set $N^+(x) = \{ y \mid xy \in E(D)\}$. Similarly, $N^-(x) = \{ y \mid yx \in E(D)\}$ is the *inset* of x. More generally, for a subdigraph A of a digraph D, we define its *outset* by $N^+(A) = \bigcup_{x \in V(A)} N^+(x) - A$ and its *inset* by $N^-(A) = \bigcup_{x \in V(A)} N^-(x) - A$. Every vertex of $N^+(A)$ is called an *out-neighbor* of A and every vertex of $N^-(A)$ is an *in-neighbor* of A. The *neighborhood* of A is defined by $N(A) = N^+(A) \cup N^-(A)$.

The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$. In addition, $D - A = D\langle V(D) - A\rangle$.

Paths and cycles in a digraph always are directed. A path from x to y is called an (x, y)-path. A k-cycle is a cycle of length k.

A *strong component* H of D is a maximal subdigraph such that for any two vertices $x, y \in V(H)$, the subdigraph H contains an (x, y)-path and a (y, x)-path. A digraph D is *strong* if it has only one strong component, and D is k-connected if for any set A of at most $k - 1$ vertices, the subdigraph $D - A$ is strong.

A digraph is *connected*, if its underlying graph is connected. In this paper, we only consider connected digraphs.

If we replace every arc xy of D by yx, then we call the resulting digraph the *converse digraph* of D.

We note that the converse digraph of a locally semicomplete digraph also is locally semicomplete.

For the proofs in this paper, we need the following known results.

THEOREM 2.1 ([3]). *A connected locally semicomplete digraph contains a hamiltonian path and every strong locally semicomplete digraph has a hamiltonian cycle.*

PROPOSITION 2.2 ([4]). *Let D be a locally semicomplete digraph and let $P_1 = x_1x_2 \cdots x_p$ and $P_2 = y_1y_2 \cdots y_q$ be two vertex-disjoint paths in D with $p \geq 2$ and $q \geq 1$. If there are two integers i and j with $1 \leq i < j \leq p$ such that x_iy_1, y_qx_j are two arcs of D, then D has an (x_1, x_p)-path P such that $V(P) = V(P_1) \cup V(P_2)$.*

THEOREM 2.3 ([3]). *Let D be a connected locally semicomplete digraph that is not strong. Then the following holds:*

(a) *If A and B are two strong components of D, then either there is no arc between them or $A \to B$ or $B \to A$.*
(b) *If A and B are two strong components of D such that A dominates B, then A and B are both semicomplete digraphs.*
(c) *The strong components of D can be ordered in a unique way $D_1, D_2, \cdots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \cdots, p-1$.*

The unique sequence $D_1, D_2, \cdots, D_p$ of the strong components of D in Theorem 2.3 (c) is called the *strong decomposition* of D.

LEMMA 2.4 ([12]). *Let T be a connected and arc-3-cyclic local tournament on n vertices. If T contains a path $P = a_1 a_2 \cdots a_k$ with $3 \leq k \leq n-1$, but there is no path from a_1 to a_k of length k, then for every vertex $v \notin V(P)$, there exist two integers $\mu(v)$ and $\nu(v)$ with $1 \leq \mu(v) < \nu(v) \leq k$ such that*

$$N^+(v) \cap V(P) = \{a_1, a_2, \cdots, a_{\mu(v)}\} \text{ and}$$
$$N^-(v) \cap V(P) = \{a_{\nu(v)}, a_{\nu(v)+1}, \cdots, a_k\}.$$

Furthermore, the subdigraph $T - V(P)$ is a tournament.

LEMMA 2.5 ([12]). *Let T be a 2-connected and arc-3-cyclic local tournament on n vertices. Suppose that T contains a path $P = a_1 a_2 \cdots a_k$ with $4 \leq k \leq n-1$ and there exist two integers μ, ν with $2 \leq \mu < \nu \leq k-1$ such that*

$$N^-(H) \cap V(P) = B \to H \to A = N^+(H) \cap V(P),$$

where $H = V(T - V(P))$, $B = \{a_\nu, a_{\nu+1}, \cdots, a_k\}$ and $A = \{a_1, a_2, \cdots, a_\mu\}$. If T contains no path from a_1 to a_k of length k, then the following statements are true.

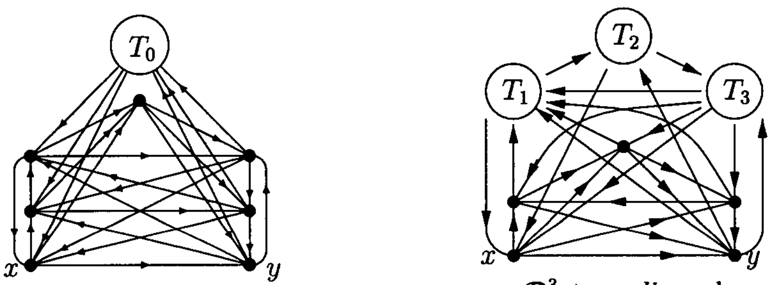
(a) *$N^+(a_\mu) \cap B = \{a_\nu\}$ or $N^-(a_\nu) \cap A = \{a_\mu\}$.*
(b) *If $N^+(a_\mu) \cap B = \{a_\nu\}$, then $T\langle B\rangle$ is a tournament containing a unique (a_ν, a_k)-path; if $N^-(a_\nu) \cap A = \{a_\mu\}$, then $T\langle A\rangle$ is a tournament containing a unique (a_1, a_μ)-path.*
(c) *The inequality $\nu \leq \mu + 2$ holds. Furthermore, if $\nu = \mu + 2$, then $A \to a_{\mu+1} \to B$ and*

$$N^+(a_\mu) \cap B = \{a_{\mu+2}\} \text{ and } N^-(a_{\mu+2}) \cap A = \{a_\mu\}.$$

3. Main results

We first determine when two prescribed vertices x, y with $y \not\to x$ are connected by an (x, y)-path of length k for all $k \geq 3$ in a 3-connected and arc-3-cyclic local tournament.

THEOREM 3.1. *Let T be a 3-connected and arc-3-cyclic local tournament on n vertices. If x and y are two distinct vertices of T and $yx \notin E(T)$, then T contains an (x, y)-path of length k for each $k \geq 3$, unless T is isomorphic to one of $\{T_8^1, T_8^2, T_8^4, T_8^5\}$ or to a $\mathcal{D}_8^1$-type digraph or to a $\mathcal{D}_8^3$-type digraph, where T_i is an arc-3-cyclic tournament for $i = 0, 1, 2, 3$.*



$\mathcal{D}_8^1$-type digraph $\qquad$ $\mathcal{D}_8^3$-type digraph

Figure 3

Proof. If x and y are not adjacent, then, by Proposition 1.2, T contains no (x, y)-path of length k for some $k \geq 2$ if and only if T is isomorphic to one of $\{T_8^1, T_8^2\}$.

It remains to show that the arc xy has a bypath of length k for all $k \geq 3$ if T is not isomorphic to one of $\{T_8^2, T_8^4, T_8^5\}$ or to a $\mathcal{D}_8^1$-type digraph or to a $\mathcal{D}_8^3$-type digraph.

We first show that xy has a bypath of length 3. Since T is a local tournament, $T\langle N^+(x)\rangle$ is a tournament. If $|N^+(x) \cap N^-(y)| \geq 2$, then, by Theorem 2.1, there is an (x, y)-path of length 3. So, we may assume that $|N^+(x) \cap N^-(y)| \leq 1$. Since T is 3-connected, x has at least three out-neighbors, and hence, there is a vertex u belonging to $N^+(x) \cap N^+(y)$. Because yu is in a 3-cycle, there is a vertex v with $u \to v \to y$. Now we see that $xuvy$ is a desired path.

Suppose that T contains (x, y)-paths of all lengths from 3 to $k-1$, but T contains no (x, y)-path of length k with $4 \leq k \leq n-1$. Let $P = a_1 a_2 \cdots a_k$ be an (x, y)-path with $x = a_1$ and $a_k = y$. According to Lemma 2.4, for every vertex v of $H = V(T - V(P))$, there are two integers $\mu(v), \nu(v)$ with $1 \leq \mu(v) < \nu(v) \leq k$ such that

$$N^+(v) \cap V(P) = \{a_1, a_2, \cdots, a_{\mu(v)}\} \text{ and}$$
$$N^-(v) \cap V(P) = \{a_{\nu(v)}, a_{\nu(v)+1}, \cdots, a_k\}.$$

Moreover, $T\langle H\rangle$ is a subtournament of T.

Assume that $k = 4$. Since T is 3-connected, every vertex of T has at least three out- and in-neighbors. It follows that $T\langle V(P)\rangle$ is a transitive tournament. Because a_2a_3 is in a 3-cycle, there is a vertex $z \in H$ with $a_3 \to z \to a_2$. But now, $a_1a_3za_2a_4$ is an (x, y)-path of length 4 and we obtain a contradiction to the initial hypothesis that T contains no (x, y)-path of length k. Therefore, $k \geq 5$ holds.

In the following proof, we do not repeat the sentence "we obtain a contradiction to the initial hypothesis" if we find an (x, y)-path of length k.

CLAIM 1. If H contains two vertices having different outsets in P or different insets in P, then T is isomorphic to a $\mathcal{D}_8^3$-type digraph.

Proof. Since the converse digraph of any $\mathcal{D}_8^3$-type digraph also is a $\mathcal{D}_8^3$-type digraph, we only need to investigate the case that H contains two vertices having different outsets in P. Let $H_i = \{\, v \,|v \in H,\ |N^+(v) \cap V(P)| = i\}$ and

$$\alpha = \min\{\, i \,|H_i \neq \emptyset\} \text{ and } \beta = \max\{\, i \,|H_i \neq \emptyset\}.$$

Then $1 \leq \alpha < \beta \leq k - 1$. Since T is 3-connected, we have $\beta \geq 3$.

Let $u \in H_\alpha$ satisfying $\nu(u) \leq \nu(u')$ for each $u' \in H_\alpha$ and let $v \in H_\beta$ satisfying $\nu(v) \leq \nu(v')$ for each $v' \in H_\beta$.

Case 1. $\nu(u) \geq \alpha + 2$.

Because of $H_i \to a_{\alpha+1}$ for all $i > \alpha$, $H_\alpha \to H_i$ for all $i > \alpha$. Since every arc from H_α to H_β is in a 3-cycle, $\beta \geq \nu(u)$ holds. Clearly, $T\langle\{a_1, a_2, \cdots, a_\beta\}\rangle$ and $T\langle\{a_{\nu(u)}, a_{\nu(u)+1}, \cdots, a_k\}\rangle$ are two subtournaments. Because $u \to a_1 \to a_k$ and every vertex a_i with $\alpha < i < \nu(u)$ is not adjacent to u, we have $a_1 \to \{a_{\alpha+1}, \cdots, a_{\nu(u)-1}\} \to a_k$.

Subcase 1.1. $\alpha \geq 2$.

Since the arc va_α is in a 3-cycle, $a_\alpha \to a_j$ for some $j \geq \nu(v) > \nu(u)$. Hence, $a_1a_{\alpha+1}a_{\alpha+2}\cdots a_{j-1}ua_2\cdots a_\alpha a_j\cdots a_k$ is an (x,y)-path of length k.

Subcase 1.2. $\alpha = 1$.

Since $|N^-(H_1)| \geq 3$ and $H_1 \to H_i$ for all $i \geq 2$, we have $3 \leq \nu(u) \leq k-2$. Note that a_1 and a_3 are adjacent. If $a_1 \to a_3$, then $a_1a_3\cdots a_{k-2}uva_2a_k$ is of length k. So we may assume that $a_3 \to a_1$. It follows that $\nu(u) = 3$.

Because T is a local tournament and $a_2 \to a_k$ and $a_2 \notin N(u)$, one can successively deduce that $a_2 \to a_i$ for $i = k-1, k-2, \cdots, 3$. If $a_1 \to a_4$, then $a_1a_4\cdots a_{k-1}uva_2a_k$ is of length k. So we consider the case when $a_1 \not\to a_4$. Since a_1 has at least three out-neighbors, the integer $\ell = \min\{i | a_1 \to a_i, 5 \leq i \leq k-1\}$ is well defined and $k \geq 6$. Note that a_3 and a_{k-1} are adjacent.

Suppose that $a_{k-1} \to a_3$. Then $a_1a_\ell\cdots a_{k-1}a_3\cdots a_{\ell-2}uva_2a_k$ is of length k.

Suppose now that $a_3 \to a_{k-1}$. If $k \geq 7$, then $a_1a_2a_5\cdots a_{k-2}uva_3a_{k-1}a_k$ is of length k. If $k=6$, then $\ell = 5$. It is obvious that we may assume $a_6 \to \{a_3, a_4\}$. Since a_3a_4 is in a 3-cycle, there is a vertex $z \in H$ with $a_4 \to z \to a_3$. Thus, $a_1a_2a_4za_3a_5a_6$ is of length 7.

Case 2. $\nu(u) = \alpha + 1$.

Subcase 2.1. $\alpha = 1$.

If H_1 contains a vertex u' with $\nu(u') > \nu(u) = 2$, then let $H_{11} = \{\, z \mid z \in H_1,\ a_2 \to z\}$ and $H_{12} = H_1 - H_{11}$. By the same arguments as above, we conclude that $H_{11} \to H_{12} \to H_i$ for all $i \geq 2$ and $a_1 \to \{a_2, a_3, \cdots, a_{\nu(u')-1}\} \to a_k$. Note again that $\beta \geq 3$.

Suppose that $k=5$. We first consider the case when $a_4 \in N^-(H_{12})$ and assume without loss of generality that $a_4 \to u'$. Note that $a_2 \to a_4$. Since a_2a_3 is in a 3-cycle, there is a vertex z with $a_3 \to z \to a_2$. If $z = a_1$ can hold, then $a_1 \to a_4$ (because of $|N^+(a_1)| \geq 3$), and hence, $a_1a_4u'va_2a_5$ is of length 5. Therefore, $a_1 \to a_3$ and $z \in H$. Now, we see that $a_1a_3a_4za_2a_5$ is of length 5. So, we consider the other case when $a_4 \notin N^-(H_{12})$. Note that $a_1 \to \{a_2, a_3, a_4\} \to a_k$. Since a_3a_4 is in a 3-cycle, we have $a_4 \to a_2$. From the fact that a_4a_5 is in a 3-cycle, we conclude that $\beta = 4$. It is easy to check that $H_\beta \to H_{11}$. Furthermore, if there is a set $H_j \in \{H_2, H_3\}$ with $H_j \neq \emptyset$, then $H_{12} \to H_j$. Since the arcs from H_{12} to H_j are in 3-cycles, there is a vertex $w \in H_j$ such that $w \in N^-(H_{11})$. It follows that w is adjacent to a_4 because of $a_4 \to H_{11}$. Clearly, $a_4 \to w$, and hence, $a_1a_4wa_2a_3a_5$ is of length 5. Therefore, $H_2 = H_3 = \emptyset$ and T is isomorphic to a $\mathcal{D}_8^3$-type digraph.

Suppose now that $k \geq 6$. Note again that $u \to u' \to v$. If $a_1 \to a_i$ for some $3 \leq i \leq 5$, then $a_1a_i \cdots a_{k-(6-i)}uu'va_2a_k$ is of length k. Therefore, we may assume that $\nu(u') = 3$ and $a_1 \to a_j$ for some j with $6 \leq j < k$. If $a_3 \to a_{k-1}$ ($a_{k-1} \to a_3$, respectively), then $a_1a_2a_4 \cdots a_{k-3}u'va_3a_{k-1}a_k$ ($a_1a_j \cdots a_{k-1}a_3 \cdots a_{j-2}u'va_2a_k$, respectively) is of length k.

In the following, we consider the case when $a_2 \to H_1$.

Since T is 3-connected, there is an arc from H_1 to H_γ for some $\gamma \geq 2$. Assume without loss of generality that uv' is such an arc. Because a_iu is in a 3-cycle for $i \geq 4$, there is a vertex a such that $u \to a \to a_i$. If $a \in H$, then $a_1a_2uaa_4 \cdots a_{k-1}a_k$ is a path of length k. If $a \notin H$, then $a = a_1$. This means that $a_1 \to a_i$ for all $i \geq 4$.

If $a_j \to a_k$ for some j with $2 \leq j \leq k-3$, then $a_1a_{j+2} \cdots a_{k-1}uv'a_2 \cdots a_ja_k$ is of length k. Hence, we have $a_k \to \{a_2, \cdots, a_{k-3}\}$. Since a_k has at least 3 in-neighbors in P, a_{k-2} dominates a_k. It follows that $N^+(H_1) \cap H \subseteq H_2$ (otherwise, $a_1a_2uv''a_3 \cdots a_{k-2}a_k$ is a path of length k for some $v'' \in H_i$ with $i \geq 3$). If there is a vertex $z \in H - H_1$ with $a_{k-1} \to z$, then $a_1a_{k-1}za_2 \cdots a_{k-2}a_k$ is of length k. It follows that $N^+(H_1) \cap H = H_2$ and $H_2 \to H_i$ for all $i \geq 3$. In particular, $H_2 \to H_\beta$ with $\beta = k - 1$. Since va_2 is in a 3-cycle, we have $k \geq 6$. But now, $a_1a_2uu'va_5 \cdots a_{k-1}a_k$ is of length k.

Subcase 2.2. $\alpha \geq 2$.

Similar to the proof above, one can prove that $\{a_{k-1}, a_k\} \to H$ if T is not isomorphic to a $\mathcal{D}_8^3$-type digraph. So, we may assume that $\{a_{k-1}, a_k\} \to H \to \{a_1, a_2\}$.

Since a_1 has at least three out-neighbors, a_1 dominates a_p for some p with $3 \leq p \leq k-1$. Assume that $3 \leq p \leq \mu(v)$. Since va_{p-1} is in a 3-cycle, there is a vertex z with $a_{p-1} \to z \to v$. If $z \in V(P)$, then $z = a_i$ for some $i \geq \nu(v)$ and $a_1a_p \cdots a_{i-1}ua_2 \cdots a_{p-1}a_i \cdots a_k$ is a path of length k. So, we consider the case when $z \in H$. By the observation

$\alpha \geq 2$, we have $p \geq 4$. Since va_{p-2} is in a 3-cycle, there is a vertex z' with $a_{p-2} \to z' \to v$.

If $z' \in H$, then the path $a_1a_2 \cdots a_{p-2}z'va_pa_{p+1} \cdots a_{k-1}a_k$ is of length k. If $z' \notin H$, then it is obvious that $z' = a_j$ for some $j \geq \nu(v)$. Now we see that $a_1a_p \cdots a_{j-1}zva_2 \cdots a_{p-2}a_j \cdots a_k$ is an (x, y)-path of length k. Hence, we may assume that $\nu(v) \leq p < k$. By the same arguments, it can be assumed that $a_q \to a_k$ for some q with $2 \leq q \leq \mu(u)$.

If $p - q = 2$, then $a_1a_p \cdots a_{k-1}uva_2 \cdots a_qa_k$ or $a_1a_p \cdots a_{k-1}vua_2 \cdots a_qa_k$ is a path of length k. So, we have $p - q \geq 3$. But now, $a_1a_p \cdots a_{k-1}va_{q+1} \cdots a_{p-2}ua_2 \cdots a_qa_k$ (if $a_{q+1} \to u$) or $a_1a_p \cdots a_{k-1}va_{q+2} \cdots a_{p-1}ua_2 \cdots a_qa_k$ (if $u \to a_{q+1}$) is a path of length k in T. □

CLAIM 2. If all vertices of H have the same outset and inset in P, then T is isomorphic to one of $\{T_8^2, T_8^4, T_8^5\}$ or to a $\mathcal{D}_8^1$-type digraph.

Proof. Let

$$\mu = \max\{\, i \mid a_i \in N^+(H)\}, \quad \nu = \min\{\, i \mid a_i \in N^-(H)\},$$

$$A = \{a_1, a_2, \cdots, a_\mu\} \text{ and } B = \{a_\nu, a_{\nu+1}, \cdots, a_k\}.$$

Then $N^-(H) \cap V(P) = B \to H \to A = N^+(H) \cap V(P)$. By the assumption that T is 3-connected, $3 \leq \mu < \nu \leq k - 2$ holds.

We consider the following two cases.

Case 1. $\nu \geq \mu + 2$.

According to Lemma 2.5 (c), $\nu = \mu + 2$ and $A \to a_{\mu+1} \to B$. Furthermore,

$$(1) \qquad N^-(a_{\mu+2}) \cap A = \{a_\mu\} \text{ and } N^+(a_\mu) \cap B = \{a_{\mu+2}\}.$$

It follows from Lemma 2.5 (b) that $T\langle A\rangle$ is a tournament containing a unique (a_1, a_μ)-path and $T\langle B\rangle$ is a tournament containing a unique $(a_{\mu+2}, a_k)$-path.

Because $a_\mu \to \{a_1, a_{\mu+2}\}$ and $\{a_\mu, a_k\} \to a_{\mu+2}$, we have $a_{\mu+2} \in N(a_1)$ and $a_\mu \in N(a_k)$, respectively. Furthermore, we conclude from (1) that $a_{\mu+2} \to a_1$ and $a_k \to a_\mu$.

Assume $\mu \geq 4$. Let z be a vertex of H. Because of $a_1 \to \{a_2, a_k\}$, a_2 and a_k are adjacent. If $a_2 \to a_k$, then the path $a_1a_{\mu+1}a_{\mu+2} \cdots a_{k-1}za_3a_4 \cdots a_\mu a_2a_k$ is of length k. Therefore, $a_k \to a_2$. From the fact that za_2 is in a 3-cycle and (1), we conclude that $a_2 \to a_j$ for some j with $\mu+2 < j < k$. But now, $a_1 a_{\mu+1} a_{\mu+2} \cdots a_{j-1} z a_3 a_4 \cdots a_\mu a_2 a_j a_{j+1} \cdots a_k$ is of length k. Hence, $\mu = 3$. Similarly, we can show that $k = \mu + 4$. It follows that $k = 7$.

Suppose that a_2 and a_5 are adjacent. Then $a_5 \to a_2$ by (1). Since $a_5 \to \{a_1, a_6\}$, a_1 and a_6 are adjacent. If $a_1 \to a_6$, then $a_1 a_6 z a_3 a_5 a_2 a_4 a_7$ is of length 7. So we have $a_6 \to a_1$. Since $a_6 z$ is in a 3-cycle, we have $a_2 \to a_6$. If $a_2 \to a_7$, then $a_1 a_4 a_6 z a_3 a_5 a_2 a_7$ is of length 7. It follows that $a_7 \to a_2$. In addition, we see that $a_6 \to a_3$.

If $|H| \geq 2$, then $a_1 a_2 a_6 z_1 z_2 a_3 a_4 a_7$ is of length 7 for any arc $z_1 z_2$ in $D\langle H \rangle$. Therefore, $|H| = 1$ and T is isomorphic to a $\mathcal{D}_8^1$-type digraph on 8 vertices.

Suppose now that a_2 and a_5 are not adjacent. Then $a_2 \to a_7$. Furthermore, $a_6 \to a_2$. Since $a_6 z$ is in a 3-cycle, we have $a_1 \to a_6$. If a_3 and a_6 are adjacent, then $a_6 \to a_3$ by (1). But now, $a_1 a_6 a_3 a_4 a_5 z a_2 a_k$ is of length 7. Hence, a_3 and a_6 are not adjacent. If $|H| \geq 2$, then the path $a_1 a_4 a_5 a_6 z z' a_2 a_7$ is of length 7, where $z' \in H$ with $z \to z'$. Hence, $|H| = 1$ and T is isomorphic to T_8^2.

Case 2. $\nu = \mu + 1$.

Since the converse digraph of T_8^2 (of T_8^4, respectively) is isomorphic to itself (to T_8^5, respectively) and the converse of any $\mathcal{D}_8^1$-type digraph also is a $\mathcal{D}_8^1$-type digraph, we may assume from Lemma 2.5 (a) that

$$(2) \qquad N^+(a_\mu) \cap B = \{a_{\mu+1}\}.$$

From Lemma 2.5 (b), $T\langle B \rangle$ is a tournament containing a unique $(a_{\mu+1}, y)$-path. Let

$$p = \max\{\, i \mid a_i \in N^-(B - a_{\mu+1})\} \text{ and}$$
$$q = \min\{\, j \mid \mu+2 \leq j \leq k,\ a_p \to a_j\}.$$

Since T is 3-connected and a_k has only one in-neighbor in $T\langle B \rangle$, we have $1 < p < \mu$. Let $F = T\langle\{a_{p+1}, \cdots, a_\mu\}\rangle$ and let $F_1, \cdots, F_\alpha$ $(\alpha \geq 1)$ be the strong decomposition of F. Note that F_i has a hamiltonian cycle if $|V(F_i)| > 1$ for $i = 1, \cdots, \alpha$. Because every arc from H to F is in a 3-cycle, $F \to a_{\mu+1}$. It is easy to check that

$$(3) \qquad F_1 \to a_i \text{ for all } i < p$$

Moreover, let $m = \max\{i|3 \leq i \leq k-1,\ a_1 \to a_i\}$ and $\ell = \max\{j|2 \leq j \leq p,\ a_j \to a_k\}$. By the same arguments above, the two integers m and ℓ are well defined.

Subcase 2.1. $\alpha \geq 2$.

Since every arc from F_1 to F_i $(i \geq 2)$ is in a 3-cycle and $F \to a_{\mu+1}$, we conclude that $a_p \to F_1 \to F_i \to a_p$.

Suppose that $\alpha \geq 3$. Since every arc from F_2 to F_3 is in a 3-cycle, there is a vertex $a_j \in N^-(F_2)$ with $1 \leq j < p$. Thus, the two paths, obtained in order of

$$a_1, \cdots, a_j, F_2, a_{\mu+1}, \cdots, a_{q-1}, z, a_{j+1}, \cdots, a_p, a_q, \cdots, a_k$$

and in order of $z, F_1, F_3, \cdots, F_\alpha, a_p$, respectively, satisfy the conditions of Proposition 2.2, and hence T contains an (x, y)-path of length k.

Suppose now that $\alpha = 2$. It is a simple matter to verify that $a_{\mu+1}$ is adjacent to every vertex of A. If there exists a vertex a_j with $1 \leq j < p$ such that $a_j \to a_{\mu+1}$, then the two paths, obtained in order of $a_1, \cdots, a_j, a_{\mu+1}, \cdots, a_{q-1}, z, a_{j+1}, \cdots, a_p, a_q, \cdots, a_k$ and in order of z, F_1, F_2, a_p, respectively, satisfy the conditions of Proposition 2.2, and hence T contains an (x, y)-path of length k. Therefore,

$$(4) \qquad a_{\mu+1} \to a_i \text{ for all } i < p\,.$$

Because of $a_{\mu+1} \to a_{\mu+2}$, $a_{\mu+2}$ and a_{p-1} are adjacent. If $a_{\mu+2} \to a_{p-1}$, then we deduce from (3) and the definition of the integer p that $a_{\mu+2} \to F_1$; if $a_{p-1} \to a_{\mu+2}$, $a_{\mu+2}$ and a_p are adjacent, and hence, we conclude from $F_2 \to a_p \to F_1$ that there exist arcs between $a_{\mu+2}$ and F, and consequently, $a_{\mu+2} \to F_1$. Hence, we have $a_{\mu+2} \to F_1$ in any case. Successively, we deduce that $a_i \to F_1$ for $i = \mu + 3, \cdots, k$.

By (3) and (4), we note that $a_m \notin V(F_1) \cup \{a_{\mu+1}\}$.

Suppose first that $\mu+2 \leq m \leq k-1$. If $\ell = p$, then the path obtained in order of $a_1, a_m, \cdots, a_{k-1}, F_1, F_2, a_{\mu+1}, \cdots, a_{m-1}, z, a_2, \cdots, a_p, a_k$ is of length k. If $\ell < p$ and $a_{k-1} \to F_2$, then the two paths, obtained in order of $a_1, a_m, \cdots, a_{k-1}, F_2, a_{\mu+1}, \cdots, a_{m-1}, z, a_2, \cdots, a_\ell, a_k$ and in order of $z, a_{\ell+1}, \cdots, a_p, F_1, a_\ell$, respectively, satisfy the conditions of Proposition 2.2, and hence T contains an (x, y)-path of length k. If $\ell < p$ and $a_{k-1} \not\to F_2$, then $a_{k-1} \notin N(F_2)$, and hence, $k = \mu + 3$. Note that $a_{\mu+1} \to a_\ell$ by (4). Now, the two paths, $a_1 a_{k-1} z a_2 \cdots a_\ell a_k$ and

$za_{\ell+1} \cdots a_p a_{p+1} \cdots a_\mu a_{\mu+1} a_\ell$ form an (x,y)-path of length k by Proposition 2.2. Therefore, a_1 has no out-neighbor in $B - \{a_k\}$.

Suppose second that $3 \le m \le p$. Since the arc za_{m-1} is in a 3-cycle and $a_{\mu+1} \to a_i$ for all $i < p$, $a_{m-1} \to a_\ell$ for some $\ell \ge \mu+2$. But now, the (x,y)-path obtained in order of $a_1, a_m, \cdots, a_p, F_1, F_2, a_{\mu+1}, \cdots, a_{\ell-1}, z, a_2, \cdots, a_{m-1}, a_\ell, \cdots, a_k$ is of length k.

Finally, we suppose that $a_m \in V(F_2)$. If F_2 contains at least two vertices, then the two paths, obtained in order of $a_1, a_m, a_{\mu+1}, \cdots, a_{q-1}, z, a_2, \cdots, a_p, a_q, \cdots, a_k$ and in order of $z, F_1, F_2 - a_m, a_p$, respectively, satisfy the conditions of Proposition 2.2, and hence T contains an (x,y)-path of length k. So, we assume that F_2 consists of the unique vertex a_m. Since a_m has at least 3 out-neighbors, a_m dominates a_i for some i with $2 \le i \le p-1$. This implies that $p \ge 3$. Now, we see that $a_1 a_m a_{\mu+1} \cdots a_{q-1} z a_{p+1} \cdots a_\mu a_2 \cdots a_p a_q \cdots a_k$ is of length k.

Subcase 2.2. $\alpha = 1$.

It is a simple matter to verify that

$$a_{\mu+1} \to a_i \text{ for all } i \le p-2 \text{ if } p \ge 3\,. \tag{5}$$

We first consider the case when $a_{\mu+2} \in N(F)$. From the definition of the integer p, we deduce that $a_{\mu+2} \to F$, and hence $a_i \to F$ for all $i \ge \mu+2$.

Suppose that $m \ge \mu+2$. If $\ell = p$, then the path $a_1 a_m \cdots a_{k-1} a_{p+1} a_{p+2} \cdots a_{m-1} z a_2 \cdots a_p a_k$ is of length k. So, we have $\ell < p$. Note by (3) that $F \to a_\ell$.

If $k \ge \mu+4$, then $a_1 a_m \cdots a_{k-1} a_{\mu+1} \cdots a_{m-1} z a_2 \cdots a_\ell a_k$ and $za_{\ell+1} \cdots a_p a_{p+1} \cdots a_\mu a_\ell$ form an (x,y)-path of length k by Proposition 2.2.

Assume thus that $k = \mu+3$. Note that $m = k-1$. If $\ell \le p-2$ or $\ell \ge 3$, then we can easily find an (x,y)-path of length k. Hence, $p = 3$ and $\ell = 2$. Clearly, we have $q = k-1$ and $a_3 \to a_1$. It is a simple matter to confirm that $|H| = 1$ and $|F| = 1$. Therefore, T has exactly 8 vertices. If $a_3 \to a_5$ and $a_2 \to a_6$, then T is isomorphic to T_8^4; if $a_3 \to a_5$ and $a_6 \to a_2$, then T is isomorphic to T_8^5. We note that $a_1 a_6 z a_3 a_4 a_2 a_7$ also is a bypath in T. If $a_5 \to a_3$ and $a_6 \to a_2$, then T is isomorphic to T_8^4; if $a_5 \to a_3$ and $a_2 \to a_6$, then T is isomorphic to T_8^5.

Suppose now that $m = \mu+1$. From (5) we conclude that $p = 2$. Since $k \ge \mu+3$ and $T\langle B\rangle$ has a unique $(a_{\mu+1}, a_k)$-path, we see that

$N^-(a_{k-1}) = \{a_2, a_{k-2}\}$, a contradiction to the assumption that T is 3-connected.

Suppose thus that $3 \leq m \leq p$. Since za_{m-1} is in a 3-cycle, $a_{m-1} \to a_t$ for some $t \geq \mu+1$. If $t \geq \mu+2$, then we can easily find an (x,y)-path of length k. So, we have $N^+(a_{m-1}) \cap B = \{a_{\mu+1}\}$. It follows by (5) that $m = p$. Since a_{k-1} has at least 3 in-neighbors, $p \geq 4$ holds. Because za_2 is in a 3-cycle, we see by (5) that $a_2 \to a_i$ for some $i \geq \mu+2$. But now, $a_1 a_p a_{p+1} \cdots a_\mu a_3 \cdots a_{p-1} a_{\mu+1} \cdots a_{i-1} z a_2 a_i \cdots a_k$ is of length k.

Now we consider the other case that there is no arc between F and $a_{\mu+2}$. Since $T\langle B\rangle$ has a unique $(a_{\mu+1}, a_k)$-path, it is easy to see that $|B| = 3$. Furthermore, $q = k$, and hence, $a_{k-1} \to a_p \to F$. Since a_{k-1} has at least 3 in-neighbors, we see that $p \geq 3$. ¿From $\{a_1, a_{k-1}\} \to a_k$ and (3), we conclude that $a_1 \to a_{k-1}$. It follows that $a_i \to a_{k-1}$ for all $i \leq p-1$. If $a_{\mu+1} \to a_j$ for some j with $2 \leq j \leq p$, then we see that

$$a_1 \cdots a_{j-1} a_{k-1} z a_{p+1} \cdots a_{\mu+1} a_j \cdots a_p a_k$$

is of length k. Thus, $a_i \to a_{\mu+1}$ for all i with $2 \leq i \leq p$. Combining this fact with (5), we have $p = 3$ and $a_{\mu+1} \to a_1$. If $a_2 \to a_k$, then $a_1 a_{k-1} a_3 a_{k-2} z a_{p+1} \cdots a_\mu a_2 a_k$ is of length k. So, we have $a_k \to a_2$. In addition, it is not difficult to check that $a_3 \to a_1$.

If $|H| \geq 2$ and $z_1, z_2 \in H$ with $z_1 \to z_2$, then $a_1 a_{k-1} z_1 z_2 a_{p+1} \cdots a_\mu a_2 a_3 a_k$ is length k. Hence, $|H| = 1$. Now we note that T is isomorphic to a $\mathcal{D}_8^1$-type digraph. □

From Claim 1 and Claim 2, the theorem is proved.

As an immediate consequence of Theorem 3.1, we obtain the following:

COROLLARY 3.2. *Let T be a 5-connected and arc-3-cyclic local tournament. Then T is generalized arc-pancyclic and every arc of T has a bypath of length m for all $m \geq 3$.*

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