J. Korean Math. Soc. **36** (1999), No. 2, pp. 257–266

SHARP MOSER-TRUDINGER INEQUALITIES

MEELAE KIM

ABSTRACT. we used Carleson and Chang's method to give another proof of the Moser-Trudinger inequality which was known as a limiting case of the Sobolev imbedding theorem and at the same time we get sharper information for the bound.

1. Introduction

Let Ω be an open bounded domain in the n-dimensional space $\mathbf{R}^n$, $n \geq 2$. Let $\dot{W}_q^1(\Omega)$ be the completion of the function class $C_0^1(\Omega)$ equipped with the norm

$$\|u\|_{\dot{W}_q^1} = \left(\int_D |\nabla u|^q dx \right)^{\frac{1}{q}} \quad \text{for all } u \in C_0^1(\Omega)$$

Then by the Sobolev imbedding theorem we have $\dot{W}_q^1(\Omega) \hookrightarrow L^p(\Omega)$, $1 < q < n$, $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.

As a limiting case of this theorem, the Moser-Trudinger inequality (see [7],[11]) was obtained for functions in $\dot{W}_n^1(\Omega)$ with resulting exponential class integrability and it provides the sharp imbedding of the space $\dot{W}_n^1(\Omega)$ into the Orlicz space $e^{L^{\frac{n}{n-1}}}$ (see [1]).

More precisely, in [7], Moser proved the following: if $u \in \dot{W}_n^1(\Omega)$ $n \geq 2$ with $\|\nabla u\|_{\dot{W}_n^1} \leq 1$ then there exists a constant c_n such that

$$\int_\Omega e^{\alpha u^p} dx \leq c_n m(\Omega), \tag{1}$$

Received July 24, 1997.
1991 Mathematics Subject Classification: 46E35.
Key words and phrases: symmetrization, limiting case of the Sobolev imbedding theorem, exponential type Sobolev inequality.
This work is partially supported by KOSEF.

where $p = \frac{n}{n-1}$, $\alpha \leq \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, $m(\Omega) = \int_\Omega dx$ and ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere.
Here, the sharpness of α_n played an important role for solving some partial differential equation problem [8],[9] and the linearized form of the inequality has been used in several different geometry problems [2],[3],[4],[10]. On the other hand, Carleson-Chang proved the existence of the extremal function of the inequality in [5]. The aim of this paper is to give another proof of the Moser inequality by using Carleson-Chang's result and at the same time we obtain a functional form of the Moser-Trudinger inequality.

2. Functional form and numerical estimates for Moser-Trudinger inequality

THEOREM 2.1. *Let Ω be a bounded domain in $\mathbf{R}^n, n \geq 2$. And let $u \in \dot{W}_n^1(\Omega)$ with*

$$\int_\Omega |\nabla u|^n dx \leq 1 \;;$$

then there exist constants A_0, A_1, A_2 which depend only on n such that

$$\frac{1}{m(\Omega)} \int_\Omega e^{\alpha_n u^p} dx \leq A_0 + A_1 e^{A_2 \int_\Omega |\nabla u|^n dx}$$

where $p = \frac{n}{n-1}$, $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, $m(\Omega) = \int_\Omega dx$ and ω_{n-1} is the $(n-1)$-dimensional surface of the unit sphere.

Particularly, when $n = 2$ or 3, we have

$$\frac{1}{m(\Omega)} \int_\Omega e^{\alpha_n u^p} dx \leq A_1 e^{A_2 \int_\Omega |\nabla u|^n dx}.$$

As earlier studies in the direction of the above theorem we found several other proof of the Moser inequality (1) in ([1]),([5]),([6]). Usually they proved the boundedness of the integral by using some constant c_n which depends only on n with the variety of proofs. In fact in [5], the value c_n in the inequality (1) is estimated to be about 4.3556, for $n = 2$. They were able to compute the value, only when $n = 2$, by using some computer experiments.

REMARK. By letting $\int_\Omega |\nabla u|^n dx = 1$ in the resulting inequalities of the theorem we can get constant bound c_n in the Moser inequality (1), for example $c_2 = 4.63$, $c_3 = 12.28$.

Our proof relies on symmetrization and a change of variable which was used in [7] to reduce the problem as an one dimensional problem. By the result of the symmetrization we obtained a rearranged function u^* of u which is defined on the ball Ω^* centered at the origin with radius R, $\int_{|x|\le R} dx = m(\Omega)$. Since u^* is radial, to change the problem as a one dimensional one we set

$$\varphi(t) = n^{\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}} u^*(|x|) \tag{2}$$

$$\frac{|x|^n}{R^n} = e^{-t}. \tag{3}$$

For our convenience, without loss of generosity assume that $m(\Omega) = m\,(\text{unit ball in } R^n)$ (*i.e.*, $R = 1$).

Thus our theorem becomes the following; if $\varphi(t)$ is a C^1-function defined on $0 \le t < \infty$ with

$$\varphi(0) = 0, \ \ \varphi'(t) \ge 0, \ \ \int_0^\infty \varphi'(t)^n dt = \delta \tag{4}$$

where $\delta \le 1$, $n \ge 2$, then there exist constants A_0, A_1, A_2 such that

$$\int_0^\infty e^{\varphi^p(t)-t} dt \le A_0 + A_1 e^{A_2 \delta}.$$

Particularly, when $n = 2$ or 3,

$$\int_0^\infty e^{\varphi^p(t)-t} dt \le A_1 e^{A_2 \delta}. \tag{5}$$

To prove the Theorem we will estimate the integral $\int e^{\varphi^p(t)-t} dt$ on each interval $[0, a)$ and $[a, \infty)$ separately by using some specific point $a \in [0, \infty)$ which satisfies the following.

CLAIM. For each φ which satisfies (4) we can choose the point a to be the first point such that

$$\varphi(a) = \left[1 - (\frac{n-1}{n})^{n-1}\right]^{\frac{1}{n}} a^{\frac{n-1}{n}}. \tag{6}$$

Proof of Claim. Suppose not, then there will be two cases;
(i) for all $t \geq 0$, $\varphi(t) < [1-(\frac{n-1}{n})^{n-1}]^{\frac{1}{n}} t^{\frac{n-1}{n}}$
(ii) for all $t \geq 0$, $\varphi(t) > [1-(\frac{n-1}{n})^{n-1}]^{\frac{1}{n}} t^{\frac{n-1}{n}}$.

In case (i) we have

$$\begin{aligned}\int_0^\infty e^{\varphi^p(t)-t}dt &\leq \int_0^\infty e^{[(1-(\frac{n-1}{n})^{n-1})^{\frac{1}{n-1}}-1]t}dt \\ &= \left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{\frac{1}{n-1}}\right]^{-1}.\end{aligned}$$

If we assume that (ii) is true, and since

$$\varphi(t) \leq \left(\int_0^t \varphi'(s)^n ds\right)^{\frac{1}{n}} t^{\frac{n-1}{n}} \quad \text{for all } t \geq 0$$

by Holder's inequality, we have

$$\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{\frac{1}{n}} \leq \left(\int_0^t \varphi'(s)^n ds\right)^{\frac{1}{n}}$$

for all $t \geq 0$. But this is a contradiction for sufficiently small t.

Now with assuming the existence of the point a, let

$$\begin{aligned}\delta_1 &= \int_0^a \varphi'(t)^n dt \\ \delta_2 &= \int_a^\infty \varphi'(t)^n dt.\end{aligned}$$

By the property (6) of a and the fact that $\varphi^n(a) \leq a^{n-1}\int_0^a \varphi'(s)^n ds$, δ_1 and δ_2 satisfies the following

$$\delta_1 \geq 1-\left(\frac{n-1}{n}\right)^{n-1} \tag{7}$$

$$\delta_2 \leq 1-\delta_1 \leq \left(\frac{n-1}{n}\right)^{n-1}. \tag{8}$$

We need the following lemma to estimate the integral $\int_a^\infty e^{\varphi^p(t)-t}dt$ in the proof of the theorem. And we will prove Lemma 2.3 by using Lemma 2.2 at the end of this section.

LEMMA 2.2. (Carleson and Chang) *Let*
$K = \{\psi\colon C^1$ *function on* $0 \le t < \infty$, $\psi(0) = 0$, $\int_0^\infty \psi'(t)^n dt \le \beta\}$ *then for each* $c > 0$ *we have*

$$\sup_K \int_0^\infty e^{c\psi(t)-t}dt < e^{(\frac{n-1}{n})^{n-1}(\frac{c^n}{n})\beta} e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}.$$

Also when $c^n\beta \to \infty$, *the inequality tends asymptotically to an equality.*

LEMMA 2.3. *For each* C^1 *function* φ *which satisfies* (4) *and with the fixed point* a *let* $\int_a^\infty \varphi'(t)^n dt = \delta_2$; *then*

$$\int_a^\infty e^{\varphi^p(t)-t}dt \le c_1 e^{c_2\delta_2} e^{[(1-(\frac{n-1}{n})^{n-1})^{\frac{1}{n-1}}-1]a}$$

where

$$c_1 = n^{\frac{2n}{n+1}} e^{\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)}$$

$$c_2 = \frac{(n+1)^{n-1}}{(n-1)^2 e}\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{\frac{1}{n-1}}.$$

Proof of the Theorem. Since $\delta_1 + \delta_2 = \delta$ and $\delta_1 \ge 1 - (\frac{n-1}{n})^{n-1}$, by the Lemma 2.3 we have

$$\int_a^\infty e^{\varphi^p(t)-t}dt \le c_1 e^{c_2(\delta_1+\delta_2)} e^{-c_2(1-(\frac{n-1}{n})^{n-1})} e^{[(1-(\frac{n-1}{n})^{n-1})^{\frac{1}{n-1}}-1]a}$$

$$= c_3 e^{c_2\delta} e^{[(1-(\frac{n-1}{n})^{n-1})^{\frac{1}{n-1}}-1]a} \tag{9}$$

where $c_3 = c_1 e^{-c_2(1-(\frac{n-1}{n})^{n-1})}$. Thus by (9) and property (6) of a, we get

$$\int_0^\infty e^{\varphi^p(t)-t}dt = \int_0^a e^{\varphi^p(t)-t}dt + \int_a^\infty e^{\varphi^p(t)-t}dt$$

$$\le \int_0^a e^{[(1-(\frac{n-1}{n})^{n-1})^{\frac{1}{n-1}}-1]t}dt$$

$$+ c_3 e^{c_2\delta} e^{[(1-(\frac{n-1}{n})^{n-1})^{\frac{1}{n-1}}-1]a}. \tag{10}$$

Since $(1-(\frac{n-1}{n})^{n-1})^{\frac{1}{n-1}} - 1 < 0$, we have

$$\int_0^\infty e^{\varphi^p(t)-t}dt \le \left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{\frac{1}{n-1}}\right]^{-1} + c_3 e^{c_2\delta}$$

for all C^1 function φ which satisfies (4) and it proves the first part of the theorem.

On the other hand, if we rewrite (10) into

$$\begin{aligned}
&\int_0^\infty e^{\varphi^p(t)-t}dt \\
&\leq \left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{\frac{1}{n-1}}\right]^{-1} \\
&\qquad + e^{[(1-(\frac{n-1}{n})^{n-1})^{\frac{1}{n-1}}-1]a}\left\{c_3e^{c_2\delta}-\left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{\frac{1}{n-1}}\right]^{-1}\right\}
\end{aligned}$$

and notice that, when $n=2,3$,

$$0\leq c_3e^{c_2\delta}-\left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{\frac{1}{n-1}}\right]^{-1}$$

for all $0\leq\delta\leq 1$ then we have

$$\int_0^\infty e^{\varphi^p(t)-t}dt\leq c_3e^{c_2\delta}.$$

Thus, we also have

$$\begin{aligned}
A_0 &= \left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{\frac{1}{n-1}}\right]^{-1} \\
A_1 &= n^{\frac{2n}{n+1}}e^{\frac{1}{n+1}(1+\frac{1}{2}+\cdots+\frac{1}{n-1})}e^{-\frac{(n+1)^{n-1}}{(n-1)^2e}(1-(\frac{n-1}{n})^{n-1})^{\frac{n}{n-1}}} \\
A_2 &= \frac{(n+1)^{n-1}}{(n-1)^2e}\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{\frac{1}{n-1}}
\end{aligned}$$

and finish the proof of theorem.

Before we use Lemma 2.2 to prove Lemma 2.3, we rewrite it for function defined on B_2 as follows; let $v\in C_0^1$ be a function defined on the unit ball B_n then for each $c > 0$ we have

$$\frac{1}{m(B_n)} \int_{B_n} e^{cv(x)} dx \leq e^{\frac{(n-1)^{n-1}}{n^{2n-1}} \frac{c^n \beta}{\omega_{n-1}}} e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}} \tag{11}$$

where $\beta = \int_{B_n} |\dot{\nabla} v|^n dx$.

Proof of Lemma 2.3 To estimate the integral $\int_a^\infty e^{\varphi^p(t)-t} dt$ which also can be recognized as the following integral

$$\frac{n}{\omega_{n-1}} \int_{|x| \leq e^{-a/n}} e^{n\omega_{n-1}^{\frac{1}{n-1}} u^{*\frac{n}{n-1}}(x)} dx$$

by (2) and (3), we will use a change of variable and the result of Lemma 2.2.
Set

$$\begin{aligned} y &= xe^{a/n} \\ g(y) &= u^*(x) - n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}} \varphi(a) \end{aligned}$$

then g is a radial function defined on the unit ball B_n with zero boundary value and having the following properties

$$\delta_2 = e^{(n-1)a} \int_{|y| \leq 1} |\nabla g|^n dy \tag{12}$$

and

$$\begin{aligned} g(y) &= n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}} (\varphi(t) - \varphi(a)) \\ &= n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}} \int_a^t \varphi'(s) ds \\ &\leq n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}} \delta_2^{\frac{1}{n}} (t-a)^{\frac{n-1}{n}} \\ &= n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}} \delta_2^{\frac{1}{n}} (-n \ln|y|)^{\frac{n-1}{n}} \end{aligned} \tag{13}$$

for all $y \in B_n$. Thus by using (13), we have

$$\begin{aligned} \int_a^\infty e^{\varphi^p(t)-t} dt &= \frac{n}{\omega_{n-1}} \int_{|x| \leq e^{-a/n}} e^{n\omega_{n-1}^{\frac{1}{n-1}} u^{*\frac{n}{n-1}}(x)} dx \\ &= \frac{n}{\omega_{n-1}} \int_{|y| \leq 1} e^{n\omega_{n-1}^{\frac{1}{n-1}} (g(y) + n^{-\frac{n-1}{n}} \omega_{n-1}^{-\frac{1}{n}} \varphi(a))^{\frac{n}{n-1}}} e^{-a} dy \\ &\leq \frac{n}{\omega_{n-1}} e^{\varphi^p(a)-a} \int_{|y| \leq 1} |y|^{-n\delta_2^{\frac{1}{n-1}}} e^{cg(y)} dy \end{aligned} \tag{14}$$

where $c = n^{\frac{2n-1}{n}} \frac{1}{n-1} \omega_{n-1}^{\frac{1}{n}} \varphi(a)^{\frac{1}{n-1}}$. In the last estimate we used the fact that $(a+b)^{\frac{n}{n-1}} \le a^{\frac{n}{n-1}} + \frac{n}{n-1} a b^{\frac{1}{n-1}} + b^{\frac{n}{n-1}}$ if $a, b > 0$. Notice that since $\delta_2 \le (\frac{n-1}{n})^{n-1}$, we have $|y|^{(n-1)-n\delta_2^{\frac{1}{n-1}}} \le 1$ for all $y \in B_n$. Thus, from (14) by using Holder's inequality we get

$$\begin{aligned}\int_a^\infty e^{\varphi^p(t)-t} dt &\le \frac{n}{\omega_{n-1}} e^{\varphi^p(a)-a} \int_{|y|\le 1} |y|^{-(n-1)} e^{cg(y)} dy \\ &\le n^{\frac{2n+1}{n+1}} \omega_{n-1}^{-\frac{1}{n+1}} e^{\varphi^p(a)-a} \left(\int_{|y|\le 1} e^{c(n+1)g(y)} dy \right)^{\frac{1}{n+1}} .\end{aligned}$$

Note that by (12) we have

$$\int_{|y|\le 1} |\nabla g|^2 dx = \delta_2 e^{(1-n)a},$$

so if we apply Lemma 2.2 (see (11)) we obtain

$$\int_a^\infty e^{\varphi^p(t)-t} dt \le n^{\frac{2n}{n+1}} e^{\varphi^p(a)-a} e^{\frac{1}{n+1}(1+\frac{1}{2}+\cdots+\frac{1}{n-1})} e^{c'}$$

where

$$c' = \frac{(n+1)^{n-1}}{n-1} \varphi(a)^{\frac{n}{n-1}} e^{(1-n)a} \delta_2 .$$

By the property (6) of a and the fact that $xe^{(1-n)x} \le 1/e(n-1)$ for $x \ge 0$,

$$c' \le \frac{(n+1)^{n-1}}{e(n-1)^2} \left(1 - \left(\frac{n-1}{n} \right)^{n-1} \right)^{\frac{1}{n-1}} \delta_2 .$$

Thus,

$$\int_a^\infty e^{\varphi^p(t)-t} dt \le c_1 e^{c_2 \delta_2} e^{[(1-(\frac{n-1}{n})^{n-1})^{\frac{1}{n-1}} - 1]a}$$

where

$$\begin{aligned} c_1 &= n^{\frac{2n}{n+1}} e^{\frac{1}{n+1}(1+\frac{1}{2}+\cdots+\frac{1}{n-1})} \\ c_2 &= \frac{(n+1)^{n-1}}{e(n-1)^2} \left(1 - \left(\frac{n-1}{n} \right)^{n-1} \right)^{\frac{1}{n-1}} .\end{aligned}$$

3. Applications of 2-dimensional results to 4-dimensional estimates for Δ

As an application of the theorem 2.1 (when $n = 2, \Omega = B_2$), we extend the inequality for the gradient on B_2 to the analogue of it for the Laplacian on B_4 under the assumption that our function is radial. We used the property of the Laplacian for the radial function and a change of variable to obtain the following result.

COROLLARY 3.1. *Let $u \in C_0^2(B_4)$ be a radial function with $\int_{B_4} |\Delta u|^2 dx \leq 1$. Then there exist $A_1'(= \pi A_1), A_2$ which depend only on n such that*

$$\frac{1}{m(B_4)} \int_{B_4} e^{24\pi^2 u^2} dx \leq A_1' e^{A_2 \int_{B_4} |\Delta u|^2 dx}.$$

REMARK. In this case (i.e., having assumption about the L_2 norm of Δu), we may not use symmetrization technique as we did in the proofs of the previous theorems. Since in general the relation between $\|\Delta u\|_2$ and $\|\Delta u^*\|_2$ is unknown. Thus, for using the result of the theorem 3.1, we restricted our u to a radial function.

REMARK. In [1], Adams showed an analogue of the Moser inequality for higher-order derivatives. Specifically, when $n = 4$, it takes the following form; let Ω be a bounded domain in $\mathbf{R}^4$, $u \in C_0^2(\Omega)$, $\int_\Omega |\Delta u|^2 dx \leq 1$ then there exists an constant c_0 such that

$$\frac{1}{m(\Omega)} \int_\Omega e^{32\pi^2 u^2} dx \leq c_0.$$

References

[1] D. R. Adams, *A sharp inequality of J. Moser for higher order derivatives,* Ann. of Math. **128** (1988), 385-398.

[2] W. Beckner, *Moser-Trudinger inequality in higher dimensions,* Int. Math. Research Notices **1** (1991), 83-91.

[3] M. S. Berger, *Riemannian structures on prescribed Gaussian curvature for compact 2-manifolds,* J. Diff. Geometry **5** (1971), 325-332.

[4] Branson, Chang, Yang, *Estimates and extremals for zeta function determinants on four-manifolds,* Comm. in Math Physics **149** (1992), 241-262.

[5] L. Carleson, S.-Y. A. Chang, *On the existence of an extremal for an inequality of J. Moser,* Bull. des Science **110** (1986), 113-127.

[6] M. Jodeit, *An inequality for the indefinite integral of a function in L^q*, Studia Math. **44** (1972), 545-554.
[7] J. Moser, *A Sharp form of an Inequality by N. Trudinger,* Indiana Univ. Math. J. **20** (1971), 1077-1092.
[8] ———, *On a nonlinear problem in differential geometry,* Dynamical systems, (M. M. Peixoto, editor), Academic Press, 1973, pp. 273-280.
[9] E. Onofri, *On the positivity of the effective action in a theory of random surfaces,* Commun. Math. Phys. **86** (1982), 321-326.
[10] B. Osgood, R. Phillips, P. Sarnak, *Extremals of determinants of Laplacians,* J.Funct. Anal. **80** (1988), 148-211.
[11] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications,* J. Math. Mech. **17** (1967), 473-483.

Department of Mathematics
Korea University
E-mail: mkim@semi.korea.ac.kr