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THE CONVERGENCE OF FINITE DIFFERENCE APPROXIMATIONS FOR SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. We consider two finite difference approximations to a singular boundary value problem arising in the study of a nonlinear circular membrane under normal pressure. It is shown that the rates of convergence are $O(h)$ and $O(h^2)$, respectively. An iterative scheme is introduced which converges to the solution of the finite difference equations. Finally the numerical experiments are given

1. Introduction

In the study of a nonlinear circular membrane under normal pressure [3,4], the following singular boundary value problem arises:

$$\begin{aligned} &-y'' - \frac{3}{x}y' - \frac{2}{y^2} = 0, \quad 0 < x < 1, \\ &y'(0) = 0, \quad y'(1) + (1-v)y(1) = 0, \quad 0 < v < 1, \end{aligned} \tag{1.1}$$

where v, $0 < v < 1$, is a constant. The existence of a unique positive solution of (1.1) has been discussed by [2,3,4,9]. Numerical solutions of this problem can be obtained by the iterative method [2] and numerical techniques [4] on the integral equation equivalent to (1.1). It is mentioned in [4] that because of singularity and the nonlinearity, difficulties are encountered if numerical solutions of (1.1) are attempted by finite difference methods. In [8], a finite difference method to a class of singular boundary value problem is introduced.

When the boundary condition at $x = 1$ is $y(1) = \lambda(> 0)$ instead of $y'(1)+(1-v)y(1) = 0$, the unique existence of a positive solution and a

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numerical solution are studied by [2,3,4,7,8,9]. In [7], a finite difference approximation to (1.1) is introduced whose rate of convergence is $O(h^2)$ and which may avoid the above difficulties stated in [4]. And the global error estimate $O(h^2)$ is better than one in [8].

In this paper, motivated by the method in [7], two finite difference approximations to (1.1), scheme I and scheme II, are considered. The rates of convergence are $O(h)$ and $O(h^2)$, respectively and both methods may avoid the difficulties stated in [4]. To obtain the solution of each finite difference equation, an iterative technique is introduced which converges monotonically to the solution of the finite difference equation. In section 2, some preliminaries are given. In section 3, two finite difference approximations, scheme I and scheme II, are introduced, and an iterative technique is given which converges monotonically to the solution of the finite difference equations. In section 4, we prove analytically that the rates of convergence of the scheme I and the scheme II are $O(h)$ and $O(h^2)$, respectively. The rates of convergence of scheme I and scheme II are verified numerically in section 5.

2. Preliminaries

To discuss the behavior of the solution of (1.1) at $x = 0$, we begin with the following lemma whose proof is straightforward.

Lemma 2.1. *Let $f \in C[0,1]$ and $f' \in C(0,1]$. If $\lim_{x\to 0+} f'(x)$ exists, then*

$$f'_+(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0+} f'(x),$$

which implies that $f'(x)$ continuous at $x = 0$.

It was shown in [9] that there exists a unique solution $Y \in C^2(0,1] \cap C^1[0,1]$ of (1.1). Thus the following lemma is obtained from Lemma 2.1 and the fact that

$$Y'(x) = -\frac{1}{x^3}\int_0^x \frac{2s^3}{Y^2(s)}ds.$$

Lemma 2.2 [7]. *Let Y be a positive solution of (1.1). Then*

(1) *$Y''_+(0)$ exists and $Y''(x)$ is continuous at $x = 0$.*

(2) $Y_+{}^{(3)}(0)(=0)$ *exists and* $Y^{(3)}(x)$ *is continuous at* $x=0$.
(3) $Y_+{}^{(4)}(0)$ *exists and* $Y^{(4)}(x)$ *is continuous at* $x=0$.

REMARK. Lemma 2.2 implies that if Y is a positive solution of (1.1) then $Y \in C^4[0,1]$.

3. Finite difference approximations

3.1. Scheme I

Let N be a positive integer, $h = \dfrac{1}{N}$, $x_i = i \cdot h$, $i = 0, 1, 2, \cdots, N$, and let y_i be the approximation of $Y(x_i)$, $i = 0,\ 1,\ 2,\ \cdots,\ N$. Consider the following finite difference approximation (scheme I):

$$
\begin{aligned}
&-8 \cdot \frac{y_1 - y_0}{h^2} - \frac{2}{y_0^2} = 0,\\
&-4 \cdot \frac{y_2 - 2y_1 + y_0}{h^2} - \frac{2}{y_1^2} = 0,\\
&-\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{3}{x_i} \cdot \frac{y_{i+1} - y_{i-1}}{2h} - \frac{2}{y_i^2} = 0,\\
&\qquad\qquad\qquad i = 2,\ 3,\ \cdots,\ N-1,\\
&-\frac{y_{N-1} - y_N}{h} + (1-v)y_N = 0.
\end{aligned} \tag{3.1.1}
$$

Let

$$
L_1 = \begin{bmatrix}
8 & -8 & 0 & & \cdots & 0\\
-4 & 8 & -4 & 0 & \cdots & 0\\
0 & -1 + \dfrac{3h}{2x_2} & 2 & -1 - \dfrac{3h}{2x_2} & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & & 0 & -1 + \dfrac{3h}{2x_{N-1}} & 2 & -1 - \dfrac{3h}{2x_{N-1}}\\
0 & & \cdots & 0 & -1 & 1 + h(1-v)
\end{bmatrix},
$$

$\tilde{y} = (y_0,\ y_1,\ \cdots,\ y_{N-1},\ y_N)^t$, and

$$N_1\tilde{\mathbf{y}} = \left(-\frac{2h^2}{y_0^2}, -\frac{2h^2}{y_1^2}, \cdots, -\frac{2h^2}{y_{N-1}^2}, 0\right)^t,$$

where $N_1\tilde{\mathbf{y}}$ and $\tilde{\mathbf{y}}$ are column vectors. Now we have a nonlinear system

$$L_1\tilde{\mathbf{y}} + N_1\tilde{\mathbf{y}} = \mathbf{0}, \tag{3.1.2}$$

where $\mathbf{0}$ is the zero vector. To solve the nonlinear system (3.1.2), we use Newton's method. So, for $m = 0, 1, 2, \cdots$, we have

$$\tilde{\mathbf{y}}^{(m+1)} = \tilde{\mathbf{y}}^{(m)} - \left(L_1 + N_1{}'\tilde{\mathbf{y}}^{(m)}\right)^{-1} \cdot \left(L_1\tilde{\mathbf{y}}^{(m)} + N_1\tilde{\mathbf{y}}^{(m)}\right), \tag{3.1.3}$$

where $N_1{}'\tilde{\mathbf{y}}$ is the diagonal matrix, $\operatorname{diag}\left[\frac{4h^2}{y_0^3}, \frac{4h^2}{y_1^3}, \cdots, \frac{4h^2}{y_{N-1}^3}, 0\right]$. Therefore, from (3.1.3), we derive

$$L_1\tilde{\mathbf{y}}^{(m+1)} + \left[N_1{}'\tilde{\mathbf{y}}^{(m)}\right]\tilde{\mathbf{y}}^{(m+1)} = \left[N_1{}'\tilde{\mathbf{y}}^{(m)}\right]\tilde{\mathbf{y}}^{(m)} - N_1\tilde{\mathbf{y}}^{(m)} \tag{3.1.4}$$

and

$$\begin{aligned} &L_1\tilde{\mathbf{y}}^{(m+1)} + N_1\tilde{\mathbf{y}}^{(m+1)} \\ &= N_1\tilde{\mathbf{y}}^{(m+1)} - N_1\tilde{\mathbf{y}}^{(m)} - N_1{}'\tilde{\mathbf{y}}^{(m)}\left[\tilde{\mathbf{y}}^{(m+1)} - \tilde{\mathbf{y}}^{(m)}\right] \\ &= \frac{1}{2}N_1{}''\xi^{(m)}\left(\left(y_j^{(m+1)} - y_j^{(m)}\right)^2\right), \end{aligned} \tag{3.1.5}$$

where $N_1{}''\tilde{\mathbf{y}}$ is the diagonal matrix, $\operatorname{diag}\left[-\frac{12h^2}{y_0^4}, -\frac{12h^2}{y_1^4}, \cdots, -\frac{12h^2}{y_{N-1}^4}, 0\right]$, and $\xi_j^{(m)}$ is between $y_j^{(m+1)}$ and $y_j^{(m)}$.

THEOREM 3.1.1 [1].

(i) *The M-matrix L_1 is an inverse positive matrix.*
(ii) *The matrix $L_1 + N_1{}'\tilde{\mathbf{y}}$ is an inverse positive matrix for any $\tilde{\mathbf{y}} > \mathbf{0}$.*

Proof. (i) Let D_i be the $(i+1)$-th leading principal minor of L_1. Then we obtain

$$D_0 = 8, \quad D_1 = 32, \quad D_2 = \left(1 + \frac{3h}{2x_2}\right) D_1, \quad \cdots,$$

$$D_{N-1} = \left(1 + \frac{3h}{2x_{N-1}}\right) D_{N-2}, \quad D_N = h(1-v)\left(1 + \frac{3h}{2x_{N-1}}\right) D_{N-2},$$

which imply that the M-matrix L_1 is an inverse positive matrix.

(ii) Since L_1 is an M-matrix and $N_1{}'\tilde{\mathbf{y}}$ is a nonnegative diagonal matrix for any $\tilde{\mathbf{y}} > \mathbf{0}$, $L_1 + N_1{}'\tilde{\mathbf{y}}$ is an inverse positive matrix. □

LEMMA 3.1.1. *If* $\mathbf{u}$ *satisfies* $L_1\mathbf{u} + N_1\mathbf{u} \geq \mathbf{0}$ *and* $\mathbf{l}$ *satisfies* $L_1\mathbf{l} + N_1\mathbf{l} \leq \mathbf{0}$, *then*

$$\mathbf{l} \leq \mathbf{u},$$

where $0 < u_i$ *and* $0 < l_i$ *for* $i = 0,\ 1,\ 2,\ \cdots,\ N$.

Proof. From the assumptions on $\mathbf{u}$ and $\mathbf{l}$, we have

$$\begin{aligned} \mathbf{0} &\leq L_1\mathbf{u} + N_1\mathbf{u} - L_1\mathbf{l} - N_1\mathbf{l} \\ &= L_1(\mathbf{u} - \mathbf{l}) + N_1(\mathbf{u} - \mathbf{l}) \\ &= (L_1 + N_1{}'\xi)(\mathbf{u} - \mathbf{l}), \end{aligned}$$

where ξ_i lies between l_i and u_i. Since $L_1 + N_1{}'\xi$ is inverse positive, $\mathbf{u} - \mathbf{l} \geq \mathbf{0}$, which completes the proof. □

LEMMA 3.1.2. *If* $\mathbf{u}$ *satisfies* $L_1\mathbf{u} + N_1\mathbf{u} \geq \mathbf{0}$, $\mathbf{y}^{(0)} > \mathbf{0}$, $L_1\mathbf{y}^{(0)} + N_1\mathbf{y}^{(0)} \leq \mathbf{0}$, *and* $\{\mathbf{y}^{(m)}\}$ *is given by* (3.1.3) *or* (3.1.4), *then*

$$\mathbf{y}^{(0)} \leq \mathbf{y}^{(1)} \leq \mathbf{y}^{(2)} \leq \cdots \leq \mathbf{y}^{(m)} \leq \cdots \leq \mathbf{u} \quad \textit{for} \quad m = 0,\ 1,\ 2,\ \cdots,$$

where $0 < u_i$ *for* $i = 0,\ 1,\ 2,\ \cdots,\ N$.

Proof. It is obvious from (3.1.3), (3.1.5) and Lemma 3.1.1. □

Let

$$\begin{aligned} &l(x) = -[\frac{(1-v)^2}{4(2-v)^2}]^{\frac{1}{3}}(x^2 - \frac{2-v}{1-v}), \\ &l_i = l(ih), i = 0, 1, 2, \cdots, N, \\ &\mathbf{l} = (l_0, l_1, l_2, \cdot, l_N)^t, \end{aligned}$$

where $h = \frac{1}{N}$. Then it is easy to show that $\mathbf{l}$ satisfies $L_1\mathbf{l} + N_1\mathbf{l} \le \mathbf{0}$. And let

$$\begin{aligned} u(x) &= -[\frac{(1-v)^2}{16}]^{\frac{1}{3}}(x^2 - \frac{3-v}{1-v}), \\ u_i &= u(ih), i = 0, 1, 2, \cdots, N, \\ \mathbf{u} &= (u_0, u_1, u_2, \cdot, u_N)^t. \end{aligned}$$

Then it is also easy to show that $\mathbf{u}$ satisfies $L_1\mathbf{u} + N_1\mathbf{u} \ge \mathbf{0}$.

THEOREM 3.1.2. *The system of equations* (3.1.2) *has a unique positive solution.*

Proof. The system of equations (3.1.2) has a positive solution from Lemma 3.1.2 and the above remark. Suppose that $\mathbf{y}$ and $\mathbf{w}$ are positive solutions of the system of equations (3.1.2) and $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then we have

$$L_1\mathbf{z} + N_1\mathbf{y} - N_1\mathbf{w} = \mathbf{0}.$$

So we obtain

$$(L_1 + {N_1}'\xi)\,\mathbf{z} = \mathbf{0},$$

where ξ_i is between y_i and w_i. Since $L_1 + {N_1}'\xi$ is an inverse positive matrix, $\mathbf{z} = \mathbf{0}$ and hence $\mathbf{y} = \mathbf{w}$. □

3.2. Scheme II

Using the same notations as given in the beginnig of Section 3.1, we consider the following finite difference approximation (scheme II):

$$\begin{aligned} &-8 \cdot \frac{y_1 - y_0}{h^2} - \frac{2}{y_0^2} = 0, \\ &-4 \cdot \frac{y_2 - 2y_1 + y_0}{h^2} - \frac{2}{y_1^2} = 0, \\ (3.2.1) \quad &-\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{3}{x_i} \cdot \frac{y_{i+1} - y_{i-1}}{2h} - \frac{2}{y_i^2} = 0, \\ &\qquad\qquad i = 2, 3, \cdots, N-1, \\ &-\frac{2y_{N-1} - 2y_N}{h^2} + \frac{2}{h}(1-v)y_N + \frac{3}{x_N}(1-v)y_N - \frac{2}{y_N^2} = 0. \end{aligned}$$

Let L_2 be the same matrix as the matrix L_1 in section 3.1 except the n-th row and let the n-th row of L_2 be $0, 0, \ldots, 0, -2, 2 + 2h(1-v) + 3h^2(1-v)$. Let

$$\tilde{\mathbf{y}} = (y_0,\ y_1,\ \cdots,\ y_{N-1},\ y_N)^t,$$

and

$$N_2\tilde{\mathbf{y}} = \left(-\frac{2h^2}{y_0^2},\ -\frac{2h^2}{y_1^2},\ \cdots,\ -\frac{2h^2}{y_{N-1}^2},\ -\frac{2h^2}{y_N^2}\right)^t,$$

where $N_2\tilde{\mathbf{y}}$ and $\tilde{\mathbf{y}}$ are column vectors. Now we have a nonlinear system

$$L_2\tilde{\mathbf{y}} + N_2\tilde{\mathbf{y}} = \mathbf{0}, \tag{3.2.2}$$

where $\mathbf{0}$ is the zero vector. So, for $m = 0,\ 1,\ 2,\ \cdots$, we have

$$\tilde{\mathbf{y}}^{(m+1)} = \tilde{\mathbf{y}}^{(m)} - \left(L_2 + N_2{}'\tilde{\mathbf{y}}^{(m)}\right)^{-1} \cdot \left(L_2\tilde{\mathbf{y}}^{(m)} + N_2\tilde{\mathbf{y}}^{(m)}\right). \tag{3.2.3}$$

where $N_2{}'\tilde{\mathbf{y}}$ is the diagonal matrix, $\operatorname{diag}\left[\frac{4h^2}{y_0^3},\ \frac{4h^2}{y_1^3},\ \frac{4h^2}{y_2^3},\ \cdots,\ \frac{4h^2}{y_N^3}\right]$. Therefore, from (3.2.3), we derive

$$L_2\tilde{\mathbf{y}}^{(m+1)} + \left[N_2{}'\tilde{\mathbf{y}}^{(m)}\right]\tilde{\mathbf{y}}^{(m+1)} = \left[N_2{}'\tilde{\mathbf{y}}^{(m)}\right]\tilde{\mathbf{y}}^{(m)} - N_2\tilde{\mathbf{y}}^{(m)} \tag{3.2.4}$$

and

$$\begin{aligned} &L_2\tilde{\mathbf{y}}^{(m+1)} + N_2\tilde{\mathbf{y}}^{(m+1)} \\ &= N_2\tilde{\mathbf{y}}^{(m+1)} - N_2\tilde{\mathbf{y}}^{(m)} - N_2{}'\tilde{\mathbf{y}}^{(m)}\left[\tilde{\mathbf{y}}^{(m+1)} - \tilde{\mathbf{y}}^{(m)}\right] \\ &= \frac{1}{2}N_2{}''\xi^{(m)}\left(\left(y_j^{(m+1)} - y_j^{(m)}\right)^2\right), \end{aligned} \tag{3.2.5}$$

where $N_2{}''\tilde{\mathbf{y}}$ is the diagonal matrix, $\operatorname{diag}\left[-\frac{12h^2}{y_0^4}, -\frac{12h^2}{y_1^4}, \cdots, -\frac{12h^2}{y_N^4}\right]$, and $\xi_j^{(m)}$ is between $y_j^{(m+1)}$ and $y_j^{(m)}$.

THEOREM 3.2.1 [1].

(i) *The M-matrix L_2 is an inverse positive matrix.*

(ii) *The matrix $L_2 + N_2'\tilde{\mathbf{y}}$ is an inverse positive matrix for any $\tilde{\mathbf{y}} > \mathbf{0}$.*

Proof. (i) Let D_i be the $(i+1)$-th leading principal minor of L_2. Then we obtain

$$\begin{aligned} &D_0 = 8, \quad D_1 = 32, \quad D_2 = \left(1 + \frac{3h}{2x_2}\right) D_1, \quad \cdots, \\ &D_{N-1} = \left(1 + \frac{3h}{2x_{N-1}}\right) D_{N-2}, \\ &D_N = \{2h(1-v) + 3h^2(1-v)\} \cdot \left(1 + \frac{3h}{2x_{N-1}}\right) D_{N-2}, \end{aligned}$$

which imply that the M-matrix L_2 is an inverse positive matrix.

(ii) The proof is the same as that of Theorem 3.1.1. □

LEMMA 3.2.1. *If $\mathbf{u}$ satisfies $L_2\mathbf{u} + N_2\mathbf{u} \geq \mathbf{0}$ and $\mathbf{l}$ satisfies $L_2\mathbf{l} + N_2\mathbf{l} \leq \mathbf{0}$, then*

$$\mathbf{l} \leq \mathbf{u},$$

where $0 < u_i$ and $0 < l_i$ for $i = 0, 1, 2, \cdots, N$.

Proof. The proof is similar to that of Lemma 3.1.1. □

LEMMA 3.2.2. *If $\mathbf{u}$ satisfies $L_2\mathbf{u} + N_2\mathbf{u} \geq \mathbf{0}$, $\mathbf{y}^{(0)} > \mathbf{0}$, $L_2\mathbf{y}^{(0)} + N_2\mathbf{y}^{(0)} \leq \mathbf{0}$, and $\{\mathbf{y}^{(m)}\}$ is given by (3.2.3) or (3.2.4), then*

$$\mathbf{y}^{(0)} \leq \mathbf{y}^{(1)} \leq \mathbf{y}^{(2)} \leq \cdots \leq \mathbf{y}^{(m)} \leq \cdots \leq \mathbf{u}, \quad \text{for} \quad m = 0, 1, 2, \cdots,$$

where $0 < u_i$ for $i = 0, 1, 2, \cdots, N$.

Proof. It is obvious from (3.2.3), (3.2.5), and Lemma 3.2.1. □

Let

$$\begin{aligned} &l(x) = -\left[\frac{(1-v)^2}{4(2-v)^2}\right]^{\frac{1}{3}} \left(x^2 - \frac{2-v}{1-v}\right), \\ &l_i = l(ih), i = 0, 1, 2, \cdots, N, \\ &\mathbf{l} = (l_0, l_1, l_2, \cdot, l_N)^t, \end{aligned}$$

where $h = \frac{1}{N}$. Then it is easy to show that $\mathbf{l}$ satisfies $L_2\mathbf{l} + N_2\mathbf{l} \leq \mathbf{0}$. And let

$$u(x) = -\left[\frac{(1-v)^2}{16}\right]^{\frac{1}{3}}\left(x^2 - \frac{3-v}{1-v}\right),$$
$$u_i = u(ih), i = 0, 1, 2, \cdot, N,$$
$$\mathbf{u} = (u_0, u_1, u_2, \cdots, u_N)^t.$$

Then it is also easy to show that $\mathbf{u}$ satisfies $L_2\mathbf{u} + N_2\mathbf{u} \geq \mathbf{0}$.

THEOREM 3.2.3. *The system of equations* (3.2.2) *has a unique positive solution.*

Proof. The proof is similar to that of Theorem 3.1.3. □

4. The convergence of finite difference approximations

4.1. Scheme I

LEMMA 4.1.1 [5,7]. *Let $Q(x_i) = Q_i$ and $E(x_i) = E_i$ be discrete functions defined on $x_0, x_1, x_2, \cdots, x_N$. Assume that there exists an $\omega > 0$ such that*

$$Q_i \leq -\omega < 0, \quad i = 0, 1, 2, \cdots, N-1.$$

Set $C = \max\left(\frac{4}{\omega}, \frac{1}{1-v}\right)$. At the grid points $x_0, x_1, x_2, \cdots, x_N$ define a difference operator L_1^h by

$$L_1^h E_0 = 8 \cdot \frac{E_1 - E_0}{h^2} + Q_0 E_0, \tag{4.1.1}$$

$$L_1^h E_1 = 4 \cdot \frac{E_2 - 2E_1 + E_0}{h^2} + Q_1 E_1, \tag{4.1.2}$$

$$L_1^h E_i = \frac{E_{i+1} - 2E_i + E_{i-1}}{h^2} + \frac{3}{x_i} \cdot \frac{E_{i+1} - E_{i-1}}{2h} + Q_i E_i, \quad i = 2, 3, \cdots, N-1, \tag{4.1.3}$$

$$L_1^h E_N = \frac{E_N - E_{N-1}}{h} + (1-v)E_N. \tag{4.1.4}$$

Then

$$|E_i| \leq C \cdot \max_{0\leq j\leq N} |L_1^h E_j|, \quad i = 0, 1, 2, \cdots, N.$$

Proof. Note that $C \geq 1$. If $\max|E_i|$ occurs for $i = N$, then

$$|E_N| \leq \frac{1}{1-v}|L_1^h E_N|.$$

Suppose that $\max|E_i|$ occurs for one of $i = 0,\ 1,\ 2, \cdots,\ N-1$. Then from the proof of Lemma 4.1 in [7], we have

$$\max_{0\leq j\leq N-1}|E_j| \leq \frac{4}{\omega}\cdot \max_{0\leq j\leq N-1}|L_1^h E_j|.$$

Thus the proof is completed. □

THEOREM 4.1.1. *Let $Y(x) \in C^4[0,1]$ be an analytic solution of the boundary value problem (1.1). Let y_i, $i = 0,\ 1,\ 2,\ \cdots,\ N$, be numerical solutions of $L_1\tilde{\mathbf{y}} + N_1\tilde{\mathbf{y}} = \mathbf{0}$ and $E_i = Y(x_i) - y_i$ be errors. Then*

$$|E_i| \leq CM_4 h,$$

where

$$M_4 = \sup_{[0,1]}\left|\frac{d^4Y}{dx^4}\right| \quad \text{and } C \text{ is a constant.}$$

Proof. By the mean value theorem and Taylor theorem, we obtain

$$\begin{aligned} 0 &= 4Y''(x_0) + \frac{2}{[Y(x_0)]^2} \\ &= 8\cdot\frac{Y(x_1)-Y(x_0)}{h^2} + \frac{2}{[Y(x_0)]^2} - Y^{(4)}(\xi_0)\cdot\frac{h^2}{3}, \end{aligned} \tag{4.1.5}$$

where $x_0 < \xi_0 < x_1$. For x_1, we have

$$\begin{aligned} 0 &= Y''(x_1) + \frac{3}{x_1}\cdot Y'(x_1) + \frac{2}{[Y(x_1)]^2} \\ &= 4Y''(x_1) + 3\left(Y''(\xi_0) - Y''(x_1)\right) + \frac{2}{[Y(x_1)]^2} \\ &= 4\cdot\frac{Y(x_2)-2Y(x_1)+Y(x_0)}{h^2} - \frac{h^2}{6}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right] \\ &\quad + 3Y^{(4)}(\xi_2)\cdot\xi_1(\xi_0 - x_1) + \frac{2}{[Y(x_1)]^2}, \end{aligned} \tag{4.1.6}$$

where $x_0 < \eta_0 < x_1 < \eta_1 < x_2$, $x_0 < \xi_0 < \xi_1 < x_1$, and $x_0 < \xi_2 < \xi_1$. And for $i = 2, 3, 4, \cdots, N-1$,

(4.1.7)

$$\begin{aligned}
0 &= Y''(x_i) + \frac{3}{x_i} \cdot Y'(x_i) + \frac{2}{[Y(x_i)]^2} \\
&= \frac{Y(x_{i+1}) - 2Y(x_i) + Y(x_{i-1})}{h^2} + \frac{3}{x_i} \cdot \frac{Y(x_{i+1}) - Y(x_{i-1})}{2h} \\
&\quad - \frac{h^2}{24}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right] - \frac{3}{x_i} \cdot \frac{h^2}{12}\left[Y^{(3)}(\xi_0) + Y^{(3)}(\xi_1)\right] \\
&\quad + \frac{2}{[Y(x_i)]^2} \\
&= \frac{Y(x_{i+1}) - 2Y(x_i) + Y(x_{i-1})}{h^2} + \frac{3}{x_i} \cdot \frac{Y(x_{i+1}) - Y(x_{i-1})}{2h} \\
&\quad - \frac{h^2}{4}\left[2Y^{(4)}(\xi_4) + \frac{Y^{(4)}(\xi_2)}{x_i}(\xi_0 - x_i) + \frac{Y^{(4)}(\xi_3)}{x_i}(\xi_1 - x_i)\right] \\
&\quad + \frac{2}{[Y(x_i)]^2} - \frac{h^2}{24}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right],
\end{aligned}$$

where $x_{i-1} < \eta_0 < x_i < \eta_1 < x_{i+1}$, $x_{i-1} < \xi_0 < \xi_2 < x_i < \xi_3 < \xi_1 < x_{i+1}$, $x_0 < \xi_4 < x_i$. And for x_N, we obtain

$$\begin{aligned}
0 &= Y'(x_N) + (1-v)Y(x_N) \\
&= \frac{Y(x_N) - Y(x_{N-1})}{h} + (1-v)Y(x_N) + \frac{h}{2}Y''(\xi_0).
\end{aligned} \tag{4.1.8}$$

where $x_{N-1} < \xi_0 < x_N$. From (3.1.1), (4.1.1), and (4.1.5) we obtain

$$L_1^h E_0 = 8 \cdot \frac{E_1 - E_0}{h^2} + Q_0 E_0 = Y^{(4)}(\xi_0) \cdot \frac{h^2}{3}.$$

From (4.1.2) and (4.1.6), we get

$$\begin{aligned}
L_1^h E_1 &= 4 \cdot \frac{E_2 - 2E_1 + E_0}{h^2} + Q_1 E_1 \\
&= \frac{h^2}{6}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right] - 3Y^{(4)}(\xi_2)\xi_1(\xi_0 - x_1).
\end{aligned}$$

And, from (4.1.3) and (4.1.7), we have

$$\begin{aligned}
L_1^h E_i =& \frac{E_{i+1} - 2E_i + E_{i-1}}{h^2} + \frac{3}{x_i} \cdot \frac{E_{i+1} - E_{i-1}}{2h} + Q_i E_i \\
=& \frac{h^2}{2} \left[2Y^{(4)}(\xi_4) + \frac{Y^{(4)}(\xi_2)}{x_i}(\xi_0 - x_i) + \frac{Y^{(4)}(\xi_3)}{x_i}(\xi_1 - x_i) \right] \\
&+ \frac{h^2}{24} \left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1) \right], \quad i = 2,\ 3,\ \cdots,\ N-1.
\end{aligned}$$

From (4.1.4) and (4.1.8)

$$L_1^h E_N = \frac{E_N - E_{N-1}}{h} + (1 - v)E_N = -\frac{h}{2} Y''(\xi_0),$$

where $F(y) = \dfrac{2}{y^2}$, $Q_i = F'(\mu_i) = -\dfrac{4}{\mu_i^3} \leq -\omega < 0$, and μ_i lies between $Y(x_i)$ and y_i. Let $M_4 = \sup\limits_{[0,1]} \left| \dfrac{d^4 Y}{dx^4} \right|$. Then we obtain

$$\begin{aligned}
|L_h E_0| &\leq \frac{h^2}{3} M_4, \\
|L_h E_1| &\leq \frac{h^2}{3} M_4 + 3h^2 M_4, \\
|L_h E_i| &\leq \frac{h^2}{12} M_4 + 2h^2 M_4, \quad i = 2,\ 3,\ \cdots,\ N-1 \\
|L_h E_N| &\leq \frac{h}{2} M_4.
\end{aligned}$$

Thus, by Lemma 4.1.1, we have

$$|E_i| \leq C M_4 h, \quad \text{for} \quad i = 0,\ 1,\ 2,\ \cdots,\ N,$$

which completes the proof. □

4.2. Scheme II

LEMMA 4.2.1. *Let $Q(x_i) = Q_i$, $E(x_i) = E_i$ be discrete functions defined on $x_0, x_1, x_2, \cdots, x_N$. Assume that there exists an $\omega > 0$ such that*

$$Q_i \le -\omega < 0, \quad i = 0, 1, 2, \cdots, N.$$

Set $C = \max\left(\frac{4}{\omega}, \frac{1}{2(1-v)}\right)$. *At the grid points $x_0, x_1, x_2, \cdots, x_N$ define a difference operator L_2^h by*

$$(4.2.1) \qquad L_2^h E_i = L_1^h E_i, \qquad i = 0, 1, 2, \cdots, N-1,$$

$$(4.2.2) \quad L_2^h E_N = \frac{2E_{N-1} - 2E_N}{h^2} - \frac{2}{h}(1-v)E_N - \frac{3}{x_N}(1-v)E_N + Q_N E_N.$$

Then

$$|E_i| \le C \cdot \max\left[\max_{0\le j\le N-1} |L_2^h E_j|,\ h|L_2^h E_N|\right], \quad i = 0, 1, 2, \cdots, N.$$

Proof. Note that $C \ge 1$. If $\max|E_i|$ occurs for $i = N$, then

$$|E_N| \le \frac{h}{2(1-v)} |L_2^h E_N|.$$

Suppose that $\max|E_i|$ occurs for one of $i = 0, 1, 2, \cdots, N-1$. Then since $L_1^h E_i = L_2^h E_i$, $i = 0, 1, 2, ;\cdot, N-1$, the remaining part of the proof is the same as that of Lemma 4.1.1. □

Since $Y(x) \in C^4[0,1]$ and $Y(1) > 0$, we may extend the positive solution of (1.1) to the interval $[0, 1+\delta]$, for sufficiently small $\delta > 0$. So we have the following theorem whose proof is the same as that of Theorem 4.1.1.

THEOREM 4.2.1. *Let $Y(x) \in C^4[0, 1+\delta]$, be an analytic solution of the boundary value problem (1.1) for sufficiently small $\delta > 0$. Let y_i be numerical solutions of $L_2\tilde{y} + N_2\tilde{y} = 0$ and $E_i = Y(x_i) - y_i$ be errors, where $i = 0, 1, 2, \cdots, N$. Then*

$$|E_i| \le C\tilde{M}_4 h^2,$$

where

$$\tilde{M}_4 = \sup_{[0,1+\delta]} \left|\frac{d^4 Y}{dx^4}\right| \quad \text{and } C \text{ is a constant.}$$

Proof. From (1.1), by the mean value theorem and Taylor theorem, we obtain

$$\begin{aligned} 0 &= Y''(x_N) + \frac{3}{x_N} \cdot Y'(x_N) + \frac{2}{[Y(x_N)]^2} \\ &= \frac{Y(x_{N+1}) - 2Y(x_N) + Y(x_{N-1})}{h^2} - (1-v)\frac{3}{x_N}Y(x_N) \\ &\quad + \frac{2}{[Y(x_N)]^2} - \frac{h^2}{24}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right], \end{aligned} \tag{4.2.3}$$

where $x_{N-1} < \eta_0 < x_N < \eta_1 < x_{N+1}$. And we obtain

(4.2.4)

$$\frac{Y(x_{N+1}) - Y(x_{N-1})}{2h} + (1-v)Y(x_N) - \frac{h^2}{6}[Y^{(3)}(\xi_0) + Y^{(3)}(\xi_1)] = 0,$$

where $x_{N-1} < \xi_0 < x_N < \xi_1 < x_{N+1}$. By substituting (4.2.4) into (4.2.3), we get

$$\begin{aligned} 0 = &\frac{2Y(x_{N-1}) - 2Y(x_N)}{h^2} - \frac{2}{h}(1-v)Y(x_N) + \frac{h}{3}[2Y^{(4)}(\xi_4) \\ &+ Y^{(4)}(\xi_2)(\xi_0 - x_N) + Y^{(4)}(\xi_3)(\xi_1 - x_N)] + \frac{2}{[Y(x_N)]^2} \\ &- \frac{h^2}{24}[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)] - (1-v)\frac{3}{x_N}Y(x_N), \end{aligned} \tag{4.2.5}$$

where $x_N < \xi_3 < x_{N+1}$, $x_0 < \xi_4 < x_N$. Therefore we have

$$\begin{aligned} L_2^h E_N &= \frac{2E_{N-1} - 2E_N}{h^2} - \frac{2}{h}(1-v)E_N - \frac{3}{x_N}(1-v)E_N + Q_N E_N \\ &= -\frac{h}{3}\left[2Y^{(4)}(\xi_4) + Y^{(4)}(\xi_2)(\xi_0 - x_N) + Y^{(4)}(\xi_3)(\xi_1 - x_N)\right] \\ &\quad + \frac{h^2}{24}[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)], \end{aligned}$$

where $F(y) = \frac{2}{y^2}$, $Q_i = F'(\mu_i) = -\frac{4}{\mu_i^3} \le -\omega < 0$, μ_i lies between $Y(x_i)$ and y_i. Since $L_2^h E_i = L_1^h E_i$, $i = 0, 1, 2, \cdot, N-1$, we obtain from the proof of Theorem 4.1.1

$$\left|L_2^h E_i\right| \le C\tilde{M}_4 h^2, i = 0, 1, 2, \cdots, N-1$$

and from (4.2.5) we get

$$|L_2^h E_N| \le \left[\frac{2h^2}{12} + \frac{4}{3}h\right] \tilde{M}_4,$$

where $\tilde{M}_4 = \sup_{[0,1+\delta]} \left|\frac{d^4Y}{dx^4}\right|$ for sufficiently small $\delta > 0$. Thus, by Lemma 4.2.1, we have

$$|E_i| \le C\tilde{M}_4 h^2, \quad \text{for} \quad i = 0,\ 1,\ 2,\ \cdots,\ N,$$

which completes the proof. □

5. Numerical experiments

5.1. Scheme I

The scheme I, proposed in section 3.1, has been implemented on an IBM PC. In the computation, we use

$$\max_{j=0,1,\cdots,N-1} \left|y^{(k+1)}(x_j) - y^{(k)}(x_j)\right| \le \mathrm{TOL} = 1.0 \times 10^{-12}$$

to stop the iteration when we solve the nonlinear system (3.1.2) by Newton's method (3.1.3) or (3.1.4). In table 1, we report the values of $\delta_{\max}(N)$ and $\delta_{\min}(N)$ for $N = 10,\ 20,\ 40,\ 80$ and $v = 0.1$, where

$$\delta_{\max}(N) = \max_{j=0,1,\cdots,N} \frac{|y_{2N}(x_j) - y_N(x_j)|}{|y_{4N}(x_j) - y_{2N}(x_j)|},$$

$$\delta_{\min}(N) = \min_{j=0,1,\cdots,N} \frac{|y_{2N}(x_j) - y_N(x_j)|}{|y_{4N}(x_j) - y_{2N}(x_j)|}$$

and y_N represents the solution of the nonlinear system (3.1.2) for the given N. And in table 2, the value of $\delta_{\max}(N)$ and $\delta_{\min}(N)$ are given for $N = 10,\ 20,\ 40,\ 80$ and $v = 0.9$. From table 1 and table 2, we see numerically that Theorem 4.1.1 is valid.

Table 1. $\delta_{\max}(N)$, $\delta_{\min}(N)$ for $N = 10,\ 20,\ 40,\ 80$ and $v = 0.1$

N	$\delta_{\max}(N)$	$\delta_{\min}(N)$
10	2.109554	2.059469
20	2.055978	2.033158
40	2.028407	2.017376
80	2.014309	2.008876

Table 2. $\delta_{\max}(N)$, $\delta_{\min}(N)$ for $N = 10,\ 20,\ 40,\ 80$ and $v = 0.9$

N	$\delta_{\max}(N)$	$\delta_{\min}(N)$
10	2.036283	2.034866
20	2.017864	2.017137
40	2.008664	2.008500
80	2.004415	2.004234

5.2. Scheme II

The scheme II, proposed in section 3.2, has also been implemented on an IBM PC. In the computation, we use

$$\max_{j=0,1,\cdots,N-1} \left| y^{(k+1)}(x_j) - y^{(k)}(x_j) \right| \le \text{TOL} = 1.0 \times 10^{-12}$$

to stop the iteration when we solve the nonlinear system (3.2.2) by Newton's method (3.2.3) or (3.2.4). In table 3 we report the values of $\delta_{\max}(N)$ and $\delta_{\min}(N)$ for $N = 10,\ 20,\ 40,\ 80$ and $v = 0.1$, where

$$\delta_{\max}(N) = \max_{j=0,1,\cdots,N} \frac{|y_{2N}(x_j) - y_N(x_j)|}{|y_{4N}(x_j) - y_{2N}(x_j)|},$$

$$\delta_{\min}(N) = \min_{j=0,1,\cdots,N} \frac{|y_{2N}(x_j) - y_N(x_j)|}{|y_{4N}(x_j) - y_{2N}(x_j)|}$$

and y_N represents the solution of the nonlinear system (3.2.2) for the given N. And in table 4 the value of $\delta_{\max}(N)$ and $\delta_{\min}(N)$ are given for $N = 10,\ 20,\ 40,\ 80$ and $v = 0.9$. From table 3 and table 4, we see numerically that Theorem 4.2.1 is valid.

Table 3. $\delta_{\max}(N)$, $\delta_{\min}(N)$ for $N = 10, 20, 40, 80$ and $v = 0.1$

N	$\delta_{\max}(N)$	$\delta_{\min}(N)$
10	4.785150	3.386142
20	4.214163	3.845060
40	4.054849	3.961117
80	4.013797	3.990280

Table 4. $\delta_{\max}(N)$, $\delta_{\min}(N)$ for $N = 10, 20, 40, 80$ and $v = 0.9$

N	$\delta_{\max}(N)$	$\delta_{\min}(N)$
10	3.988212	3.767755
20	3.997707	3.962608
40	3.999464	3.990738
80	3.999965	3.997585

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