Bull. Korean Math. Soc. 36 (1999), No. 4, pp. 823–828

SOME MODULES IN CATEGORY $\mathcal{O}^g$ OF GENERALIZED KAC-MOODY ALGEBRAS

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Abstract. We extend the notion of category $\mathcal{O}^g$ for Kac-Moody algebras to generalized Kac-Moody algebras and prove some analogues of results for Kac-Moody algebras.

1. Introduction

The category $\mathcal{O}$ of representations of Kac-Moody algebras and its subcategory $\mathcal{O}^g$ for Kac-Moody algebras were introduced in [2]. The objects in $\mathcal{O}^g$ are modules in $\mathcal{O}$ for which the highest weights of all irreducible subquotients lie in a certain translated cone. One of important properties about these categories is that it is invariant under tensoring with irreducible modules $L(\theta)$, whose highest weight θ is dominant integral. This property has been used to define translation functors and applying these functors to Verma modules it has been proved that the structure of Verma modules depends only on the "chamber" of the Weyl group W to which its highest weight belongs.

In this note we will extend the study of these categories to generalized Kac-Moody algebras and prove some analogues of the results in [2]. We proceed here in manners close to those in [2], but we need some additional arguments due to the presence of simple imaginary roots.

In Section 3 we will prove that the category $\mathcal{O}^g$ for genalized Kac-Moody algebras is stable under tensoring with irreducible modules $L(\theta)$, whose highest weight θ is dominant integral.

We denote by $Z_{>0}$ the set of positive integers and $Z_{\geq 0}$ the set of nonnegative integers.

Received October 2, 1998.

1991 Mathematics Subject Classification: 17B10, 17B65.

Key words and phrases: generalized Kac-Moody algebras, Verma modules, category $\mathcal{O}^g$.

2. Preliminaries

In this section we recall the definitions and some elementary facts about generalized Kac-Moody algebras ([1] and [3]).

A generalized Kac-Moody algebra $G(A)$ (= GKM algebra) is a contragredient Lie algebra over $\mathbb{C}$ associated to a symmetrizable real square matrix $A = (a_{ij})_{i,j,\in I}$ indexed by a finite or countably infinite set I which satisfies the conditions:

(C1) either $a_{ii} = 2$, or $a_{ii} \leq 0$ for $i \in I$;
(C2) $a_{ij} \leq 0$ if $i \neq j$;
(C3) $a_{ij} \in Z$ for $j \in I$ if $a_{ii} = 2$.

When A is a generalized Cartan matrix, $G(A)$ is just a Kac- Moody algebra. Let H be a Cartan subalgebra of G and H^* its algebraic dual. Let Π be the set of simple roots $\{\alpha_i,\ i \in I\}$, Π^v be the set of simple coroots $\{\alpha_i^v,\ i \in I\}$,and I_{re} $(resp.,\ I_{im})$ the subset $\{i \in I | a_{ii} = 2\ (resp.,\ a_{ii} \leq 0)\}$ of the indexing set I. For $\alpha_i \in \Pi$ we call α_i real (resp., imaginary) simple root when $i \in I_{re}$ (resp., I_{im}). Let Δ be the set of all roots, Δ^+ $(resp.,\ \Delta^-)$the set of positive $(resp.,$ negative) roots. We let Δ_{re} $(resp.,\ \Delta_{im})$ be the set of real (resp., imaginary) roots. For any real root α, we denote by r_α the reflection of H^* with respect to α. Recall that the Weyl group W of a GKM algebra $G(A)$ is by definition the subgroup of $GL(H^*)$ generated by reflections r_{α_i}, $i \in I_{re}$. We let D be the diagonal matrix with entries q_i which makes the matrix DA symmetric. As in the case of Kac-Moody algebras there exists a symmetric bilinear form (,) on H^* satisfying the condition: $(\lambda, \alpha_i) = \lambda(q_i \alpha_i^v)$.

We call $\lambda \in H^*$ is integral if $\lambda(\alpha_i^v) \in Z$ for any $i \in I_{re}$. We denote by $P^+ = \{\lambda \in H^* | (\lambda, \alpha_i) \geq 0$ for all $i \in I$ and $\lambda(\alpha_i^v) \in Z_{\geq 0}$ for all $i \in I_{re}\}$ the set of dominant integrals. We put $D = \{\lambda \in H^* | (\lambda, \alpha_i) \geq 0$ for all $i \in I_{re}\}$.

Fix an element ρ in H^* such that $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$. Here, (,) is a nondegenerate bilinear form on H^*. We put $\Delta_{im}^+ = (\Delta_{im} \cap \Delta^+)$ and $\Delta_{im}^- = (\Delta_{im} \cap \Delta^-)$.

The following two propositions are well known [3].

PROPOSITION 2.1.

(1) *All roots are integral.*
(2) *W preserves the set of integral elements.*
(3) *For any simple real root α_i, $\Delta_+ - \{\alpha_i\}$ is r_{α_i} invariant.* *Furthermore:*

$$W(\Delta_{re}) = \Delta_{re}$$
$$W(\Delta_{im}) = \Delta_{im}$$
$$W(\Delta_{im}^+) = \Delta_{im}^+$$
$$W(\Delta_{im}^-) = \Delta_{im}^-.$$

PROPOSITION 2.2. *Let λ be integral, and let U be a $W-$ invariant subset of H^*, all of whose elements are of the form $\lambda - \sum_{i\in I} n_i\alpha_i, n_i \in Z_{>0}$. Then every element of U is conjugate to an element in D.*

COROLLARY 2.3. *Every element of Δ_{im}^- is conjugate to an element in D.*

Proof. Put $\lambda = 0, U = \Delta_{im}^-$. Apply Proposition 2.2 and Proposition 2.1. □

We now recall from [2] or [4, chap. 9] the definition of the category $\mathcal{O}$ whose objects are $G(A)$ modules M satisfying:

(1) M is $H-$semisimple with finite dimensional weight spaces.
(2) There exist finitely many elements $\mu_1, \mu_2, \cdots \mu_k \in H^*$ such that any weight of M (μ is a weight iff the weight space $M_\mu \neq 0$) belongs to some $D(\mu_i)$, where $D(\mu_i) = \{\mu_i - \gamma | \gamma \in Q_+ = \sum Z_{\geq 0}\alpha_i\}$.

An important class of modules in $\mathcal{O}$ is the class of highest modules, in particular Verma modules. We denote by $M(\lambda)$ the Verma modules with highest weight λ. Any Verma module $M(\lambda)$ has a unique irreducible quotient, we denote it by $L(\lambda)$.By the same argument in [2] one has the following propositions.

PROPOSITION 2.4 ([2, Proposition 3.2]). *Let $M \in \mathcal{O}$ and $\lambda \in H^*$. Then there exists an increasing filtration $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$ of submodules of M and a subset J of $\{1, \cdots, t\}$ such that*

(i) *For $j \in J, M_j/M_{j-1} \simeq L(\lambda_j)$ for some $\lambda_j \geq \lambda$ and*

(ii) *for* $j \notin J$ *and any* $\mu \geq \lambda, (M_j/M_{j-1})_\mu = (0)$.

We call such a filtration a *local composition series* of M at λ and refer to $\{M_j/M_{j-1}\}_{j\in J}$ as the irreducible subquotients occurring in it. We call these subquotients the components of M. The following proposition describes the components of Verma modules.

PROPOSITION 2.5 ([2, Theorem 3.6]). *Let* $\lambda, \mu \in H^*$. *Then* $L(\mu)$ *is a component of* $M(\lambda)$ *iff the ordered pair* $\{\lambda, \mu\}$ *has the following condition:*

(*) There exist a sequence $\phi_1, \phi_2, \cdots, \phi_k$ of positive roots and a sequence $n_1, n_2, \cdots, n_k$ of positive integers such that
(i) $\lambda - \mu = \sum_{i=1}^{i=k} n_i\phi_i$,
(ii) $2(\lambda + \rho - n_1\phi_1 - \cdots - n_{j-1}\phi_{j-1}, \phi_j) = n_j(\phi_j, \phi_j), \quad \forall\, 1 \leq j \leq k.$

3. Subcategory $\mathcal{O}^g$

In this section we extend the notion of the category $\mathcal{O}^g$ for Kac-Moody algebras to arbitrary symmetrizable GKM algebras and prove the category $\mathcal{O}^g$ is stable under tensoring with irreducible highest weight module whose highest weight is dominant integral.

DEFINITION 3.1. Let $C = \{\lambda \in H^* | (\lambda, \alpha_i) \geq 0,$ for all simple roots $\alpha_i\}$. Put $K = \cup_{w\in W} w(C)$.

Now we set $K^g = -\rho + K, C^g = -\rho + C$.

PROPOSITION 3.2. (1) *K is a convex $W-$invariant cone, consisting of the elements λ in H^* satisfying the followings:*
(1) *the set $\{\alpha \in \Delta^+ | (\lambda, \alpha) < 0\}$ is finite, and $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta^+_{im}$.*
(2) *Every orbit of W in K contains a unique element of C.*

Proof. Let X' be the set of all elements satisfying the properties in (1). Clearly $K \subset X'$. Conversely let $\lambda \in X'$. Then the set $\Phi_\lambda = \{\alpha \in \Delta^+ | (\lambda, \alpha) < 0\}$ is finite and $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta^+_{im}$. When $\Phi_\lambda = \emptyset, (\lambda, \alpha) \geq 0$ for any $\alpha \in \Delta^+$. In particular $(\lambda, \alpha_i) \geq 0$ for all simple roots α_i. Hence, $\lambda \in C \subset K$.

When $\Phi_\lambda \neq \emptyset$, there is a positive root α such that $(\lambda, \alpha) < 0$. By definition of X', α is a positive real root, put $\alpha = \sum k_i \alpha_i$, where $k_i \in Z_{\geq 0}$ and α_i's are simple roots. Since $\lambda \in X'$, we have $(\lambda, \alpha_i) \geq 0$ for all simple imaginary root. Thus there exists some simple real root α_i such that $(\lambda, \alpha_i) < 0$. By Proposition 2.1 we have $|\Phi_{r_i(\lambda)}| < |\Phi_\lambda|$. We use induction on $|\Phi_\lambda|$ to complete the proof.

For (2) one modifies the proof of Φ_λ in [4, Proposition 3.12] slightly.□

We define the category $\mathcal{O}^g$ as follows: The object of $\mathcal{O}^g$ are those modules $M \in \mathcal{O}$ all of whose components have highest weights in K^g.

LEMMA 3.3. *Let $\lambda \in K^g$ and $\{\lambda, \mu\}$ satisfy the condition $(*)$. Then $\mu \in K^g$.*

Proof. The definition of K^g implies that $\lambda + \rho \in K$. Referring to Proposition 2.1 we may assume there exists $\beta \in \Delta^+$ such that $\lambda - \mu = n\beta$ and $2(\lambda+\rho, \beta) = n(\beta, \beta)$ for some positive integer n. When $(\beta, \beta) \geq 0$, β is real and $\mu+\rho = r_\beta(\lambda+\rho) \in K$. Hence $\mu \in K^g$. When $(\beta, \beta) < 0$, β is a positive imaginary root. By Proposition 2.2, $-\beta$ is conjugate to an element $\sigma(-\beta)$ in D, i.e., $(\sigma(-\beta), \alpha_i) \geq 0$ for all $i \in I_{re}$. We want to show $(\sigma(-\beta), \alpha_j) \geq 0$ for all $j \in I_{im}$. Proposition 2.1 implies $\sigma(-\beta) \in \Delta_{im}^-$. We put $\sigma(-\beta) = -\sum_{i \in I} n_i \alpha_i$, where each $n_i \in Z_{>0}$. By definition of (,) for any $j \in I_{im}$, we have $(\alpha_j, \alpha_j) \leq 0$ and $(\alpha_i, \alpha_j) \leq 0$ for $i \neq j$. Hence $\sigma(-\beta, \alpha_j)) = -\sum_{i \neq j} n_i(\alpha_i, \alpha_j) - (\alpha_j, \alpha_j) \geq 0$. This shows $-\beta \in K$. This with Proposition 3.2 implies that $\mu+\rho = \lambda+\rho-n\beta \in K$. Hence, $\mu \in K^g$. □

REMARK. It is immediate from Lemma 3.3 and Proposition 3.2 that if $\lambda \in K^g$ then for any component $L(\mu)$ of $M(\lambda)$ we have $\mu \in K^g$. In particular, $M(\lambda) \in \mathcal{O}^g$.

The following can be obtained immediately from Proposition 2.1.

COROLLARY 3.4. *Let $\lambda \in K^g$ and $\{\lambda, \mu\}$ satisfy the condition (*). Then there exist positive imaginary roots $\beta_1, \beta_2, \cdots, \beta_k$ such that $\mu + \rho = \sigma(\lambda + \rho - n_1\beta_1 + n_2\beta_2 + \cdots + n_k\beta_k)$ for some $\sigma \in W$ and some n_i's $\in Z_{\geq 0}$.*

LEMMA 3.5. *Let θ be a dominant integral weight. Then for any $\lambda \in K^g$ and any weight θ_i of $L(\theta)$ we have $\lambda + \theta_i \in K^g$.*

Proof. Since θ is dominant integral, $(\theta, \alpha_i) \geq 0$ for all $i \in I$ and $(\theta, \alpha_i) \in Z$ when $i \in I_{re}$. Let U be the set of weights of the irreducible highest module $L(\theta)$. Note that U is W- invariant and all of its elements are of the form $\theta - \sum_{i \in I} n_i \alpha_i$. By Proposition 2.2, θ_i being an element of U is conjugate to an element ξ in D. Also, since $\xi \in U$, we put $\xi = \theta - \sum_{i \in I} n_i \alpha_i$. Now it follows from the fact that θ is dominant that $(\xi, \alpha_j) \geq 0$ for all $j \in I_{im}$. Hence $\xi \in C$ and $\theta_i \in W(C) = K$. Since K is convex, $\lambda + \theta_i \in K^g$. □

PROPOSITION 3.6. *If $M \in \mathcal{O}^g$ then $M \otimes L(\theta) \in \mathcal{O}^g$.*

Proof. It is clear that $M \otimes L(\theta) \in \mathcal{O}$. Let λ be the highest weight of a component of $M \otimes L(\theta)$. We have to show that λ is in K^g. Following the proof of Proposition 5.9 in [2], one gets a local composition series of $M \otimes L(\theta)$ at λ such that its subquotients having a weight $\nu \geq \lambda$ are irreducible subquotients of $M(\lambda_j + \theta_i)$ for some $\lambda_j \in K^g$ and weight θ_i of $L(\theta)$. By Lemma 3.4, $\lambda_j + \theta_i \in K^g$. Hence by the remark, $M(\lambda_j + \theta_i) \in \mathcal{O}^g$. □

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