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CONSTRUCTIONS FOR SPARSE ROW-ORTHOGONAL MATRICES WITH A FULL ROW

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ABSTRACT. In [4], it was shown that an n by n orthogonal matrix which has a row of nonzeros has at least

$$(\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1}$$

nonzero entries. In this paper, the matrices achieving these bounds are constructed. The analogous sparsity problem for m by n row-orthogonal matrices which have a row of nonzeros is conjectured.

1. Introduction

At the 1990 SIAM Linear Algebra meeting, M. Fiedler asked:

How sparse can an n by n orthogonal matrix (whose rows and columns cannot be permuted to give a matrix which is a direct sum of matrices) be?

The assumption precluding direct sums is necessary, since otherwise the answer is trivially n. Fiedler's question is answered in [1] (see also [5]), where it is shown that each n by n orthogonal matrix which is not direct summable has at least $4n-4$ nonzero entries, and that for $n \geq 2$, there exist such orthogonal matrices with exactly $4n-4$ nonzero entries. Recently, the n by n orthogonal matrices with exactly $4n-4$ nonzero entries were constructed in [2]. The analogous sparsity problem for m by n row-orthogonal matrices under two natural notions of irreducibility which extends the work in [1, 5] was studied in [3].

And also, it was studied in [4], the question of how sparse an n by n orthogonal matrix which has a column of nonzeros can be. In

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particular, it was shown that such an n by n orthogonal matrix has at least

$$(\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1} \tag{1}$$

nonzero entries, and matrices achieving these bounds are constructed and characterized, and are related to orthogonal matrices arising from the Haar wavelet.

Note that if A is an n by n orthogonal matrix with a row of nonzeros then A has also at least the number of nonzero entries in (1).

In this paper, we get another constructions for the n by n orthogonal matrices which have a full row and have exactly nonzero entries in (1), where a vector is *full* if each of its entries is nonzero. Furthermore, the analogous sparsity problem for m by n row-orthogonal matrices with a full row is conjectured.

For a matrix A, we denote the number of nonzero entries in A by $\#(A)$.

2. Constructions for the sparsest orthogonal matrices with a full row

An m by n matrix is *row-orthogonal* provided each of its rows is nonzero, and its rows are pairwise orthogonal.

We begin by describing a way to build row-orthogonal matrices from smaller row-orthogonal matrices. Let

$$X = \begin{bmatrix} \widehat{X} \\ \mathrm{x}^T \end{bmatrix}$$

be an s by t row-orthogonal matrix and let

$$Y = \begin{bmatrix} \mathrm{y}^T \\ \widehat{Y} \end{bmatrix}$$

be an k by l row-orthogonal matrix, where $\widehat{X}$ is $(s-1)$ by t matrix and $\widehat{Y}$ is $(k-1)$ by l matrix. Define $X \Diamond Y$ to be the $(s+k-1)$ by $(t+l)$ matrix

$$X \Diamond Y = \begin{bmatrix} \widehat{X} & O \\ \mathrm{x}^T & \mathrm{y}^T \\ O & \widehat{Y} \end{bmatrix}.$$

Certainly, $X\Diamond Y$ is a row-orthogonal matrix. We can extend this construction to use any number of row-orthogonal matrices by defining $X\Diamond Y\Diamond Z$ as $(X\Diamond Y)\Diamond Z$. This construction can be used in a recursive manner to construct m by n row-orthogonal matrices.

Now, we describe a way of constructing an n by n orthogonal matrices having a full row and exactly $(\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1}$ nonzero entries. This is a different manner from the one used in [4].

LEMMA 2.1. *Let*

$$X = \begin{bmatrix} \widehat{X} \\ \mathrm{x}^T \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} \mathrm{y}^T \\ \widehat{Y} \end{bmatrix}$$

be an r by r orthogonal matrix and a s by s orthogonal matrix respectively where $\widehat{X}$ is $(r-1)$ by r matrix and $\widehat{Y}$ is $(s-1)$ by s matrix. Then

$$A = \begin{bmatrix} X & \Diamond & Y \\ \mathrm{x}^T & & -\mathrm{y}^T \end{bmatrix} \tag{2}$$

is an n by n row-orthogonal matrix where $r + s = n$. Thus the matrix, $\widehat{A}$, obtained from A by normalizing the row r and the row n of A is an n by n orthogonal matrix with the same zero pattern as A.

Proof. Since $X\Diamond Y$ is an $(n-1)$ by n row-orthogonal matrix, it is sufficient to show that the row r and the row n of A are orthogonal each other. Indeed,

$$[\mathrm{x}^T \ \mathrm{y}^T][\mathrm{x}^T \ -\mathrm{y}^T]^T = ||\mathrm{x}^T||^2 - ||\mathrm{y}^T||^2 = 1 - 1 = 0.$$

Thus the proof is completed. □

Note that if both x^T and y^T in Lemma 2.1 are full rows then $\widehat{A}$ is an n by n orthogonal matrix with a full row.

Throughout in this paper, we define $\rho(n)$ by

$$\rho(n) = (\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1}.$$

THEOREM 2.2. *Let*

$$X = \begin{bmatrix} \widehat{X} \\ x^T \end{bmatrix}$$

be an r by r orthogonal matrix with the full row x^T which has $\rho(r)$ nonzero entries, and let

$$Y = \begin{bmatrix} y^T \\ \widehat{Y} \end{bmatrix}$$

be a s by s orthogonal matrix with the full row y^T which has $\rho(s)$ nonzero entries, where $r + s = n$. If

$$2^{\lfloor \log_2 n \rfloor - 1} \le r, s \le 2^{\lfloor \log_2 n \rfloor} \tag{3}$$

then

$$A = \begin{bmatrix} X & \diamondsuit & Y \\ & & \\ x^T & & -y^T \end{bmatrix}$$

is an n by n row-orthogonal matrix with a full row which has $\rho(n)$ nonzero entries. Thus the matrix, $\widehat{A}$, obtained from A by normalizing the row r and the row n of A is an n by n orthogonal matrix with the same zero pattern as A.

Proof. There exist r and s satisfying (3) and $r + s = n$, since we may take $r = \lfloor \frac{n}{2} \rfloor$ and $s = \lfloor \frac{n+1}{2} \rfloor$. From Lemma 2.1, A is an n by n row-orthogonal matrix with a full row. It is easy to show that

$$\begin{cases} \lfloor \log_2 r \rfloor = \lfloor \log_2 s \rfloor - 1 = \lfloor \log_2 n \rfloor - 1 & \text{if } n = 2^k - 1, \\ \lfloor \log_2 r \rfloor = \lfloor \log_2 s \rfloor = \lfloor \log_2 n \rfloor - 1 & \text{otherwise.} \end{cases}$$

Thus if $n \neq 2^k - 1$ then

$$\begin{aligned} \#(A) &= \#(X) + \#(Y) + \#([x^T \;\; -y^T]) \\ &= (\lfloor \log_2 r \rfloor + 3)r - 2^{\lfloor \log_2 r \rfloor + 1} + (\lfloor \log_2 s \rfloor + 3)s - 2^{\lfloor \log_2 s \rfloor + 1} + n \\ &= (\lfloor \log_2 n \rfloor + 2)(r + s) - 2 \cdot 2^{\lfloor \log_2 n \rfloor} + n \\ &= (\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1}. \end{aligned}$$

Let $n = 2^k - 1$. Then we take $r = \lfloor \frac{n}{2} \rfloor$ and $s = \lfloor \frac{n+1}{2} \rfloor$. Since $\lfloor \log_2 n \rfloor = \lfloor \log_2 (2^k - 1) \rfloor = k - 1$, we have

$$s = \left\lfloor \frac{n+1}{2} \right\rfloor = 2^{k-1} = 2^{\lfloor \log_2 n \rfloor}.$$

Thus

$$\begin{aligned}
\#(A) &= \#(X) + \#(Y) + \#([\mathrm{x}^T \ \ -\mathrm{y}^T]) \\
&= (\lfloor \log_2 r \rfloor + 3)r - 2^{\lfloor \log_2 r \rfloor + 1} + (\lfloor \log_2 s \rfloor + 3)s - 2^{\lfloor \log_2 s \rfloor + 1} + n \\
&= (\lfloor \log_2 n \rfloor + 2)(r + s) - 3 \cdot 2^{\lfloor \log_2 n \rfloor} + s + n \\
&= (\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1},
\end{aligned}$$

which completes the proof. □

Since $\rho(n) = 4n - 4$ for $n = 2, 3, 4$, from the result in [1], for each $n = 2, 3, 4$ we know zero patterns, B_n, of n by n orthogonal matrices with a full row which have $\rho(n)$ nonzero entries. That is,

$$B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

For $n = 5$, since

$$B_3 \Diamond B_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

by lemma 2.1

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{4}$$

is a zero pattern of 5 by 5 sparse orthogonal matrix with a full row which has $\rho(5) = 17$ nonzero entries.

Furthermore, from the result in [2], since, for each $n = 2, 3, 4$, we can get n by n orthogonal matrices with the same zero patterns as B_2, B_3, and B_4 respectively, we get a 5 by 5 orthogonal matrix with the same zero pattern as (4).

For example, let $n = 9$. From (3), since $4 \le r, s \le 8$, we take $r = 4$ and $s = 5$. Let X be a 4 by 4 orthogonal matrix with the full row which has $\rho(4) = 12$, and let Y be a 5 by 5 orthogonal matrix with the full row which has $\rho(5) = 17$. Take

$$X = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

$$Y = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} X & \Diamond & Y \\ \mathrm{x}^T & & -\mathrm{y}^T \end{bmatrix},$$

and

$$\widehat{A}=\begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & & & & & \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & & & O & & \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & & & & & \\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\
 & & & & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
 & & & & \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0 \\
 & O & & & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
 & & & & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}}
\end{bmatrix}$$

is a 9 by 9 sparse orthogonal matrix with the full row which has $\rho(9) = 38$ nonzero entries.

By these recursive manners, we can construct sparse n by n orthogonal matrices with a full row which have $\rho(n)$ nonzero entries.

3. Conjecture for sparse row-orthogonal matrices with a full row

We consider the case that A is an m by n row-orthogonal matrix with a full row.

Let

$$X = \begin{bmatrix} \widehat{X} \\ \mathrm{x}^T \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} \widehat{Y} \\ \mathrm{y}^T \end{bmatrix}$$

be an r by r matrix and a s by s matrix, respectively. Then both

$$X \Diamond Y = \begin{bmatrix} \widehat{X} & O \\ \mathrm{x}^T & \mathrm{y}^T \\ O & \widehat{Y} \end{bmatrix}$$

and

$$A = \begin{bmatrix} \widehat{X} & O \\ O & \widehat{Y} \\ \mathrm{x}^T & \mathrm{y}^T \end{bmatrix}$$

are $(r+s-1)$ by $(r+s)$ matrices and have the same nonzero entries. It is clear that A is an row-orthogonal matrix with the full row if and only if both X and Y are square orthogonal matrix with the full row x^T and with the full row y^T, respectively.

We define an m by n matrix A with $m \leq n$ to be *indecomposable* provided A does not contain a zero submatrix whose dimensions sum to n. It is not difficult to verify that if both $\widehat{X}$ and $\widehat{Y}$ are non-square indecomposable row-orthogonal matrices, then so is their direct sum $\widehat{X} \oplus \widehat{Y}$.

For each $i = 1, 2, \ldots, n-m+1$, let

$$X_{p_i} = \begin{bmatrix} \widehat{X}_{p_i} \\ \\ \mathrm{x}_{p_i}^T \end{bmatrix}$$

be a p_i by p_i orthogonal matrix with the full row $\mathrm{x}_{p_i}^T$ which has $\rho(p_i)$ nonzero entries where

$$\rho(p_i) = (\lfloor \log_2 p_i \rfloor + 3)p_i - 2^{\lfloor \log_2 p_i \rfloor + 1}.$$

Define

$$(5) \qquad A = \begin{bmatrix} \widehat{X}_{p_1} & O & O & O \\ O & \widehat{X}_{p_2} & O & O \\ O & O & \ddots & O \\ O & O & O & \widehat{X}_{p_{n-m+1}} \\ \mathrm{x}_{p_1}^T & \mathrm{x}_{p_2}^T & \cdots & \mathrm{x}_{p_{n-m+1}}^T \end{bmatrix}$$

where $\widehat{X}_{p_i}$ is a $(p_i - 1)$ by p_i row-orthogonal matrix, and

$$(6) \qquad 2^{\lfloor \log_2 (\frac{n}{n-m+1}) \rfloor} \leq p_i \leq 2^{\lfloor \log_2 (\frac{n}{n-m+1}) \rfloor + 1}$$

and

$$(7) \qquad p_1 + p_2 + \cdots + p_{n-m+1} = n.$$

Certainly, A is an m by n indecomposable, row-orthogonal matrix with the full row.

There exists p_i's satisfying (6) and (7), since we may assume $p_1 \leq p_2 \leq \cdots \leq p_{n-m+1}$ and we may take

$$p_1 = \left\lfloor \frac{n}{n-m+1} \right\rfloor, \quad p_2 = \left\lfloor \frac{n+1}{n-m+1} \right\rfloor, \ldots,$$
$$p_{n-m+1} = \left\lfloor \frac{n+(n-m)}{n-m+1} \right\rfloor.$$

For example, let A be a 17 by 19 row-orthogonal matrix with the form in (5). From (6) since $4 \leq p_i \leq 8$, (p_1, p_2, p_3)'s satisfying $p_1 + p_2 + p_3 = 19$ are (4,7,8), (5,7,7), (6,6,7), and A has the following forms respectively:

$$(8) \quad \begin{bmatrix} \widehat{X}_4 & O & O \\ O & \widehat{X}_7 & O \\ O & O & \widehat{X}_8 \\ \mathrm{x}_4^T & \mathrm{x}_7^T & \mathrm{x}_8^T \end{bmatrix}, \text{ or } \begin{bmatrix} \widehat{X}_5 & O & O \\ O & \widehat{X}_7 & O \\ O & O & \widehat{X}_7 \\ \mathrm{x}_5^T & \mathrm{x}_7^T & \mathrm{x}_7^T \end{bmatrix} \text{ or } \begin{bmatrix} \widehat{X}_6 & O & O \\ O & \widehat{X}_6 & O \\ O & O & \widehat{X}_7 \\ \mathrm{x}_6^T & \mathrm{x}_6^T & \mathrm{x}_7^T \end{bmatrix}$$

where for each $i = 1, 2, 3$,

$$\begin{bmatrix} \widehat{X}_{p_i} \\ \\ \mathrm{x}_{p_i}^T \end{bmatrix}$$

is a p_i by p_i orthogonal matrix with the full row $\mathrm{x}_{p_i}^T$ which has $\rho(p_i)$ nonzero entries. These matrices are determined from Theorem 2.2. It is easy to compute that $\#(A) = 71$ for the matrices in (8). But note that if

$$A = \begin{bmatrix} \widehat{X}_3 & O & O \\ O & \widehat{X}_8 & O \\ O & O & \widehat{X}_8 \\ \mathrm{x}_3^T & \mathrm{x}_8^T & \mathrm{x}_8^T \end{bmatrix}$$

then $\#(A) = 72$. This means that the condition (6) for p_i's is necessary to get sparse row-orthogonal matrices with a full row.

Now, we determine the number of nonzero entries of A in (5). We claim

$$\#(A) = (k+3)n - (n-m+1)2^{k+1}$$

where

$$k = \left\lfloor \log_2 \left(\frac{n}{n-m+1} \right) \right\rfloor .$$

Since $2^k \le p_i \le 2^{k+1}$ for each $i = 1, 2, \ldots, n-m+1$,

$$\lfloor \log_2 p_i \rfloor = \begin{cases} k & \text{if } 2^k \le p_i < 2^{k+1} \\ k+1 & \text{if } p_i = 2^{k+1}. \end{cases}$$

Thus if $2^k \le p_i < 2^{k+1}$ for each $i = 1, 2, \ldots, n-m+1$, then

$$\begin{aligned} \#(A) &= \#(X_{p_1}) + \#(X_{p_2}) + \cdots + \#(X_{p_{n-m+1}}) \\ &= (k+3)(p_1 + p_2 + \cdots + p_{n-m+1}) - (n-m+1)2^{k+1} \\ &= (k+3)n - (n-m+1)2^{k+1}. \end{aligned}$$

Let $p_i = 2^{k+1}$ for $i = j, j+1, \ldots, n-m+1$. Since $p_j + p_{j+1} + \cdots + p_{n-m+1} = (n-m-j+2)2^{k+1}$,

$$\begin{aligned} \#(A) &= \#(X_{p_1}) + \#(X_{p_2}) + \cdots + \#(X_{p_{n-m+1}}) \\ &= (k+3)(p_1 + p_2 + \cdots + p_{j-1}) - (j-1)2^{k+1} \\ &\quad + (k+4)(p_j + p_{j+1} + \cdots + p_{n-m+1}) - (n-m-j+2)2^{k+2} \\ &= (k+3)n - (n-m+1)2^{k+1}. \end{aligned}$$

In the above example, *i.e.*, if A is a 17 by 19 row-orthogonal matrix with the full row in (8) then $k = 2$, and thus $\#(A) = 5 \cdot 19 - 3 \cdot 2^3 = 71$.

Thus, for positive integers m and n with $m \le n$, if $f(m,n)$ denote the least number of nonzero entries in an m by n indecomposable, row-orthogonal matrix with a full row then we conclude that

$$f(m,n) \le (k+3)n - (n-m+1)2^{k+1} \tag{9}$$

where

$$k = \left\lfloor \log_2 \left(\frac{n}{n-m+1} \right) \right\rfloor .$$

And we have the following conjecture.

CONJECTURE. For positive integers m and n with $m \leq n$, let $f(m,n)$ denote the least number of nonzero entries in an m by n indecomposable, row-orthogonal matrix with a full row, then the equality holds in (9). Furthermore, the equality holds in (9) if and only if, up to row and column permutations, the matrix is A in (5).

Note that if A is an n by n indecomposable, orthogonal matrix with a full row, from [4], since

$$\#(A) \geq (\lfloor \log_2 n \rfloor + 3)n - 2^{\lfloor \log_2 n \rfloor + 1},$$

this conjecture holds for $m = n$. Thus this conjecture is a generalization of the result in [4].

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