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DISCRETE SOBOLEV ORTHOGONAL POLYNOMIALS AND SECOND ORDER DIFFERENCE EQUATIONS

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Abstract. Let $\{R_n(x)\}_{n=0}^{\infty}$ be a discrete Sobolev orthogonal polynomials (DSOPS) relative to a symmetric bilinear form

$$\phi(p,q)=\int_{\mathbb{R}} pq d\mu_0+\int_{\mathbb{R}} \Delta p \Delta q\, d\mu_1,$$

where $d\mu_0$ and $d\mu_1$ are signed Borel measures on $\mathbb{R}$. We find necessary and sufficient conditions for $\{R_n(x)\}_{n=0}^{\infty}$ to satisfy a second order difference equation

$$\ell_2(x)\Delta\nabla y(x)+\ell_1(x)\Delta y(x)=\lambda_n y(x)$$

and classify all such $\{R_n(x)\}_{n=0}^{\infty}$. Here, Δ and ∇ are forward and backward difference operators defined by $\Delta f(x)=f(x+1)-f(x)$ and $\nabla f(x)=f(x)-f(x-1)$.

1. Introduction

Let $\mathcal{P}$ be the space of real polynomials in a single variable and $\deg(\pi)$ the degree of any $\pi(x)\in\mathcal{P}$ with the convention that $\deg(0)=-1$. By a polynomial system (PS), we mean a sequence of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\deg(\phi_n)=n$, $n\geq 0$.

Any bilinear form $\phi(\cdot,\cdot)$ defined on $\mathcal{P}\times\mathcal{P}$ is called quasi-definite (respectively, positive-definite) if the double sequence (called the moments of $\phi(\cdot,\cdot)$)

$$\phi_{mn}:=\phi(x^m,x^n)\quad (m \text{ and } n\geq 0)$$

satisfy the Hamburger condition

$$\Delta_n(\phi):=det[\phi_{i,j}]_{i,j=0}^{n}\neq 0\quad (\text{respectively, } \Delta_n(\phi)>0)$$

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for each $n \geq 0$. It is well known ([7]) that if the bilinear form $\phi(\cdot,\cdot)$ is quasi-definite (respectively, positive-definite), there is a PS $\{R_n(x)\}_{n=0}^{\infty}$ such that

$$\phi(R_m, R_n) = K_n\delta_{mn}, \quad m \text{ and } n \geq 0, \tag{1.1}$$

where K_n is a non-zero (respectively, a positive) constant and vice versa. In this case, each $R_n(x)$ is uniquely determined up to a non-zero constant multiple and we call $\{R_n(x)\}_{n=0}^{\infty}$ a (generalized) orthogonal polynomial system (OPS) relative to $\phi(\cdot,\cdot)$.

We now let Δ and ∇ be the forward and backward difference operators defined by

$$\Delta f(x) = f(x+1) - f(x) \quad \text{and} \quad \nabla f(x) = f(x) - f(x-1)$$

and consider a symmetric bilinear form on $\mathcal{P} \times \mathcal{P}$ given by

$$\phi(p,q) := \int_{\mathbb{R}} p(x)q(x)\, d\mu_0(x) + \int_{\mathbb{R}} \Delta p(x)\Delta q(x)\, d\mu_1(x), \tag{1.2}$$

where $d\mu_i(x)$ $(i = 1, 2)$ is a signed Borel measure on the real line $\mathbb{R}$. When $\phi(\cdot,\cdot)$ in (1.2) is quasi-definite, we call any PS $\{R_n(x)\}_{n=0}^{\infty}$ satisfying (1.1) a discrete Sobolev orthogonal polynomial system (DSOPS). When $d\mu_1(x) \equiv 0$, $\{R_n(x)\}_{n=0}^{\infty}$ is just an ordinary OPS relative to $d\mu_0(x)$.

In this work, we first find necessary and sufficient conditions for a DSOPS relative to $\phi(\cdot,\cdot)$ in (1.2) to satisfy the second order difference equation

$$L[y](x) = \ell_2(x)\Delta\nabla y + \ell_1(x)\Delta y = \lambda_n y, \tag{1.3}$$

where $\ell_2(x) = \ell_{22}x^2 + \ell_{21}x + \ell_{20}$ and $\ell_1(x) = \ell_{11}x + \ell_{10}$ are polynomials and λ_n is the eigenvalue parameter given by

$$\lambda_n = \ell_{22}n(n-1) + \ell_{11}n, \quad n \geq 0.$$

We then classify all such DSOPS's which generalize the discrete classical orthogonal polynomial systems, that is, OPS's relative to $d\mu_0(x)$, which are eigenfunctions of the difference equation (1.3). The general theory of discrete classical OPS's is rather well developed, for which we refer to [3, 5, 6, 8]. Similar problem, where the difference operator is replaced by the differential operator, is handled in [7].

2. Polynomials satisfying difference equations

Due to Boas' ([1]) or Duran's ([2]) theorem on the moment problem, any linear functional σ on $\mathcal{P}$, which we call a moment functional, can be represented as an the integral of the form

$$\langle \sigma, \pi \rangle = \int_{\mathbf{R}} \pi(x)\, d\mu(x), \quad (\pi \in \mathcal{P})$$

or

$$\langle \sigma, \pi \rangle = \int_{\mathbf{R}} \pi(x) w(x)\, dx, \quad (\pi \in \mathcal{P}),$$

where $\mu(x)$ is a function of bounded variation on $\mathbb{R}$ and $w(x)$ is a C^∞-function in the Schwartz space of rapidly decaying functions. Hence, in studying DSOPS's the symmetric bilinear form in (1.2) can be replaced by

$$\phi(p, q) = \langle \sigma, pq \rangle + \langle \tau, \Delta p \Delta q \rangle, \tag{2.1}$$

where σ and τ are moment functionals. As we shall see later, it is much more convenient to use moment functionals instead of their integral representations as in (1.2). We call a DSOPS $\{P_n(x)\}_{n=0}^\infty$ relative to $\phi(\cdot,\cdot)$ in (2.1) with $\tau = 0$ to be an OPS relative to σ.

For a PS $\{P_n(x)\}_{n=0}^\infty$, we call any moment functional σ satisfying

$$\langle \sigma, P_0 \rangle \neq 0 \quad \text{and} \quad \langle \sigma, P_n \rangle = 0, \quad n \geq 1$$

a canonical moment functional for $\{P_n(x)\}_{n=0}^\infty$, which is unique up to a non-zero constant multiple. A PS $\{P_n(x)\}_{n=0}^\infty$ is called a weak orthogonal polynomial system (WOPS) if there is a non-zero moment functional σ such that $\langle \sigma, P_m P_n \rangle = 0$ for $m \neq n$. Note that if $\{R_n(x)\}_{n=0}^\infty$ is a WOPS relative to σ or a DSOPS relative to $\phi(\cdot,\cdot)$ in (2.1), then σ must be a canonical moment functional for $\{R_n(x)\}_{n=0}^\infty$.

For a moment functional σ, a polynomial $\pi(x)$, and a real constant a, $\Delta\sigma$, $\nabla\sigma$, $\pi\sigma$ and $\tau_a\sigma$ are moment functionals defined by

$$\begin{aligned}
&\langle \Delta\sigma, \psi \rangle = -\langle \sigma, \nabla\psi \rangle, \quad \langle \nabla\sigma, \psi \rangle = -\langle \sigma, \Delta\psi \rangle \\
&\langle \pi\sigma, \psi \rangle = \langle \sigma, \pi\psi \rangle \\
&\langle \tau_a\sigma, \psi \rangle = \langle \sigma, \tau_{-a}\psi \rangle = \langle \sigma, \psi(x+a) \rangle, \quad (\psi \in \mathcal{P}).
\end{aligned}$$

For convenience we denote $\tau_a\sigma$ by $\sigma(x-a)$. Then, the followings are easy consequences of definitions.

LEMMA 2.1. *Let $\pi(x)$ be a polynomial and σ a moment functional. Then*

(i) *for any $a \in \mathbb{R}$, $\sigma(x-a) = 0$ if and only if $\sigma = 0$;*
(ii) *$\Delta\sigma = 0$ (or $\nabla\sigma = 0$) if and only if $\sigma = 0$;*
(iii) *$\Delta(\pi\sigma) = (\Delta\pi)\sigma + \pi(x+1)\Delta\sigma = \pi\Delta\sigma + \sigma(x+1)\Delta\pi$;*
(iv) *when σ is quasi-definite, $\pi\sigma = 0$ if and only if $\pi(x) \equiv 0$.*

By a direct calculation, it is easy to see that the difference equation (1.3) has a unique monic polynomial solution of degree n for each $n \geq 0$ except possibly for a finite number of values of n and for those exceptional values of n, there is either no polynomial solution of degree n or infinitely many monic polynomial solutions of degree n.

DEFINITION 2.1. ([4]) The difference operator $L[\cdot]$ in (1.3) (or the equation (1.3) itself) is called admissible if $\lambda_m \neq \lambda_n$ for $m \neq n$.

LEMMA 2.2. *For the difference equation* (1.3), *the followings are equivalent:*

(i) *$L[\cdot]$ in (1.3) is admissible;*
(ii) *$s_n := \ell_{22}n + \ell_{11} \neq 0$ for $n \geq 0$;*
(iii) *For each $n \geq 0$, the difference equation* (1.3) *has a unique monic polynomial solution of degree n.*

Proof. See Lemma 2.4 in [6]. □

To discuss the orthogonality of polynomials satisfying the difference equation (1.3), we need the following which involves discrete moment equations.

LEMMA 2.3. *If the difference equation* (1.3) *has a PS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions, then any canonical moment functional σ of $\{P_n(x)\}_{n=0}^{\infty}$ satisfies*

$$\Delta(\ell_2\sigma) - \ell_1\sigma = 0 \tag{2.2}$$

or equivalently

$$\begin{aligned} s_n\sigma_{n+1} + (n(2n-1)\ell_{22} + n\ell_{21} + n\ell_{11} + \ell_{10})\sigma_n + n\ell_2(n-1)\sigma_{n-1}, \\ = 0 \qquad n \geq 1, \end{aligned} \tag{2.3}$$

where

$$\sigma_n := \langle \sigma, x^{[n]} \rangle, \quad n \geq 0, \quad \sigma_n = 0 \quad \text{for} \quad n < 0$$

and $x^{[n]}$ are factorial polynomials:

$$x^{[0]} = 1, \qquad x^{[n]} := x(x-1)\cdots(x-n+1).$$

Moreover if the difference equation (1.3) *is admissible, then the equation* (2.2) *or equivalently* (2.3) *has a unique linearly independent solution.*

Proof. See Lemma 2.5 and Lemma 2.6 in [6] (see also [3]). □

We call (2.2) the discrete moment equation for the difference equation (1.3). By Favard's theorem, any monic PS $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS if and only if $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a three term recurrence relation

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x),\ n \geq 0 \quad (P_{-1}(x) = 0),$$

where $c_n \neq 0$, $n \geq 1$. In particular, $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to a positive-definite moment functional if and only if $c_n > 0$, $n \geq 1$. In this case, if we let σ be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$ with $\langle \sigma, 1\rangle = 1$ and $c_0 = 1$, then $\langle \sigma, P_n^2\rangle = c_n c_{n-1} \cdots c_0$, $n \geq 0$.

The next theorem gives a necessary and sufficient condition for the difference equation (1.3) to have an OPS as solutions.

THEOREM 2.4. *The difference equation* (1.3) *has a monic OPS $\{P_n(x)\}_{n=0}^{\infty}$ as solutions if and only if*

(i) $s_n \neq 0$, $n \geq 0$;
(ii) $\ell_2(-\frac{t_n}{s_{2n}}) \neq 0$, *where* $t_n = \ell_{22}n^2 + (\ell_{21} + \ell_{11})n + \ell_{10}$, $n \geq 0$.

In this case, the coefficients of the three term recurrence relation are

$$b_n = \frac{nt_{n-1}}{s_{2n-2}} - \frac{(n+1)t_n}{s_{2n}} + n, \quad n \geq 0,$$

and

$$c_n = -\frac{ns_{n-2}}{s_{2n-3}s_{2n-1}}\ell_2\Big(-\frac{t_{n-1}}{s_{2n-2}}\Big), \quad s_{-1} = 1, \quad n \geq 1.$$

Proof. See Theorem 3.5 in [6]. □

Note that for any moment functional σ satisfying (2.2) and any polynomial $p(x)$, we have

$$L[p](x)\sigma = \Delta[(\nabla p)\ell_2\sigma].$$

Now assume that the equation (1.3) is admissible, that is, $s_n \neq 0$, $n \geq 0$ and let σ be a canonical moment functional of the unique monic PS

$\{P_n(x)\}_{n=0}^{\infty}$ of solutions of the equation (1.3). Then

$$\lambda_n\langle\sigma, P_mP_n\rangle = \langle L[P_n]\sigma, P_m\rangle = \langle\Delta[(\nabla P_n)\ell_2\sigma], P_m\rangle$$
$$= -\langle\ell_2\sigma, \nabla P_n\nabla P_m\rangle = \lambda_m\langle\sigma, P_mP_n\rangle, \quad m \text{ and } n \geq 0$$

so that $\langle\sigma, P_mP_n\rangle = 0$ for $m \neq n$, that is, $\{P_n(x)\}_{n=0}^{\infty}$ is a WOPS. But, $\{P_n(x)\}_{n=0}^{\infty}$ need not be an OPS in general, unless the condition (ii) in Theorem 2.4 is satisfied.

The condition (i) in Theorem 2.4 is just the admissibility of $L[\cdot]$ (cf. Lemma 2.2). Hence, if the equation (1.3) has an OPS $\{P_n(x)\}_{n=0}^{\infty}$ as solutions, then $\{P_n(x)\}_{n=0}^{\infty}$ must be orthogonal relative to any non-zero solution σ of the discrete moment equation (2.2).

Now, we are ready to classify all OPS's satisfying the difference equation (1.3) (cf. [6]). There are four cases, up to a real linear change of variable, to be considered according to the root system of $\ell_2(x)$:

$$\ell_2(x) = x(A-x),\ x^2+\zeta^2,\ x,\ 1,$$

where A and $\zeta(>0)$ are real numbers. In case $\ell_2(x) \equiv 0$, it is easy to see that the equation (1.3) cannot have an OPS as solutions (see Remark 3.1).

In the following classification, we always let b_n and c_n be the coefficients of the three term recurrence relation satisfied by a monic PS of solutions of the corresponding difference equation.

Case 1: $\ell_2(x) = x(A-x)$. In this case, set

$$\alpha+\gamma = A, \quad \alpha+\beta+2 = -\ell_{11}, \quad \text{and} \quad (\beta+1)(\gamma-1) = \ell_{10}$$

so that the equation (1.3) becomes

$$\begin{aligned}(2.4)\qquad & x(\gamma+\alpha-x)\Delta\nabla y + [(\beta+1)(\gamma-1)-(\alpha+\beta+2)x]\Delta y \\ & = -n(n+\alpha+\beta+1)y.\end{aligned}$$

By Theorem 2.4, (2.4) has a monic OPS $\{h_n^{(\alpha,\beta)}(x,\gamma)\}_{n=0}^{\infty}$ as solutions if and only if

$$\alpha,\ \beta,\ -\gamma,\ \alpha+\beta+1,\ \alpha+\beta+\gamma \notin \mathbb{Z}^- := \{-1,-2,\cdots\}.$$

In this case,

$$b_n = \frac{n(n-\gamma)(n+\beta)}{2n+\alpha+\beta} - \frac{(n+1)(n-\gamma+1)(n+\beta+1)}{2n+\alpha+\beta+2} + n, \quad n \geq 0,$$

and

$$c_n = \frac{n(n+\alpha+\beta)(n+\alpha)(n+\beta)(\gamma-n)(n+\alpha+\beta+\gamma)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \quad n \geq 1.$$

Note that $\{h_n^{(\alpha,\beta)}(x,\gamma)\}_{n=0}^{\infty}$ cannot be a positive-definite OPS since $c_n < 0$ for n large.

REMARK 2.1. It is worth to note that the constants α, β, and γ need not be real but $\{h_n^{(\alpha,\beta)}(x,\gamma)\}_{n=0}^{\infty}$ is always a real PS. For example, the following difference equation

$$x(1-x)\Delta\nabla y - (1+x)\Delta y = -n^2 y$$

has a real OPS $\{h_n^{(\alpha,\beta)}(x,\gamma)\}_{n=0}^{\infty}$ as solutions by Theorem 2.4, where $\alpha = -i$, $\beta = -1+i$, and $\gamma = 1+i$.

REMARK 2.2. If $\alpha+\beta+1 \notin \mathbb{Z}^-$, then the difference equation (2.4) is admissible and so has a unique monic PS $\{h_n^{(\alpha,\beta)}(x,\gamma)\}_{n=0}^{\infty}$ as solutions, which is a WOPS. In particular, if either $\alpha > -1$, $\beta > -1$, and $\gamma = N$ or $\alpha < 1-N$, $\beta < 1-N$, and $\gamma = N$ for some positive integer N, then $\{h_n^{(\alpha,\beta)}(x,N)\}_{n=0}^{\infty}$ has the finite orthogonality:

$$\langle \sigma, [h_n^{(\alpha,\beta)}(x,N)]^2 \rangle = \begin{cases} \text{positive}, & 0 \leq n \leq N-1 \\ 0, & n \geq N, \end{cases}$$

where σ is a canonical moment functional of $\{h_n^{(\alpha,\beta)}(x,N)\}_{n=0}^{\infty}$. In the literature([8]), $\{h_n^{(\alpha,\beta)}(x,N)\}_{n=0}^{N-1}$ are known as Hahn polynomials of type 1 if $\alpha > -1$ and $\beta > -1$ and Hahn polynomials of type 2 if $\alpha < 1-N$ and $\beta < 1-N$.

Case 2: $\ell_2(x) = x^2 + \zeta^2$ $(\zeta > 0)$. In this case, the equation (1.3) becomes

$$(x^2+\zeta^2)\Delta\nabla y + (ax+b)\Delta y = (n+a-1)ny. \tag{2.5}$$

By Theorem 2.4, (2.5) has a monic OPS $\{\check{h}_n^{(a,b)}(x,\zeta)\}_{n=0}^{\infty}$ as solutions if and only if $a-1 \notin \mathbb{Z}^-$. In this case,

$$b_n = \frac{n[(n-1)^2+a(n-1)+b]}{2n+a-2} - \frac{(n+1)[n^2+an+b]}{2n+a} + n, \quad n \geq 0$$

and

$$c_n = \frac{-n(n+a-2)}{(2n+a-3)(2n+a-1)}\left(\frac{[(n-1)^2+a(n-1)+b]^2}{(2n+a-2)^2}+\zeta^2\right), \quad n \geq 1.$$

Hence, $\{\check{h}_n^{(a,b)}(x,\zeta)\}_{n=0}^{\infty}$ cannot be a positive definite OPS.

Case 3: $\ell_2(x) = x$. In this case, the equation (1.3) can be parameterized as

$$\text{(2.6)} \qquad x\Delta\nabla y + [a-(1-\mu)x]\Delta y = (\mu-1)ny.$$

By Theorem 2.4, (2.6) has a monic OPS as solutions if and only if $\mu \neq 1$ and $\mu n + a \neq 0$, $n = 0, 1, \cdots$.

If $\mu = 0$ and $a \neq 0$, then the difference equation (2.6) becomes

$$x\Delta\nabla y + (a-x)\Delta y = -ny,$$

which has a monic OPS $\{c_n^{(a)}(x)\}_{n=0}^{\infty}$, known as Charlier polynomials ([8]), as solutions. In this case,

$$b_n = n + a, \quad n \geq 0, \quad \text{and} \quad c_n = an, \quad n \geq 1.$$

Note that $\{c_n^{(a)}(x)\}_{n=0}^{\infty}$ is a positive-definite OPS only for $a > 0$.

If $\mu \neq 0$, then by setting $a = \gamma\mu$, the difference equation (2.6) becomes

$$x\Delta\nabla y + [\gamma\mu - (1-\mu)x]\Delta y = (\mu-1)ny,$$

which has a monic OPS $\{m_n^{(\gamma,\mu)}(x)\}_{n=0}^{\infty}$, known as Meixner polynomials ([8]), as solutions if and only if $\mu \neq 0, 1$ and $\gamma - 1 \notin \mathbb{Z}^-$. In this case,

$$b_n = \frac{1}{1-\mu}((1+\mu)n + \gamma\mu), \quad n \geq 0$$

and

$$c_n = \frac{\mu n}{(1-\mu)^2}(n+\gamma-1), \quad n \geq 1.$$

Note that $\{m_n^{(\gamma,\mu)}(x)\}_{n=0}^{\infty}$ is a positive-definite OPS only for $\mu > 0$, $\mu \neq 1$, and $\gamma > 0$. In the literature ([8]), Meixner polynomials $\{m_n^{(\gamma,\mu)}(x)\}_{n=0}^{\infty}$ is usually introduced with $0 < \mu < 1$ and $\gamma > 0$. However, $\{m_n^{(\gamma,\mu)}(x)\}_{n=0}^{\infty}$

is also a positive-definite OPS for $\mu > 1$ and $\gamma > 0$, which is orthogonal with respect to a discrete weight function

$$w(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)\mu^k}{k!\Gamma(\gamma)} \delta(-\gamma - k).$$

REMARK 2.3. When $\mu \neq 1$ and $a = -\mu N$ for some positive integer N, we set $p = \frac{\mu}{\mu-1}$ so that the equation (2.6) becomes

$$(2.7) \qquad x\Delta\nabla y(x) + \frac{Np - x}{q}\Delta y(x) = -\frac{n}{q}y(x) \quad (q \neq 0,\, p+q=1).$$

The equation (2.7) is admissible so that has a unique monic WOPS $\{k_n^{(p)}(x,N)\}_{n=0}^{\infty}$ as solutions, which has the finite orthogonality:

$$\langle \sigma, [k_n^{(p)}(x,N)]^2 \rangle = \begin{cases} \text{positive}, & 0 \le n \le N \\ 0, & n \ge N+1. \end{cases}$$

In the literature ([8]), $\{k_n^{(p)}(x,N)\}_{n=0}^{N}$ are known as Krawchuk polynomials.

Case 4: $\ell_2(x) = 1$. In this case, the equation (1.3) becomes

$$(2.8) \qquad \Delta\nabla y + (\ell_{11}x + \ell_{10})\Delta y = \ell_{11}ny.$$

By a real linear change of variable: $x \mapsto -x - \frac{1+\ell_{10}}{\ell_{11}}$ and using $\Delta - \nabla = \Delta\nabla$, the difference equation (2.8) can be transformed into

$$x\Delta\nabla y + (a-x)\Delta y = -ny \quad (a = -1/\ell_{11}),$$

which has Charlier polynomials $\{c_n^{(a)}(x)\}_{n=0}^{\infty}$ as solutions (cf. Case 3).

Lastly in this section, we examine more closely the orthogonality of polynomial solutions of the difference equation (1.3). For any monic PS $\{P_n(x)\}_{n=0}^{\infty}$, we can write $P_{n+1}(x)$ as

(2.9)

$$P_{n+1}(x) = (x - \xi_n^n)P_n(x) - \xi_n^{n-1}P_{n-1}(x) - \sum_{k=0}^{n-2} \xi_n^k P_k(x) \qquad (n \ge 1),$$

where $\xi_1^0 = \xi_1^{-1} = 0$ and $P_{-1}(x) \equiv 0$.

LEMMA 2.5. *Assume that the difference equation (1.3) has a monic PS $\{P_n(x)\}_{n=0}^{\infty}$ as solutions. Let $N \ge 0$ be the largest integer such that $\lambda_N = 0$. Then*

(i) $\xi_n^0 = 0$ *if* $n \geq 2$ *and* $n+1 \neq N$

and

(ii) *for any moment functional solution* σ *of the discrete moment equation* (2.2),

$$\langle \sigma, P_n^2 \rangle = \xi_n^{n-1} \cdot \xi_{n-1}^{n-2} \cdots \xi_{N+1}^{N} \langle \sigma, P_N^2 \rangle, \qquad n \geq N+1, \tag{2.10}$$

where ξ_n^0 *and* ξ_n^{n-1} *are the constants given in* (2.9).

Proof. First note that $\lambda_m = \lambda_n$ for $m \neq n$ if and only if $m + n = N$. Hence we have for $m \neq n$ and $m + n \neq N$ (cf. Proposition 3.2),

$$\langle \sigma, P_m P_n \rangle = 0 \tag{2.11}$$

for any solution σ of the discrete moment equation (2.2).

(i) Let σ be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then σ satisfies the discrete moment equation (2.2) by Lemma 2.3. If we apply σ to the equation (2.9), we obtain for $n \geq 2$

$$\begin{aligned} 0 = \langle \sigma, P_{n+1} \rangle &= \langle \sigma, xP_n \rangle - \xi_n^n \langle \sigma, P_n \rangle - \xi_n^{n-1} \langle \sigma, P_{n-1} \rangle - \sum_{k=0}^{n-2} \xi_n^k \langle \sigma, P_k \rangle \\ &= \langle \sigma, xP_n \rangle - \xi_n^0 \langle \sigma, P_0 \rangle \end{aligned}$$

since $\langle \sigma, P_n \rangle = 0$ for $n \geq 1$. Hence we have (i) since $\langle \sigma, P_0 \rangle \neq 0$ and $\langle \sigma, xP_n \rangle = \langle \sigma, P_1 P_n \rangle = 0$ for $n \geq 2$ and $n + 1 \neq N$.

(ii) Now let σ be any solution of (2.2). If we multiply the equation (2.9) by $P_{n-1}(x)$ and apply σ, then we obtain from (2.11)

$$0 = \langle \sigma, P_{n+1} P_{n-1} \rangle = \langle \sigma, P_n^2 \rangle - \xi_n^{n-1} \langle \sigma, P_{n-1}^2 \rangle - \xi_n^0 \langle \sigma, P_0 P_{n-1} \rangle$$

for $n \geq N + 1$. If $n > N + 1$, then $\langle \sigma, P_0 P_{n-1} \rangle = 0$ by (2.11). If $n = N + 1$ and $N \geq 1$, then $\xi_{N+1}^0 = 0$ by (i). Finally if $N = 0$, then $\xi_1^0 = 0$. Therefore, we have

$$\langle \sigma, P_n^2 \rangle = \xi_n^{n-1} \langle \sigma, P_{n-1}^2 \rangle, \qquad n \geq N+1,$$

from which (ii) follows immediately. □

We are now ready to prove the following result which we need later in section three and is interesting in its own right.

THEOREM 2.6. *Assume that the difference equation* (1.3) *has a monic PS* $\{P_n(x)\}_{n=0}^{\infty}$ *as solutions. If* $\{P_n(x)\}_{n=0}^{\infty}$ *is not an OPS, then for any* *solution σ of the discrete moment equation* (2.2), *there is an integer $m \geq 0$ such that*

$$\langle \sigma, P_n^2 \rangle = 0 \quad \textit{for all} \quad n \geq m+1.$$

Proof. If $\lambda_n \neq 0$, $n \geq 1$, then $L[\cdot]$ is admissible and $\{P_n(x)\}_{n=0}^{\infty}$ is a WOPS. Consequently, $\xi_k^{k-1} = 0$ for some $k \geq 1$ by Lemma 2.5 since $\{P_n(x)\}_{n=0}^{\infty}$ is not an OPS. Hence $\langle \sigma, P_n^2 \rangle = 0$, $n \geq k$ by (2.10). Now, we assume that $\lambda_N = 0$ for some $N \geq 1$. For each $n \geq 0$, let $P_n(x) = \sum_{k=0}^{n} \eta_k^n x^{[k]}$ ($\eta_n^n = 1$) be a monic polynomial of degree n. Then $P_n(x)$ satisfies the difference equation (1.3) if and only if

(2.12)
$$(\lambda_n - \lambda_k)\eta_k^n = (k+1)[\ell_{22}k^2 + \ell_{21}k + \ell_{11}k + \ell_{10}]\eta_{k+1}^n + \sum_{j=k}^{n} \eta_{j+2}^n (j+2)(j+1)(-1)^{j-k} P(j, j-k), \quad 0 \leq k \leq n,$$

where $\eta_{n+1}^n = \eta_{n+2}^n = 0$ and $P(n,m) = n(n-1)\cdots(n-m+1)$. Since $\lambda_{N+1} - \lambda_k \neq 0$ for $0 \leq k \leq N$ and $\lambda_{N+2} - \lambda_k \neq 0$ for $0 \leq k \leq N+1$, the equation (2.12) is uniquely solvable for η_{N-1}^{N+1}, η_N^{N+1}, η_N^{N+2}, and η_{N+1}^{N+2}. Actually, we have

$$\eta_n^{n+1} = \frac{(n+1)t_n}{s_{2n}}, \quad (n = N, N+1)$$

and

$$\eta_{n-1}^{n+1} = \frac{n(n+1)(t_{n-1}t_n + ns_{2n})}{2s_{2n-1}s_{2n}}, \quad (n = N, N+1).$$

Note that since $\lambda_N = 0$ or equivalently $s_{N-1} = 0$, $s_{2N-1} \neq 0$, $s_{2N} \neq 0$ and $s_{2N+1} \neq 0$. Hence, we have

$$\xi_{N+1}^{N} = \frac{-(N+1)s_{N-1}}{s_{2N-1}s_{2N+1}} \ell_2\left(\frac{-t_N}{s_{2N}}\right) = 0$$

and the conclusion follows by (2.10). □

3. Classification of discrete Sobolev orthogonal polynomials

From here on, we shall consider DSOPS's relative to a symmetric bilinear form (2.1). We first obtain necessary and sufficient conditions for a DSOPS relative to $\phi(\cdot,\cdot)$ to satisfy a second order difference equation (1.3).

THEOREM 3.1. *For a bilinear form $\phi(\cdot,\cdot)$ in (2.1), the followings are all equivalent.*

(i) *The difference operator $L[\cdot]$ in (1.3) is symmetric on polynomials relative to $\phi(\cdot,\cdot)$, that is,*

$$\phi(L[p],q)=\phi(p,L[q]),\quad (p,q\in\mathcal{P}). \tag{3.1}$$

(ii) *The moment functionals σ and τ satisfy the functional equations*

$$\Delta(\ell_2\sigma)-\ell_1\sigma=0 \tag{3.2}$$

and

$$\Delta(\ell_2\tau)-[\Delta(\ell_2+\ell_1)+\ell_1]\tau=0. \tag{3.3}$$

(iii) *The moments of σ and τ given by $\sigma_n:=\langle\sigma,x^{[n]}\rangle$ and $\tau_n=\langle\tau,x^{[n]}\rangle$ satisfy for $n\geq 0$,*

$$\begin{aligned}(\ell_{22}n+\ell_{11})\sigma_{n+1}&+[\ell_{22}n(2n-1)+(\ell_{21}+\ell_{11})n+\ell_{10}]\sigma_n\\ &+n[\ell_{22}(n-1)^2+\ell_{21}(n-1)+\ell_{20}]\sigma_{n-1}=0\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}(\ell_{22}n+2\ell_{22}+\ell_{11})\tau_{n+1}&+[\ell_{22}n(2n-1)+(\ell_{21}+2\ell_{22}+\ell_{11})n\\ &+\ell_{22}+\ell_{21}+\ell_{11}+\ell_{10}]\tau_n+n[\ell_2(n)+\ell_1(n)]\tau_{n-1}=0,\end{aligned} \tag{3.5}$$

where $\sigma_n=0$ and $\tau_n=0$ if $n<0$.

Furthermore if $\phi(\cdot,\cdot)$ is quasi-definite and $\{R_n(x)\}_{n=0}^{\infty}$ is a DSOPS relative to $\phi(\cdot,\cdot)$, then the statements (i), (ii), and (iii) are also equivalent to

(iv) *$\{R_n(x)\}_{n=0}^{\infty}$ satisfies the difference equation (1.3).*

Proof. (i)$\Leftrightarrow$(ii): It is easy to check the following identities: for $p,q\in\mathcal{P}$,

$$\phi(L[p],q)=\langle L^+[q\sigma],p\rangle-\langle L^+[\nabla[(\Delta q)\tau]],p\rangle,$$

$$\phi(p,L[q])=\langle L[q]\sigma,p\rangle-\langle\nabla[(\Delta L[q])\tau],p\rangle,$$

where $L^+[\cdot]$ is an adjoint of $L[\cdot]$ defined by

$$L^+[y]=\Delta\nabla(\ell_2 y)-\nabla(\ell_1 y).$$

Hence, the equation (3.1) is equivalent to

$$L^+[q\sigma] - L^+[\nabla[(\Delta q)\tau]] - L[q]\sigma + \nabla[(\Delta L[q])\tau] = 0, \quad (q \in \mathcal{P}),$$

which can be written out as

$$\begin{aligned}(3.6)\quad &[\Delta(\ell_2\tau) - (\Delta(\ell_2+\ell_1)+\ell_1)\tau]\Delta^2\nabla q + [(\ell_2+\ell_1)\nabla\tau - \ell_1\tau]\Delta\nabla^2 q \\ &+ \nabla[(\ell_2+\ell_1)\nabla\tau - \ell_1\tau]\nabla^2 q + 2\Delta[(\ell_2+\ell_1)\nabla\tau - \ell_1\tau]\Delta\nabla q \\ &+ [\Delta\nabla[(\ell_2+\ell_1)\nabla\tau - \ell_1\tau] + (\nabla(\ell_2\sigma) - \ell_1\sigma + \nabla(\ell_1\sigma)]\nabla q \\ &+ [\Delta(\ell_2\sigma) - \ell_1\sigma]\Delta q + \nabla[\Delta(\ell_2\sigma) - \ell_1\sigma]q = 0.\end{aligned}$$

We can see that the condition (3.6) is equivalent to the fact that σ and τ satisfy (3.2) and (3.3) since $\Delta(\ell_2\tau) - [\ell_1 + \Delta(\ell_1+\ell_2)]\tau = 0$ if and only if $(\ell_2+\ell_1)\nabla\tau - \ell_1\tau = 0$ and $\nabla(\ell_2\sigma) - \ell_1\sigma + \nabla(\ell_1\sigma) = 0$ if and only if $\Delta(\ell_2\sigma) = \ell_1\sigma$ by Lemma 2.1.
(ii)⇔(iii): The equivalence of (3.2) and (3.4) is proved in Lemma 2.3 and the equivalence of (3.3) and (3.5) can be proved similarly.

We now assume that $\{R_n(x)\}_{n=0}^\infty$ is a DSOPS relative to $\phi(\cdot,\cdot)$.
(i)⇒(iv): Since $L[R_n](x)$ is a polynomial of degree $\le n$, we may write

$$L[R_n](x) = \sum_{k=0}^{n} C_{nj}R_j(x)$$

for some real constant C_{nj}, $j = 0, 1, \cdots, n$. Then for $0 \le k \le n-1$,

$$C_{nk}\phi(R_k, R_k) = \sum_{k=0}^{n} C_{nj}\phi(R_j, R_k) = \phi(L[R_n], R_k) = \phi(R_n, L[R_k]) = 0,$$

since $\deg(L[R_k]) \le k$. Hence, $C_{nk} = 0$ for $k = 0, 1, \cdots, n-1$ so that

$$L[R_n](x) = C_{nn}R_n(x) = \lambda_n R_n(x)$$

by comparing the coefficients of x^n on both sides.
(iv)⇒(i): (3.1) follows immediately from

$$\phi(L[R_n], R_m) = \lambda_n\phi(R_n, R_m) = \lambda_m\phi(R_n, R_m) = \phi(R_n, L[R_m])$$

since $\{R_n\}_{n=0}^\infty$ is a basis of $\mathcal{P}$. □

When $\tau = 0$, the equivalences of (i), (ii), (iii), and (iv) in Theorem 3.1 are proved in [6] (see also [3]).

REMARK 3.1. When $\ell_2(x) \equiv 0$, the difference equation (1.3) reduces to the first order difference equation

$$\ell_1(x)\Delta y = \ell_{11} n y, \tag{3.7}$$

which can have a PS as solutions only when $\ell_{11} \neq 0$. In this case, the general solutions of the moment equations (3.2) and (3.3) are

$$\sigma = c_1 \delta(x + \ell_{10}/\ell_{11}) \quad \text{and} \quad \tau = c_2 \delta(x + \ell_{10}/\ell_{11}),$$

where c_1 and c_2 are arbitrary constants. Then the corresponding symmetric bilinear form $\phi(\cdot,\cdot)$ cannot be quasi-definite. Hence, the equation (3.7) cannot have a DSOPS as solutions. Similarly, we can also show that if $\ell_2(x) + \ell_1(x) \equiv 0$, then the equation (1.3) can not have a DSOPS as solutions.

REMARK 3.2. If we act Δ on both sides of (1.3), then $z(x) = \Delta y(x)$ satisfies

$$\ell_2 \Delta \nabla z + [\ell_1 + \Delta(\ell_1 + \ell_2)]\Delta z = (\lambda_n - \Delta \ell_1) z. \tag{3.8}$$

Note that (3.3) is the discrete moment equation for the difference equation (3.8).

PROPOSITION 3.2. *If $L[p] = \lambda p$ and $L[q] = \mu q$ for some $p, q \in \mathcal{P}$ and $\lambda \neq \mu$, then*

$$\phi(p, q) = 0$$

for any solutions σ and τ of the discrete moment equations (3.2) *and* (3.3) *respectively.*

Proof. It immediately follows from the fact (cf. Theorem 3.1)

$$(\lambda - \mu)\phi(p, q) = \phi(L[p], q) - \phi(p, L[q]). \qquad \square$$

We are now ready to classify all DSOPS's relative to the bilinear form $\phi(\cdot,\cdot)$ in (2.1) satisfying the difference equation (1.3). In the following, we shall assume $\ell_2(x) \not\equiv 0$ and $\ell_2(x) + \ell_1(x) \not\equiv 0$ (see Remark 3.1). Concerning the symmetric bilinear form $\phi(\cdot,\cdot)$ in (2.1), there arise the following three cases:

Type A: σ is quasi-definite;

Type B: Both σ and τ are not quasi-definite;

Type C: σ is not quasi-definite but τ is quasi-definite.

We now consider three cases individually.

Type A: σ is quasi-definite.

It is well known (cf. Proposition 2.5 in [5]) that if $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical OPS relative to σ satisfying the equation (1.3), then $\{\nabla P_n(x)\}_{n=0}^{\infty}$ is also an OPS relative to $\ell_2(x)\sigma$. In this case, $\{\Delta P_n(x)\}_{n=0}^{\infty}$ is also an OPS relative to $\ell_2(x+1)\sigma(x+1) = (\ell_2(x) + \ell_1(x))\sigma$ since $\Delta f(x) = \nabla f(x+1)$ and $(\ell_2\sigma)' = \ell_1\sigma$.

THEOREM 3.3. *If the difference equation (1.3) has a DSOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot,\cdot)$ as solutions and σ is quasi-definite, then $\{R_n(x)\}_{n=0}^{\infty}$ must be a discrete classical OPS relative to σ and either $\tau = 0$ or τ is also quasi-definite.*

Proof. Since σ is a canonical moment functional of $\{R_n(x)\}_{n=0}^{\infty}$, $\{R_n(x)\}_{n=0}^{\infty}$ must be an OPS so that a discrete classical OPS relative to σ when σ is quasi-definite. Then $\{\Delta R_n(x)\}_{n=0}^{\infty}$ is also an OPS relative to $\tilde{\tau} = (\ell_2(x) + \ell_1(x))\sigma$ and satisfy the equation (3.8). Hence, τ and $\tilde{\tau}$ satisfy the moment equation (3.3), which is uniquely solvable (up to a constant factor) by Lemma 2.3 so that $\tau = a\tilde{\tau}$ for some constant a. Thus, $\tau = 0$ if $a = 0$ or τ is quasi-definite if $a \neq 0$. □

Type B: Both σ and τ are not quasi-definite.

Here, we will show that any DSOPS relative to a bilinear form $\phi(\cdot,\cdot)$ in (2.1) cannot satisfy a second order difference equation of the form (1.3).

THEOREM 3.4. *Let $\{R_n(x)\}_{n=0}^{\infty}$ be a DSOPS relative to a quasi-definite bilinear form $\phi(\cdot,\cdot)$ in (2.1). If both σ and τ are not quasi-definite, then $\{R_n(x)\}_{n=0}^{\infty}$ cannot satisfy a difference equation of the form (1.3).*

Proof. Assume that $\{R_n(x)\}_{n=0}^{\infty}$ satisfies the difference equation (1.3). Then by Theorem 3.1, σ and τ must be non-trivial solutions of the moment equations (3.2) and (3.3) respectively. Hence, $\{R_n(x)\}_{n=0}^{\infty}$ cannot be an OPS since σ is not quasi-definite. On the other hand, $\{\Delta R_n(x)\}_{n=1}^{\infty}$ is a PS satisfying the difference equation (3.8) and τ satisfies the corresponding moment equation (3.3). Hence, $\{\Delta R_n(x)\}_{n=1}^{\infty}$ cannot be an OPS since τ is not quasi-definite. Then, by Theorem 2.6, we have

$$\langle \sigma, R_n^2 \rangle = \langle \tau, (\Delta R_n)^2 \rangle = 0$$

for all n large enough and so

$$\phi(R_n, R_n) = \langle \sigma, R_n^2 \rangle + \langle \tau, (\Delta R_n)^2 \rangle = 0$$

for all n large enough, which contradicts the fact that $\{R_n(x)\}_{n=0}^{\infty}$ is a DSOPS relative to $\phi(\cdot,\cdot)$. □

Type C: σ is not quasi-definite but τ is quasi-definite.

THEOREM 3.5. *Assume that the difference equation* (1.3) *has a PS* $\{R_n(x)\}_{n=0}^{\infty}$ *of solutions, which is a DSOPS relative to the bilinear form* $\phi(\cdot,\cdot)$ *in* (2.1). *If* τ *is quasi-definite, then*

(i) $\{\Delta R_n(x)\}_{n=1}^{\infty}$ *is a discrete classical OPS relative to* τ *and satisfies the equation* (3.8);
(ii) $\{R_n(x)\}_{n=0}^{\infty}$ *is a WOPS relative to* σ;
(iii) $(\ell_2 + \ell_1)\sigma = a\tau$ *for some constant* a *so that either* $(\ell_2 + \ell_1)\sigma = 0$ *or* $(\ell_2 + \ell_1)\sigma$ *is quasi-definite.*

Proof. (i) Let $\{Q_n(x)\}_{n=0}^{\infty}$ be the monic OPS relative to τ. Since τ satisfies the moment equation (3.3), $\{Q_n(x)\}_{n=0}^{\infty}$ is a discrete classical OPS satisfying the difference equation (3.8) (cf. Theorem 3.1). Hence, the equation (3.8) must be admissible. On the other hand, the PS $\{\Delta R_{n+1}(x)\}_{n=0}^{\infty}$ also satisfies the difference equation (3.8). Hence, $\Delta R_{n+1}(x) = C_n Q_n(x)$, $n \geq 0$ by Lemma 2.2, where C_n is the coefficient of x^n in $\Delta R_{n+1}(x)$.
(ii) It follows from the orthogonalities of $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot,\cdot)$ and $\{\Delta R_n(x)\}_{n=1}^{\infty}$ relative to τ.
(iii) Since the equation (3.8) is admissible, the moment equation (3.3) has only one linearly independent solution. Since τ and $(\ell_2+\ell_1)\sigma$ satisfy the moment equation (3.3), we have $(\ell_2 + \ell_1)\sigma = a\tau$ for some constant a so that either $(l_2 + l_1)\sigma = 0$ if $a = 0$ or $(l_2 + l_1)\sigma$ is quasi-definite if $a \neq 0$. □

As in section two, we may assume that

$$\ell_2(x) = x(A - x), \quad x^2 + \zeta^2, \quad x, \quad 1 \qquad (A \text{ real and } \zeta > 0).$$

In each case, we look for conditions that the equation (3.2) has no quasi-definite moment functional solution and the equation (3.3) has a quasi-definite moment functional solution.

Case C.1. $\ell_2(x) = x(A-x)$. In this case, the difference equation (1.3) can be written as

$$(3.9) \qquad x(\gamma+\alpha-x)\Delta\nabla y + [(\beta+1)(\gamma-1)-(\alpha+\beta+2)x]\Delta y = -n(n+\alpha+\beta+1)y.$$

Then the corresponding discrete moment equations are

$$(3.10) \qquad \Delta[x(\gamma+\alpha-x)\sigma] = [(\beta+1)(\gamma-1)-(\alpha+\beta+2)x]\sigma$$

$$(3.11) \qquad \Delta[x(\gamma+\alpha-x)\tau] = [(\beta+2)(\gamma-2)-(\alpha+\beta+4)x]\tau.$$

The equation (3.10) has no quasi-definite moment functional solution if and only if $\alpha \in \mathbb{Z}^-$ or $\beta \in \mathbb{Z}^-$ or $\alpha+\beta+1 \in \mathbb{Z}^-$ or $\gamma \in \mathbb{Z}^+$ or $\alpha+\beta+\gamma \in \mathbb{Z}^-$. The equation (3.11) has a quasi-definite moment functional solution if and only if $\alpha+1 \notin \mathbb{Z}^-$, $\beta+1 \notin \mathbb{Z}^-$, $\alpha+\beta+2 \notin \mathbb{Z}^-$, $\gamma-1 \notin \mathbb{Z}^+$ and $\alpha+\beta+\gamma+1 \notin \mathbb{Z}^-$. Hence, there are five cases:

(i) $\alpha = -1$, $\beta+1$, $1-\gamma$, and $\beta+\gamma \notin \mathbb{Z}^-$;
(ii) $\beta = -1$, α, $1-\gamma$, and $\alpha+\gamma \notin \mathbb{Z}^-$;
(iii) $\gamma = 1$, α, β, and $\alpha+\beta+2 \notin \mathbb{Z}^-$;
(iv) $\alpha+\beta = -2$, $\alpha, \beta \notin \mathbb{Z}^-$, and $\gamma \notin \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^-$;
(v) $\alpha+\beta+\gamma = -1$, $\alpha, \beta, \alpha+\beta+1 \notin \mathbb{Z}^-$, and $\gamma \notin \mathbb{Z}^+ := \{1, 2, \cdots\}$.

We now have:

THEOREM 3.6.

(i) *If $\alpha = -1$, $\beta+1, 1-\gamma$, and $\beta+\gamma \notin \mathbb{Z}^-$, then the difference equation (3.9) always has a DSOPS $\{h_n^{(-1,\beta)}(x,\gamma)\}_{n=0}^{\infty}$ as solutions, which are orthogonal relative to*
   (a) *$\phi_A(p,q) = Ap(\gamma-1)q(\gamma-1) + \langle W^{(0,\beta+1)}(x,\gamma-1), \Delta p \Delta q\rangle$ if $\beta \neq -1$, where A is any non-zero constant;*
   (b) *$\phi_{A,B}(p,q) = Ap(0)q(0) + Bp(\gamma-1)q(\gamma-1) + \langle W^{(0,0)}(x,\gamma-1), \Delta p\Delta q\rangle$ if $\beta = -1$, $\gamma \neq 1$, and $h_1^{(-1,-1)}(x,\gamma) = x+a$, where a, A, and B are arbitrary constants with $A+B \neq 0$, $Aa^2 + B(\gamma-1+a)^2+1 \neq 0$, and $Aa + B(\gamma-1+a) = 0$;*
   (c) *$\phi_{A,B}(p,q) = Ap(0)q(0) - B[p(0)q'(0)+p'(0)q(0)] + \langle W^{(0,0)}(x,0), \Delta p\Delta q\rangle$ if $-\beta = \gamma = 1$ and $h_1^{(-1,-1)}(x,1) = x+a$, where a, A, and B are arbitrary constants with $A \neq 0$, $Ab^2 - 2Bb + 1 \neq 0$, and $Aa - B = 0$.*

(ii) *If $\beta = -1$, $\alpha, 1-\gamma$, and $\alpha+\gamma \notin \mathbb{Z}^-$, then the difference equation (3.9) always has a unique DSOPS $\{h_n^{(\alpha,-1)}(x,\gamma)\}_{n=0}^{\infty}$ as solutions,* *which are orthogonal relative to*

$$\phi_A(p,q) = Ap(0)q(0) + \langle W^{(\alpha+1,0)}(x,\gamma-1), \Delta p \Delta q\rangle,$$

where A is an arbitrary nonzero constant.

(iii) *If* $\gamma = 1$, α, β, *and* $\alpha+\beta+2 \notin \mathbb{Z}^-$, *then the difference equation* (3.9) *always has a DSOPS* $\{h_n^{(\alpha,\beta)}(x,1)\}_{n=0}^{\infty}$ *as solutions, which are orthogonal relative to*

(a) $\phi_A(p,q) = Ap(0)q(0) + \langle W^{(\alpha+1,\beta+1)}(x,0), \Delta p \Delta q\rangle$ *if* $\alpha+\beta \neq -2$, *where A is an arbitrary non-zero constant;*

(b) $\phi_{A,B}(p,q) = Ap(0)q(0) + Bp(\alpha+1)q(\alpha+1) + \langle W^{(\alpha+1,\beta+1)}(x,0), \Delta p \Delta q\rangle$ *if* $\alpha+\beta = -2$, *and* $h_1^{(\alpha,\beta)}(x,1) = x+a$, *where* a, A, *and* B *are arbitrary constants with* $A+B \neq 0$, $AB(\alpha+1)^2+A+B \neq 0$, *and* $a(A+B)+(\alpha+1)B = 0$.

(iv) *If* $\alpha+\beta = -2$, $\alpha, \beta \notin \mathbb{Z}^-$, *and* $\gamma \notin \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^-$, *then the difference equation (3.9) has no DSOPS as solutions.*

(v) *If* $\alpha+\beta+\gamma = -1$, $\alpha, \beta, -\gamma$, *and* $\alpha+\beta+1 \notin \mathbb{Z}^-$, *then the difference equation* (3.9) *always has a DSOPS* $\{h_n^{(\alpha,\beta)}(x,\gamma)\}_{n=0}^{\infty}$ *as solutions, which are orthogonal relative to*

$$\phi_A(p,q) = Ap(\alpha+\gamma)q(\alpha+\gamma) + \langle W^{(\alpha+1,\beta+1)}(x,\gamma-1), \Delta p \Delta q\rangle,$$

where A is an arbitrary nonzero constant.

Here, $W^{(\alpha,\beta)}(x,\gamma)$ *is the canonical moment functional of the monic discrete classical OPS* $\{h_n^{(\alpha,\beta)}(x,\gamma)\}_{n=0}^{\infty}$ *with* $\langle W^{(\alpha,\beta)}(x,\gamma), 1\rangle = 1$.

Before giving the proof, we note that in cases (a) of (i), (ii), (a) of (iii), and (v) above, the equation (3.9) is admissible so that each $h_n^{(\alpha,\beta)}(x,\gamma)$, $n \geq 0$, is unique. However, in other cases except (iv), the equation (3.9) is not admissible but $h_n^{(\alpha,\beta)}(x,\gamma)$, $n \neq 1$ is unique and for $n = 1$, $h_1^{(\alpha,\beta)}(x,\gamma) = x + a$ satisfies the equation (3.9) for any constant a.

Proof of Theorem 3.6. We prove only (b) of (i) and (iv) since all the other cases can be proved similarly. When $\alpha = -1, \beta = -1$ and $\gamma \neq 1$, the equation (3.9) becomes

$$x(\gamma-1-x)\Delta\nabla y = -n(n-1)y, \tag{3.12}$$

which is not admissible. However, it is easy to see that the equation (3.12) has a unique monic polynomial solution $h_n^{(-1,-1)}(x,\gamma)$ for $n \neq 1$

and for $n = 1$, any polynomial $h_1^{(-1,-1)}(x) = x + a$ with arbitrary constant a is a solution of (3.12). Then we have from (2.4) and (3.12)

$$\Delta h_n^{(-1,-1)}(x,\gamma) = n h_{n-1}^{(0,0)}(x,\gamma-1), \quad n \geq 1.$$

On the other hand, the general solutions of the moment equations (3.10) and (3.11) are

$$\sigma = d_1\delta(x) + d_2\delta(x-\gamma+1) \quad \text{and} \quad \tau = d_3 W^{(0,\beta+1)}(x,\gamma-1)$$

where d_i $(i = 1,2,3)$ are arbitrary constants. Hence, by Proposition 3.2,

(3.13)

$$\phi_{A,B}(h_m^{(-1,-1)}(x,\gamma), h_n^{(-1,-1)}(x,\gamma)) = \begin{cases} A+B & \text{if } m=n=0 \\ Aa^2 + B(\gamma-1+a)^2 + 1 & \text{if } m=n=1 \\ n^2\langle W^{(0,\beta+1)}(x,\gamma-1), h_{n-1}^{(0,0)}(x,\gamma-1)^2\rangle \neq 0 & \text{if } m=n\geq 2 \\ Aa + B(\gamma-1+a) & \text{if } m=0, n=1 \\ 0 & \text{if } m\neq n,\ m\geq 2 \end{cases}$$

since $h_n^{(-1,-1)}(x,\gamma) = 0$, $n \geq 2$ at $x = 0$ and $\gamma - 1$ and $\{h_n^{(0,0)}(x,\gamma-1)\}_{n=0}^{\infty}$ is a discrete classical OPS relative to $W^{(0,\beta+1)}(x,\gamma-1)$ (see Case 1 in Section 2). Hence (b) of (i) follows immediately from (3.13).

Now, let $\alpha + \beta = -2$, $\alpha, \beta \notin \mathbb{Z}^-$, and $\gamma \notin \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^-$. Then the equation (3.9) become

$$x(\gamma+\alpha-x)\Delta\nabla y + (\beta+1)(\gamma-1)\Delta y = -n(n-1)y. \tag{3.14}$$

In this case, the equation (3.14) has no polynomial solution of degree 1. □

Case C.2. $\ell_2(x) = x^2 + \zeta^2$ $(\zeta > 0)$. In this case, the difference equation (1.3) can be written as

$$(x^2+\zeta^2)\Delta\nabla y + (ax+b)\Delta y = n(n+a-1)y. \tag{3.15}$$

Then the corresponding discrete moment equations are

$$\Delta[(x^2+\zeta^2)\sigma] = (ax+b)\sigma \tag{3.16}$$

$$\Delta[(x^2+\zeta^2)\tau] = [(a+2)x + a + b + 1]\tau. \tag{3.17}$$

The equation (3.16) has no quasi-definite moment functional solution if and only if $a - 1 \in \mathbb{Z}^-$. The equation (3.17) has a quasi-definite moment functional solution if and only if $a + 1 \notin \mathbb{Z}^-$. Hence, there are two cases:

$a = 0$ or $a = -1$. If $a = 0$, $b \neq 0$ or $a = -1$, then the difference equation (3.15) has no polynomial solution of degree 1 or 2, respectively. When $a = b = 0$, we have:

THEOREM 3.7. *If $a = b = 0$, then difference equation (3.15) always has a DSOPS $\{\check{h}_n^{(0,0)}(x,\zeta)\}_{n=0}^{\infty}$ as solutions, which are orthogonal relative to*

$$\phi_{A,B}(p,q) = A\langle\sigma^{(1)}, pq\rangle + B\langle\sigma^{(2)}, pq\rangle + \langle W^{(2,1)}(x,\zeta), \Delta p \Delta q\rangle,$$

if $\check{h}_1^{(0,0)}(x,\zeta) = x + a$, where a, A, and B are arbitrary constants with $A \neq 0$, $A - 2A\zeta^2 - 2B^2\zeta^2 \neq 0$, and $Aa - B\zeta = 0$. Here, $\sigma^{(1)}$ and $\sigma^{(2)}$ are moment functionals defined by

$$\langle\sigma^{(1)}, x^n\rangle = (\zeta i)^n + (-\zeta i)^n \quad \text{and} \quad \langle\sigma^{(2)}, x^n\rangle = [(\zeta i)^n - (-\zeta i)^n]i,$$
$$n \geq 0 \ (i = \sqrt{-1})$$

and $W^{(2,1)}(x,\zeta)$ is the canonical moment functional of the monic discrete classical OPS $\{\check{h}_n^{(2,1)}(x,\zeta)\}_{n=0}^{\infty}$ with $\langle W^{(2,1)}(x,\zeta), 1\rangle = 1$.

Case C.3. $\ell_2(x) = x$. In this case, the difference equation (1.3) can be written as

$$x\Delta\nabla y + [a - (1-\mu)x]\Delta y = -n(1-\mu)y. \tag{3.18}$$

Then the corresponding discrete moment equations are

$$\Delta(x\sigma) = [a - (1-\mu)x]\sigma \tag{3.19}$$

$$\Delta(x\tau) = [a + \mu - (1-\mu)x]\tau. \tag{3.20}$$

The equation (3.19) has no quasi-definite moment functional solution if and only if $\mu = 1$ or $a = -\mu n$ for some $n \in \{0, 1, \cdots\}$. The equation (3.20) has a quasi-definite moment functional solution if and only if $\mu \neq 1$ and $a \neq -\mu(n+1)$ for all $n \in \{0, 1, \cdots\}$. Hence, there is only one case: $a = 0$ and $\mu \neq 0, 1$.

THEOREM 3.8. *If $a = 0$ and $\mu \neq 0, 1$, then the difference equation (3.18) always has a unique DSOPS $\{m_n^{(0,\mu)}(x)\}_{n=0}^{\infty}$ as solutions, which are orthogonal relative to*

$$\phi_A(p,q) = Ap(0)q(0) + \langle W^{(1,\mu)}(x), \Delta p \Delta q\rangle,$$

where A is any non-zero constant and $W^{(1,\mu)}(x)$ is the canonical moment functional of the monic discrete classical OPS $\{m_n^{(1,\mu)}(x)\}_{n=0}^{\infty}$ with $\langle W^{(1,\mu)}(x), 1\rangle = 1$.

Proofs of Theorem 3.7 and Theorem 3.8 are essentially the same as that of Theorem 3.6.

Case C.4. $\ell_2(x) = 1$. In this case, the difference equation (1.3) can be written as

$$\Delta\nabla y + (ax + b)\Delta y = -ny. \tag{3.21}$$

Then the corresponding discrete moment equations are

$$\Delta\sigma = (ax + b)\sigma \tag{3.22}$$

$$\Delta\tau = (ax + a + b)\tau. \tag{3.23}$$

The equation (3.22) has no quasi-definite moment functional solution if and only if $a - 1 \in \mathbb{Z}^-$. The equation (3.23) has a quasi-definite moment functional solution if and only if $a \notin \mathbb{Z}^-$. Hence, there is only one case: $a = 0$, for which the difference equation (3.21) has no polynomial solution of degree 1.

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