J. Korean Math. Soc. **36** (1999), No. 2, pp. 317–332

OPTIMAL PROBLEM FOR RETARDED SEMILINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we deal with the optimal control problem for the semilinear functional differential equations with unbounded delays. We will also establish the regularity for solutions of the given system. By using the penalty function method we derive the optimal conditions for optimality of an admissible state-control pairs.

1. Introduction

The problem to be controlled is considered by the following evolution equation with delay terms:

$$(1.1)\qquad \begin{cases} \dfrac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) + \displaystyle\int_{-h}^{0} a(s)A_2x(t+s)ds \\ \qquad\qquad + f(t,x(t)) + Bu(t) \\ x(0) = g^0, \quad x(s) = g^1(s), \quad -h \le s < 0 \end{cases}$$

The cost function is denoted by

$$J(x,u) = \int_0^b L(t,x(t),u(t))dt$$

where the pair (x,u) belongs to a separable Banach space. The object of this paper is to minimize $J(x,u)$ satisfying (1.1).

Received April 13, 1998.
1991 Mathematics Subject Classification: Primary 35B37; Secondary 93C20.
Key words and phrases: retarded semilinear differential equations, optimal pairs, penalty function, optimal conditions.
This work was supported by the Basic Science Research Institute Program, Ministry Education, 1996, Project No. BSRI-97-1410.

First, we deal with the regularity for solutions of the retarded functional differential equation (1.1) with the Lipschitz continuity of nonlinear term $f(t, x(t))$ and give the variation of constant formula for the mild solutions of (1.1)

Next, by using the regularity for solutions of (1.1) we derive the existence of optimal pairs for $J(x, u)$, and obtain necessary optimality conditions for (1.1) which are described by the fundamental solution of (1.1). To derive necessary conditions for optimality of an admissible state-control pair , we use the penalty function method. Here, following the methods of Papageorgiou [6,7] we extend his works to the retarded semilinear functional differential equations (1.1).

2. Retarded semilinear differential equations

Let H and V be Hilbert spaces such that $V \subset H \subset V^*$. Therefore, for the sake of simplicity, we may regard that $||u||_* \leq |u| \leq ||u||$ for all $u \in V$ where the notations $|\cdot|$, $||\cdot||$ and $||\cdot||_*$ denote the norms of H, V and V^* respectively, as usual. Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\text{(2.1)} \qquad \text{Re } a(u,u) \geq c_0 ||u||^2 - c_1 |u|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let A_0 be the operator associated with a sesquilinear form

$$\text{(2.2)} \qquad (A_0 u, v) = -a(u, v), \quad u, \ v \in V.$$

Then the operator A_0 is a bounded linear from V to V^*. The operators A_1 and A_2 are bounded linear operators from V to V^* such that they map $D(A_0)$ into H. We may assume that $(D(A_0), H)_{1/2,2} = V$ satisfying

$$\text{(2.3)} \qquad ||u|| \leq C_1 ||u||_{D(A_0)}^{1/2} |u|^{1/2}$$

for some constant $C_1 > 0$ where $(D(A_0), H)_{\theta,p}$ denotes the real interpolation space between $D(A_0)$ and H. The function $a(\cdot)$ is assumed to be a real valued Hölder continuous in $[-h, 0]$ and the controller operator B is a bounded linear operator from some Banach space Y to H.

Let f be a nonlinear mapping from $\mathcal{R} \times V$ into H. We assume that for any x_1, $x_2 \in V$ there exists a constant $L > 0$ such that

$$|f(t,x_1) - f(t,x_2)| \leq L||x_1 - x_2|| \tag{2.4}$$

$$f(t,0) = 0. \tag{2.5}$$

We may assume that (2.1) holds for $c_1 = 0$ as noting that $A_0 + c_1$ is an isomorphism from V to V^* if $c_1 \neq 0$.

Identifying the antidual of H with H we may consider $V \subset H \subset V^*$. The realization of A_0 in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V : A_0 u \in H\}$$

is also denoted by A_0. It is known that A_0 generates an analytic semigroup in both H and V^*. Replacing intermediate space F in the paper [3] with the space H, we can derive the results of G. Blasio, K. Kunisch and E. Sinestrari [3] regarding term by term to deduce our semilinear system as is seen in Theorem 2.1 in [8].

PROPOSITION 2.1. *Under the above assumptions for the nonlinear mapping f, there exists a unique solution x of* (1.1) *such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $g = (g^0, g^1) \in Z = H \times L^2(-h,0;V)$. Moreover, there exists a constant C such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||u||_{L^2(0,T;Y)}),$$

where

$$||\cdot||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} = \max\{||\cdot||_{L^2(0,T;V)}, ||\cdot||_{W^{1,2}(0,T;V^*)}\}.$$

We first consider the fundamental solution of retarded system. The fundamental solution $W(t)$ of the equation (1.1) is defined as follows:

$$\frac{d}{dt}W(t) = A_0W(t) + A_1W(t-h) + \int_{-h}^{0} a(s)A_2W(t+s)ds, \ t > 0,$$
$$W(0) = I, \quad W(s) = 0, \ s \in [-h,0).$$

Since we are assuming that $a(\cdot)$ is Hölder continuous, as is seen in [10] the fundamental solution exists. Let A_0 generate an analytic semigroup $G(t)$ on H. Then as is seen in [10], the fundamental solution $W(t)$ is a unique solution of

$$W(t) = G(t) + \int_0^t G(t-s)(A_1 W(s-h) + \int_{-h}^0 a(\tau)A_2 W(\tau+s)d\tau)ds, \quad t \geq 0, \tag{2.6}$$

$$W(s) = 0, \ s \in [-h, 0). \tag{2.7}$$

It is also known that $W(t)$ is strongly continuous and $AW(t)$ and $dW(t)/dt$ are strongly continuous except at $t = nh$, $n = 0,\ 1,\ 2,\ \cdots$. Therefore we may assume that

$$|W(t)| \leq M, \quad t \geq 0$$

where M is a constant. The solution of (1.1) is expressed by

$$x(t) = W(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)ds + \int_0^t W(t-\tau)\{f(\tau, x(\tau)) + Bu(\tau)\}d\tau,$$

$$U_t(s) = W(t-s-h)A_1 + \int_{-h}^s W(t-s+\sigma)a(\sigma)A_2 d\sigma$$

in the sense of [5].

PROPOSITION 2.2. *Let $f \in L^2(0,T;H)$ and $x(t) = \int_0^t W(t-s)f(s)ds$. Then there exists a constant C such that*

$$||x||_{L^2(0,T;V)} \leq C\sqrt{T}||f||_{L^2(0,T;H)}.$$

Proof. By the similar way of Theorem 2.3 of [3] it holds that

$$||x||_{L^2(0,T;D(A_0))} \leq C_T||f||_{L^2(0,T;H)}. \tag{2.8}$$

By using Hölder inequality,

$$\begin{aligned}||x||^2_{L^2(0,T;H)} &= \int_0^T \left|\int_0^t W(t-s)f(s)ds\right|^2 dt \\ &\le M^2 \int_0^T \left(\int_0^t |f(s)|ds\right)^2 dt \\ &\le M^2 \int_0^T t \int_0^t |f(s)|^2 dsdt \\ &\le M^2 \frac{T^2}{2} \int_0^t |f(s)|^2 ds.\end{aligned}$$

Therefore

$$||x||_{L^2(0,T;H)} \le MT||f||_{L^2(0,T;H)}. \tag{2.9}$$

Combining (2.8) and (2.9) we have that

$$||x||^2_{L^2(0,T;V)} \le C_T MT||f||^2_{L^2(0,T;H)}.$$

□

3. Existence of optimal pairs

In what follows we assume that the each embedding $D(A_0) \subset V \subset H$ is compact. We consider the optimal control problem of (1.1). Let $\mathcal{W}$ be the set of all admissible state-control pairs. The associated cost function is given by

$$J(x,u) = \int_0^b L(t, x(t), u(t))dt.$$

We will find a state-optimal element $(x,u) \in \mathcal{W}$ such that

$$J(x,u) = \inf_{(x',u')\in\mathcal{W}} J(x',u') = m \tag{P}$$

satisfying (1.1) a.e. $0 < t \le T$ and $u(t) \in U(t)$ a.e., $u(\cdot)-$measurable.

The state-control (x, u) is called to be admissible if it satisfies the constraint (P) mentioned above. We assume that there exists $(x, u) \in \mathcal{W}$ such that $J(x, u) < \infty$.

Let $T = [0, b]$ and Y be a separable, reflexive Banach space as considered the control space.

We need following hypotheses.

H(L). $L : T \times H \times Y \to R \cup \{\infty\}$ is an integrand such that
  (1) $(t, x, u) \to L(t, x, u)$ is measurable
  (2) $(x, u) \to L(t, x, u)$ is lower semicontinuous
  (3) $L(t, x, \cdot)$ is convex
  (4) $\varphi(t) - M(|x| + \|u\|) \le L(t, x, u)$ a.e. with $\varphi \in L^1(T), M > 0$

H(U). $t \to U(t)$ is a measurable multifunction such that $|U(t)| = \sup\{\|u\|_Y : u \in U(t)\} \le M$ a.e. on T and $U(t)$ is a nonempty, closed and convex subset of Y. (Here $t \to U(t)$ measurable means that for all $z \in Y$, $t \to d_Y(z, U(t)) = \inf\{\|z - v\|_Y : v \in U(t)\}$ is measurable)

The control space will be modeled by a separable, reflexive Banach space Y. We assume that $t \mapsto U(t)$ has nonempty, weakly compact and convex values in Y, it is measurable and that

$$|U(t)| = \sup_{z \in U(t)} ||z|| \in L^2(0, \infty).$$

We denote by U_{ad} the set of admissible controls given by

$$U_{ad} = \{u \in L^2(0, T; Y) : u(t) \in U(t),\ a.e.\}. \tag{3.1}$$

It is known that U_{ad} is weakly compact as in Theorem 2.1 in [7] and by the Eberlein-Smulian theorem, is sequentially weakly compact.

Let $Z = H \times L^2(-h, 0; V)$ be the state space and be a product Hilbert space with the norm

$$||g||_Z = (|g^0|^2 + \int_{-h}^0 ||g^1(s)||^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in Z.$$

Let $g \in Z$ and $x(t; g, f, u)$ be a solution of the equation (1.1) associated with nonlinear term f and control Bu at time t.

THEOREM 3.1. *Let $x_u(t) = x(t; g, f, u)$. Then the mapping $u \mapsto x_u$ is compact from $L^2(0,T;Y)$ to $L^2(0,T;H)$.*

Proof. We define the solution mapping S from $L^2(0,T;Y)$ to $L^2(0, T;H)$ by

$$(Su)(t) = x_u(t), \quad u \in L^2(0,T;Y).$$

From (2.6) and Proposition 2.1 it follows

$$||Su||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} = ||x_u|| \le C\{||g||_Z + ||Bu||_{L^2(0,T;H)}\}.$$

Hence if u is bounded in $L^2(0,T;Y)$, then so is x_u in $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$. Since V is compactly embedded in H by assumption, the embedding $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H)$ is also compact in view of Theorem 2 of J. P. Aubin [2]. Hence, the mapping $u \mapsto Su = x_u$ is compact from $L^2(0,T;Y)$ to $L^2(0,T;H)$. □

To prove the existence of admissible state-control pairs satisfying the constraints of (P) we apply the method of Theorem 1 of Ref. [1] to our following result.

THEOREM 3.2. *Under hypotheses H(L) and H(U), there exists an admissible state-control pair $(x,u) \in \mathcal{W}$ such that* inf $J(x,u) = m$

Proof. We denote by $S(x_0)$ the set trajectories of (1.1). Let $\{(x_n, u_n)\} \subset S(x_0) \times U_{ad}$ be minimizing sequence for the problem (P). We may assume that $w-\lim u_n = u$ in U_{ad} and $w-\lim x_n = x$ in $L^2 \cap W^{1,2}$. Since $S(x_0)$ is bounded in $L^2 \cap W^{1,2}$ and $L^2 \cap W^{1,2} \subset L^2(0,T;H)$ compactively it holds $x_n \to x$ strongly in $L^2(0,T;H)$. Thus as in Theorem 1 in [1], we have that

$$J(x,u) \le \underline{\lim}\, J(x_n, u_n) = m.$$

From now on, we show that $(x,u) \in \mathcal{W}$, i.e., (x,u) is the desired optimal state-control pair. Let $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{B}$ be the Nemitsky operators corresponding to the maps A, f and B, which are defined by

$$(\mathcal{A}x)(\cdot) = A_0x(\cdot), \quad (\mathcal{F}u)(\cdot) = f(\cdot, x_u), \quad \text{and} \quad (\mathcal{B}u)(\cdot) = Bu(\cdot).$$

Denote by $((\cdot,\cdot))_*$ the duality bracket for the pair $(L^2(0,T;V), L^2(0,T;V^*))$ and by $((\cdot,\cdot))$ the inner product for $L^2(0,T;H)$.

In virtue of Theorem 3.1 it is easily seen that $\mathcal{F}$ is a compact mapping from $L^2(0,T;Y)$ to $L^2(0,T;H)$. If we set

$$(\mathcal{A}_1 x)(t) = A_1 x(t-h) \quad \text{and} \quad (\mathcal{A}_2 x)(t) = \int_{-h}^{0} a(s) A_2 x(t+s)ds$$

then $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) is a bounded operator from $L^2(0,T;V)$(resp. $L^2(0,T;V)$) to $L^2(-h,T-h;V^*)$(resp. $L^2(-h,T;V^*)$).

For every $p \in L^2(0,T;V)$ we consider

$$\begin{aligned} ((\dot{x}_n, p))_* =& ((\mathcal{A}x_n, p))_* + ((\mathcal{A}_1 x_n, p))_* + ((\mathcal{A}_2 x_n, p))_* \\ &+ ((\mathcal{F}u_n, p)) + ((\mathcal{B}u_n, p)). \end{aligned} \tag{3.2}$$

Passing to the limit as $n \to \infty$ in (3.2), it follows

$$((\dot{x}, p))_* = ((\mathcal{A}x, p))_* + ((\mathcal{A}_1 x, p))_* + ((\mathcal{A}_2 x, p))_* + ((\mathcal{F}u, p)) + ((\mathcal{B}u, p)).$$

and hence since p is arbitrary, we have

$$\begin{aligned} \frac{d}{dt}x(t) =& A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} a(s) A_2 x(t+s)ds \\ &+ f(t, x_u(t)) + Bu(t), \quad \text{a.e.} \quad 0 < t \le T, \\ x(0) =& g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0). \end{aligned}$$

Thus, we have that $J(x,u) = m$, i.e., (x,u) is the desired optimal state-control pair. □

4. Optimality conditions

To derive necessary conditions for optimality of an admissible state-control pair, we use the penalty function method. Let $X = L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ and $(\widehat{x}, \widehat{u}) \in X \times L^2(Y)$ be an optimal pair for (P). For every $\delta > 0$, let

$$\begin{aligned} C_\varepsilon = \Big\{ (x,u) \in X \times U_{ad} \; ; \; x(0) = 0, \quad x(s) \equiv 0, \\ \|x - \widehat{x}\| \le \varepsilon \quad \text{and} \quad \|u - \widehat{u}\| \le \varepsilon \Big\} \end{aligned}$$

where $U_{ad} = \{v \in L^2(0,T;Y); v(t) \in U(t) \ a.e.\}$. We introduce the following penalty function defined on $X \times L^2(0,T;Y)$:

$$\begin{aligned} J_\varepsilon(x,u) =& J(x,u) + \|x-\widehat{x}\|_X^2 + \|u-\widehat{u}\|^2_{L^2(0,T;Y)} \\ &+ \|x - \int_0^t W(t-s)\{f(s,x(s)) + Bu(s)\}ds\|^2 \end{aligned}$$

where $(x,u) \in C_\varepsilon$ and $(\widehat{x},\widehat{u}) \in X \times L^2(0,T;Y)$ is the optimal pair for $J(x,u)$. Let us consider the following equation

$$\tag{4.1} \begin{cases} \dfrac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) \\ \qquad\qquad + \displaystyle\int_{-h}^0 a(s)A_2x(t+s)ds + f(t) \\ x(0) = g^0, \quad x(s) = g^1(s), \quad -h \le s < 0 \end{cases}$$

By virtue of Theorem 3.3 of [3] we have the following result on the equation (4.1).

PROPOSITION 4.1. 1) *Let* $F = (D(A_0),H)_{\frac{1}{2},2}$. *For* $(\phi^0,\phi^1) \in F \times L^2(-h,0;D(A_0))$ *and* $f \in L^2(0,T;H)$, $T > 0$, *there exists a unique solution* x *of* (4.1) *belonging to*

$$L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;H) \subset C([0,T];F)$$

and satisfying

$$\tag{4.2} \begin{aligned} &||x||_{L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)} \\ &\le C_1(||\phi^0||_F + ||\phi^1||_{L^2(-h,0;D(A_0))} + ||f||_{L^2(0,T;H)}), \end{aligned}$$

where C_1 *is a constant depending on* T.
2) *Let* $(\phi^0,\phi^1) \in H \times L^2(-h,0;V)$ *and* $f \in L^2(0,T;V^*)$, $T > 0$. *Then there exists a unique solution* x *of* (4.1) *belonging to*

$$L^2(-h,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$\tag{4.3} \begin{aligned} &||x||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \\ &\le C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}), \end{aligned}$$

where C_1 is a constant depending on T.

For the sake of simplicity we take the initial data $(g^0, g^1) = (0, 0)$ and let the operator $\mathcal{Q}$ from $L^2(0,T;H)$ to $L^2(0,T;V)$ defined by

$$(\mathcal{Q}f)(t) = \int_0^t W(t-s)f(s)ds \quad f \in L^2(0,T;H) \tag{4.4}$$

LEMMA 4.1. *The operator $\mathcal{Q}$ defined by (4.4) is compact and $\|\mathcal{Q}\| \leq C\sqrt{T}$ where the constant C is in* Proposition 2.2.

Proof. The mild solution of (4.1) with the initial value $(g^0, g^1) = (0, 0)$ is represented by

$$(\mathcal{Q}f)(t) = x(t) = \int_0^t W(t-s)f(s)ds$$

and from (4.2)

$$\|\mathcal{Q}f\|_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \leq C_1\|f\|_{L^2(0,T;H)}$$

Hence if f is bounded in $L^2(0,T;H)$ then so is $\mathcal{Q}f$ in $L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)$. Noting that assumption that $D(A_0)$ is compactly embedded in X. We know that the embedding

$$L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$$

is compact. Thus we may consider that the operator $\mathcal{Q}$ from $L^2(0,T;H)$ to $L^2(0,T;V)$ is compact in a sense composition of operators. □

The penalty function defined on $X \times L^2(Y)$ is as follows

$$\begin{aligned} J_\varepsilon(x,u) =& J(x,u) + \|x - \widehat{x}\|^2 + \|u - \widehat{u}\|^2 \\ &+ \|x - \mathcal{Q}(\mathcal{F}x + \mathcal{B}u)\|^2_{L^2(0,T;V)} \end{aligned}$$

for $(x,u) \in C_\varepsilon$. Noting that $J_\varepsilon(\widehat{x},\widehat{u}) = J(\widehat{x},\widehat{u}) = m$, we have

$$\inf J_\varepsilon \leq m.$$

If $(y,v) \in C_\varepsilon$, we will say that (y,v) is ε-admissible. Now we will characterize to minimize $J_\varepsilon(\cdot,\cdot)$ over all ε-admissible pairs.

PROPOSITION 4.2. *Let us assume H(L) and H(U) in section 3. Then for every $\varepsilon > 0$ the minimum of J_ε over C_ε is attained at a pair $(x_\varepsilon, u_\varepsilon) \in C_\varepsilon$.*

Proof. We will prove that for every $\varepsilon > 0$, $m < J_\varepsilon(x,u)$ for all $\|x - \widehat{x}\|_X = \varepsilon$ and all $u \in U_{ad}$, $\|u - \widehat{u}\|_{L^2(0,T;Y)} = \varepsilon$. Indeed, if this dose not the case , then there is a sequence $\varepsilon_k \to 0$ and a sequence $(x_k, u_k) \in X$ such that $\|x_k - x\| = \varepsilon_k$, $\|u_k - u\| = \varepsilon_k$ and

$$J_{\varepsilon_k}(x_k, u_k) \le m,$$

i.e.,

$$J(x_k, u_k) + \|x_k - x\|^2 + \|u_k - u\|^2 \le m - \|x_k - \mathcal{Q}(\mathcal{F}x_k + \mathcal{B}u_k)\|. \tag{4.5}$$

Recall that $U_{ad} = \{v \in L^2(0,T;Y); v(t) \in U(t) \ a.e.\}$, the set C_ε is bounded and hence, weakly compact subset of $X \times L^2(0,T;Y)$. Thus we may assume that $x_k \to x$ weakly in X and $u_k \to u$ weakly in $\mathcal{U}_{ad}$. The adjoint operator $\mathcal{Q}^*$ is given by

$$(\mathcal{Q}^*\phi)(t) = \int_t^b W^*(s-t)\phi(s)ds.$$

Let $x^* \in L^2(0,T;V^*)$. Then $B^*\mathcal{Q}^*x^*$ belongs to $L^2(0,T;Y^*)$. Hence

$$\langle \mathcal{Q}Bu_k, x^* \rangle = \langle u_k, B^*\mathcal{Q}^*x^* \rangle \to (u, B^*\mathcal{Q}^*x^*)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. By assumption of nonlinear term f it holds that $\mathcal{QF}x_n \to \mathcal{QF}x$ in $L^2(0,T;V)$. Thus from (4.5) we have

$$\begin{aligned} \|x - \mathcal{Q}(\mathcal{F}x - \mathcal{B}u)\| &\le \liminf \|x_k - \mathcal{Q}(\mathcal{F}x_k - \mathcal{B}u_k)\| \\ &\le -[J(x_k, u_k) + \|x_k - x\|^2 + \|u_k - u\|^2] + m \\ \Rightarrow x(t) = \mathcal{Q}(\mathcal{F}x - \mathcal{B}u)(t) \ a.e. \ \ & x(0) = 0, \ x(s) = 0 \ \ s \in [-h, 0) \end{aligned}$$

From the above inequality we have (x,u) is an admissible state-control pair and hence

$$m \le J(x,u).$$

On the other hand, from inequality (4.5) above we see that

$$J(x,u) \leq m - 2\varepsilon^2.$$

Thus we have a contradiction and so the fact explained in the beginning of the proof is complete.

Next, since $J_\varepsilon(\cdot,\cdot)$ is weakly lower semicontinuous on the C_ε, we can find $(x_\varepsilon, u_\varepsilon) \in C_\varepsilon$ such that $J_\varepsilon(x_\varepsilon, u_\varepsilon) = \inf_{C_\varepsilon} J_\varepsilon(x,u)$. In virtue of in the beginning of the proof, we have that

$$\|x_\varepsilon - \widehat{x}\| < \varepsilon \ , \ \|u_\varepsilon - \widehat{u}\| < \varepsilon. \qquad \square$$

To derive the ε-optimality conditions we need the following stronger hypotheses:

H(f_1)

(1) $f(\cdot, x)$ is measurable and $f(t,\cdot)$ is sequentially weakly continuous .

(2) $f(t,\cdot)$ is continuously Gateaux differentiable

$$|\partial_x f(t,x)| < M_f \quad \text{for all} \quad x \in V$$

where $\partial_x f(t,x)$ is the Gateaux derivative of $f(t,x)$ in the second argument for (t,x).

H(L_1)

(1) $t \to L(t,x,u)$ is measurable.

(2) $(x,u) \to L(t,x,u)$ is continuous and convex in u

(3) $(x,u) \to L(t,x,u)$ is Gateaux differentiable and the Gateaux derivative $\partial_x L(t,x,u)$, $\partial_u L(t,x,u)$ belongs to $L^2(0,T;H)$, $L^2(0,T;Y^*)$ respectively, and

$$\max\{\|\partial_x L(t,x,u)\|, \|\partial_u L(t,x,u)\|\} \\ \leq \theta_1(t) + \theta_2(t)(|x|^2 + \|u\|^2) \ a.e.$$

with $\theta_1 \in L^1(T;R^+), \theta_2 \in L^\infty(T;R^+)$.

LEMMA 4.2. *Let us assume* $H(f_1)$, $H(L_1)$. *If* $P_\varepsilon(t) = 2(x_\varepsilon - \mathcal{Q}(\mathcal{F}x_\varepsilon - \mathcal{B}u)(t))$ *then*

(i) $P_\varepsilon(t) - \partial_x f(t, x_\varepsilon)^* \mathcal{Q}^* P_\varepsilon(t) + \partial_x L(t, x_\varepsilon, u_\varepsilon) + 2(x_\varepsilon - \widehat{x}) = 0$ a.e.

(ii)
$$\int_0^b (\partial_u L(t, x_\varepsilon(t), u_\varepsilon(t)) + 2(u_\varepsilon(t) - \widehat{u}(t)) + B^* \mathcal{Q}^* P_\varepsilon(t)(u(t) - u_\varepsilon(t)) \geq 0$$

for every $u \in U_{ad}$.

Proof. Let $w \in X$. By virtue of Proposition 4.2, there exists $r > 0$ such that the pair $(x_\varepsilon + \lambda w, u_\varepsilon)$ is still ε-admissible. Define

$$\Phi(\lambda) = J_\varepsilon(x_\varepsilon + \lambda w, u_\varepsilon), \quad |\lambda| < r.$$

Since H(L_1) be satisfied, the cost J_ε is also Gateaux differentiable, and hence we know that the necessary optimality condition is given by the variational inequality $\Phi'(0) = 0$. Therefore we have

$$\begin{aligned}
0 = \Phi'(0) &= \lim(\Phi(\lambda) - \Phi(0))/\lambda \\
&= \lim\{J_\varepsilon(x_\varepsilon + \lambda w, u_\varepsilon) - J_\varepsilon(x_\varepsilon, u_\varepsilon)\}/\lambda \\
&= \int_0^b (\partial_x L(t, x_\varepsilon(t), u(t)), w(t))\, dt + 2\int_0^b (x_\varepsilon - \widehat{x}(t), w(t))dt \\
&\quad + 2\int_0^b (x_\varepsilon(t) - \mathcal{Q}(\mathcal{F}x_\varepsilon - Bu)(t), w(t) - \mathcal{Q}\partial_x f(t, x_\varepsilon)w(t))dt.
\end{aligned}$$

Set $P_\varepsilon(t) = 2(x_\varepsilon - \mathcal{Q}(\mathcal{F}x_\varepsilon - \mathcal{B}u)(t))$. Then

$$\int_0^b (P_\varepsilon(t) - \partial_x f(t, x)^* \mathcal{Q}^* P_\varepsilon(t) + \partial_x L(t, x_\varepsilon, u_\varepsilon) + 2(x_\varepsilon - \widehat{x}), w(t)) = 0$$

Since w is arbitrary in X the proof of (i) is complete. Next, let $(x_\varepsilon, u_\varepsilon)$ be the optimal for the ε-penalized problem. Then we have

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\left[J_\varepsilon(x_\varepsilon, u_\varepsilon + \lambda(u - u_\varepsilon)) - J_\varepsilon(x_\varepsilon, u_\varepsilon)\right] \geq 0$$

for every $u \in \mathcal{U}_{ad}$. Thus it follows

$$\int_0^b \left(\partial_u L(t, x_\varepsilon(t), u_\varepsilon(t) + 2(u_\varepsilon(t) - \widehat{u}) + B^* \mathcal{Q}^* P_\varepsilon(t), u(t) - u_\varepsilon(t)\right) \geq 0$$

Therefore, the part (ii) of the Lemma is complete. □

THEOREM 4.1. *Let us assume $H(f_1)$, $H(L_1)$. Then there exists $P \in X$*

(i) $P(t) - \partial f(t,x)^* \mathcal{Q}^* P(t) + \partial_x L(t, \widehat{x}, \widehat{u}) = 0$ *where* $(\widehat{x}, \widehat{u})$ *is an optimal state-control pair.*

(ii) (*Minimum principle*)

$$\inf_{u \in U(t)} (\partial_u L(t, x(t), u(t)) + B^* \mathcal{Q}^* P(t), u - \widehat{u}(t)) = 0 \quad \text{a.e.}$$

Proof. Let us consider the behavior of the ε-optimal pairs $(x_\varepsilon, u_\varepsilon)$ and of the variables P_ε as ε tends to zero.

Recall that $(x_\varepsilon, u_\varepsilon) \in C_\varepsilon$. Since $\varepsilon \to 0+$, we have $x_\varepsilon \to \widehat{x}$ in X and $u_\varepsilon \to u$ in $L^2(0,T;Y)$. By the properties of fundamental solution of $W(t)$ we have $P_\varepsilon \to P$ in X .

Next, from (ii) of Lemma, we have

$$\int_0^b (\partial_u L(t, x(t), u(t)) + B^* \mathcal{Q}^* P(t), u(t) - \widehat{u}(t)) dt \geq 0, \quad u \in \mathcal{U}_{ad}.$$

To obtain the pointwise minimum principle, we follow the and of proof of Theorem 4.1 [1].

Let us assume that there exists $E \subseteq T$ measurable such that $\lambda(E) > 0$ where $\lambda(\cdot)$ stands for Lebesgue measure on T and for all $t \in E$

$$\inf_{u \in U(t)} (\partial_u L(t, x(t), u(t)) + B^* \mathcal{Q}^* P(t), u - \widehat{u}(t)) < 0.$$

Set $h(t,u) = (\partial_u L(t, x(t), u(t)) + B^* \mathcal{Q}^* P(t), u - u(t)) < 0$. Since $U(\cdot)$ is measurable and $u \to h(t,u)$ is continuous it follows $h(\cdot,\cdot)$ is jointly measurable. Apply Aumann's selection theorem(Ref. 9, Theorem 3) to

$$\{(t,u) \in T \times Y : h(t,u) < 0\} \cap G_r U \in B(E) \times B(Y),$$

where $B(E)$ is Borel σ-field of E, there exists $u_0 : E \to Y$ measurable such that $h(t, u_0(t)) < 0$ and $u_0(t) \in U(t)$ for $t \in T$. Let

$$\overline{u}(t) = \begin{cases} u_0(t) & \text{if } t \in E \\ \widehat{u}(t) & \text{if } t \in E^c. \end{cases}$$

Then $\overline{u} \in \mathcal{U}_{ad}$ and

$$\int_0^b (\partial_u L(t, x(t), u(t)) + B^* Q^* P(t), \overline{u}(t) - \widehat{u}(t))dt < 0,$$

it is contradictory to the integral minimum principle. □

REMARK. $\big(\partial_u L(t, x_\varepsilon(t), u_\varepsilon(t)), (u(t)-u_\varepsilon(t))\big)$ denotes the directional derivative of the convex integrand $L(t, x_\varepsilon(t), \cdot)$ at the point $u_\varepsilon(t)$, in the direction $u(t) - u_\varepsilon(t)$. From convex analysis , we know that there exists

$$v_\varepsilon(\cdot) \in L^2(0, T; V^*)$$

such that

$$v_\varepsilon(t) \in \partial_u(t, x_\varepsilon(t), u_\varepsilon(t)) \text{ a.e.}$$

and

$$(\partial_u L(t, x_\varepsilon(t), u_\varepsilon(t)), u(t) - u_\varepsilon(t)) = (v_\varepsilon(t), u(t) - u_\varepsilon(t)) \text{ a.e.}$$

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