Bull. Korean Math. Soc. **36** (1999), No. 4, pp. 809–822

EVALUATION FORMULAS OF CONDITIONAL YEH-WIENER INTEGRALS

J. M. Ahn, K. S. Chang, S. K. Kim and I. Yoo

Abstract. In this paper, we introduce conditional Yeh-Wiener integrals for generalized conditioning functions including vector-valued functions. And also we establish various evaluation formulas of conditional Yeh-Wiener integrals for generalized conditioning functions.

1. Introduction

J. Yeh [15] introduced the concept of conditional Wiener integrals $E[F|X]$ of F given X as a function on the value space of X and derived a Fourier transform inversion formula for computing conditional Wiener integrals. Using this formula, he obtained some very useful results including a Kac-Feynman integral equation and a Cameron-Martin translation theorem [15,16]. Using Yeh's inversion formula, Chang and Chang [4] evaluated the conditional Wiener integral of F given $X(x) = (x(t_1), \cdots, x(t_n))$ where $0 = t_0 < t_1 < t_2 < \cdots < t_n \leqq T$. And Chang [3] also extended the above results to those for conditional Yeh-Wiener integral of F given $X(x) = (x(s_1, t_1), \cdots, x(s_m, t_n))$ where $0 = s_0 < s_1 < \cdots < s_m \leqq S$ and $0 = t_0 < t_1 < \cdots < t_n \leqq T$. But Yeh's inversion formulas for conditional Wiener and Yeh-Wiener integrals were very complicated to apply when conditioning functions were vector-valued.

In [7], Park and Skoug introduced a very simple formula for the conditional Wiener integral of F given $X(x) = (x(t_1), \cdots, x(t_n))$. In particular, they expressed the conditional Wiener integral directly in terms of an ordinary Wiener integral. Also using this formula, they

Received March 8, 1999.
1991 Mathematics Subject Classification: 28C20.
Key words and phrases: conditional Yeh-Wiener integral, stochastic integral.

generalized the Kac-Feynman formula and obtained a conditional version of Cameron-Martin translation theorem involving vector-valued conditioning functions. Moreover they extended the conditional Wiener integral theory in [7] to the conditional Yeh-Wiener integral theory with vector-valued conditioning functions [8].

The purpose of this paper is to develop an useful formula to convert conditional Yeh-Wiener integrals of generalized conditioning functions into ordinary Yeh-Wiener integrals and then to obtain various evaluation formulas of conditional Yeh-Wiener integrals for these generalized conditioning functions.

2. Preliminaries

Let $(C_2(Q), \mathcal{Y}, m_y)$ denote Yeh-Wiener space where $C_2(Q)$ is the space of all continuous functions x on $Q = [0,S] \times [0,T]$ such that $x(0,t) = x(s,0) = 0$ for every $(s,t) \in Q$.

Let $\{\alpha_{ij}\}$ be a complete orthonormal basis for $L_2(Q)$. Now we define the corresponding stochastic integrals

$$\gamma_{ij}(x) = \int_Q \alpha_{ij}(s,t)\, \tilde{d}x(s,t)$$

for $i = 1, 2, \cdots$, and $j = 1, 2, \cdots$ where $\int_Q \alpha_{ij}(s,t)\, \tilde{d}x(s,t)$ denotes the Paley-Wiener-Zygmund integral discussed in [2,4]. Then $\{\gamma_{ij}\}$ forms a set of independent standard Gaussian random variables on $C_2(Q)$ with

$$E[x(s,t)\gamma_{ij}(x)] = \int_0^s \int_0^t \alpha_{ij}(u,v)\, dudv \equiv \beta_{ij}(s,t). \tag{2.1}$$

For each m and $n \in \mathbb{N}$, let $X_{mn} : C_2(Q) \longrightarrow \mathbb{R}^{mn}$ be the conditioning function defined by

$$X_{mn}(x) = (\gamma_{11}(x), \cdots, \gamma_{1n}(x), \cdots, \gamma_{m1}(x), \cdots, \gamma_{mn}(x)). \tag{2.2}$$

Let $\mathcal{F}^*{}_{mn} = \{X_{mn}{}^{-1}(B) | B \in \mathcal{B}^{mn}\}$ where $\mathcal{B}^{mn}$ denotes the $\sigma-$algebra of Borel sets in $\mathbb{R}^{mn}$. Then by the definition of conditional expections (see Doob [6], Tucker [10] and Yeh [13]), for each $F \in L_1(C_2(Q), m_y)$,

$$\begin{aligned}(2.3)\qquad \mu(B) &\equiv \int_{X_{mn}^{-1}(B)} F(x) m_y(dx)\\ &= \int_{X_{mn}^{-1}(B)} E[F|\mathcal{F}^*{}_{mn}] m_y(dx)\\ &= \int_B E[F(x)|X_{mn}(x) = \vec{\xi}] P_{X_{mn}}(d\vec{\xi})\end{aligned}$$

where $P_{X_{mn}}(B) \equiv m_y(X_{mn}^{-1}(B))$ and $E[F(X)|X_{mn}(x) = \vec{\xi}]$ is a Lebesgue measurable function of $\vec{\xi}$ which is unique up to null sets in $\mathbb{R}^{mn}$.

Let $X_k{}^*$ be the conditioning function defined by

$$(2.4)\qquad X_k{}^*(x) = (\gamma_{11}(x), \cdots, \gamma_{1\,k-1}(x), \gamma_{2\,k-2}(x), \cdots \gamma_{k-1\,1}(x))$$

for $k = 2, 3, \cdots$ and let $X_\infty : C_2(Q) \to \mathbb{R}^\infty$ be defined by

$$(2.5)\qquad X_\infty(x) = (\gamma_{11}(x), \gamma_{12}(x), \gamma_{21}(x), \gamma_{13}(x), \gamma_{22}(x), \gamma_{31}(x), \cdots).$$

Then $X_k{}^*$ is equal to X_{mn} if $mn = \frac{k(k-1)}{2}$. Let $\mathcal{F}_k{}^*$ be the σ-algebra generated by the sets $\{X_k{}^{*-1}(B)|B \in \mathcal{B}^{\frac{k(k-1)}{2}}\}$ and let $\mathcal{F}^*$ be the σ-algebra generated by $\bigcup_{k=1}^\infty \mathcal{F}_k{}^*$. Then $\{\mathcal{F}_k{}^*\}$ is an increasing sequence of σ-algebras of Yeh-Wiener measurable sets, hence for Yeh-Wiener integrable function F, $\{E\big[F|\mathcal{F}_k{}^*\big]\}$ is an martingale sequence. Thus $E|E[F|\mathcal{F}_k{}^*]| \le E[|F||\mathcal{F}_k{}^*] \le E|F|$, for every k, and so by the martingale convergence theorem,

$$(2.6)\qquad \lim_{k\to\infty} E[F|\mathcal{F}_k{}^*] = E[F|\mathcal{F}^*]$$

almost surely and for every $A \in \bigcup_{k=2}^\infty \mathcal{F}_k{}^*$,

$$(2.7)\qquad \lim_{k\to\infty} \int_A E\big[F(x)|\mathcal{F}_k{}^*\big] m_y(dx) = \int_A E[F(x)|\mathcal{F}^*] m_y(dx).$$

From this and definition of conditional expectations, it follows that for every Borel set B in $\mathbb{R}^n$

(2.8)
$$\int_B E[F(x)|\gamma_{ij}(x)=\xi_{ij}\, i=1,2,\cdots\, j=1,2,\cdots]P_{X_\infty}(d\vec{\xi})$$
$$=\lim_{k\to\infty}\int_B E[F(x)|\gamma_{ij}(x)=\xi_{ij}\ \ i+j\le k]P_{X_{k^*}}(d\vec{\xi}),$$

where

(2.9)
$$P_{X_{k^*}}(d\vec{\xi})=\prod_{l=2}^{k}\prod_{i+j=l}\left\{(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{{\xi_{ij}}^2}{2}\right)d\xi_{ij}\right\},$$
$$P_{X_\infty}(d\vec{\xi})=\prod_{l=2}^{\infty}\prod_{i+j=l}\left\{(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{{\xi_{ij}}^2}{2}\right)d\xi_{ij}\right\}.$$

3. A Simple Fomular for Conditional Yeh-Wiener Integrals

Let $\{\alpha_{ij}\}$ and $\{\gamma_{ij}(x)\}$ be as in Section 2. Now we define the projection map $P_{(mn)}$ from $L_2(Q)$ into $L_2(Q)$ by

(3.1)
$$P_{(mn)}h(s,t)=\sum_{i=1}^{m}\sum_{j=1}^{n}(h,\alpha_{ij})\alpha_{ij}(s,t).$$

For $x\in C_2(Q)$ and $\vec{\xi}=(\xi_{11},\cdots,\xi_{1n},\cdots,\xi_{m1},\cdots,\xi_{mn})$, let

(3.2)
$$x_{(mn)}(s,t)=\int_Q P_{(mn)}I_{[0,s]\times[0,t]}(u,v)\,\tilde{d}x(u,v)$$
$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\gamma_{ij}(x)\int_0^s\int_0^t\alpha_{ij}(u,v)\,dudv$$

and

$$\vec{\xi}_{(mn)}(s,t)=\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}\int_{[0,s]\times[0,t]}\alpha_{ij}(u,v)\,dudv$$

where $I_{[0,s]\times[0,t]}$ is the indicator function of $[0,s]\times[0,t]$ for $(s,t)\in Q$. Similarly, we define the projection map $P_\infty : L_2(Q)\to L_2(Q)$ by

$$P_\infty h(s,t)=\sum_{k=2}^{\infty}\sum_{i+j=k}(h,\alpha_{ij})\alpha_{ij}(s,t).$$

For $x\in C_2(Q)$ and $\vec{\xi}=(\xi_{11},\xi_{12},\xi_{21},\xi_{13},\xi_{22},\xi_{31},\cdots)$, let

$$\begin{aligned} x_\infty(s,t)&=\int_Q P_\infty\, I_{[0,s]\times[0,t]}(u,v)\,dx(u,v) \\ &=\sum_{k=2}^{\infty}\sum_{i+j=k}\gamma_{ij}(x)\int_0^s\int_0^t\alpha_{ij}(u,v)\,dudv \end{aligned} \tag{3.3}$$

and

$$\vec{\xi}_\infty(s,t)=\sum_{k=1}^{\infty}\sum_{i+j=k}\xi_{ij}\int_0^s\int_0^t\alpha_{ij}(u,v)\,dudv.$$

THEOREM 3.1. *If $\{x(s,t)|(s,t)\in Q\}$ is the standard Yeh-Wiener process, then the processes $\{x(s,t)-x_\infty(s,t)|(s,t)\in Q\}$ and $\gamma_{ij}(x)$ are independent for $i=1,\cdots$ $j=1,\cdots$. Also, $\{x(s,t)-x_{(mn)}(s,t)|(s,t)\in Q\}$ and $\gamma_{ij}(x)$ are independent for $i=1,\cdots,m$ and $j=1,\cdots,n$.*

Proof. For each i and j, we know that $E[\gamma_{ij}(x)(x(s,t)-x_\infty(s,t))]=0$. Since both $\gamma_{ij}(x)$ and $x(s,t)-x_\infty(s,t)$ are Gaussian, it follows that they are independent. The second argument follows in the same manner. □

COROLLARY 3.2. *The processes $\{x(s,t)-x_\infty(s,t)|(s,t)\in Q\}$ and $\{x_\infty(s,t)|\ (s,t)\in Q\}$ are independent, and so are $\{x(s,t)-x_{(mn)}(s,t)|\ (s,t)\in Q\}$ and $\{x_{(mn)}(s,t)|\ (s,t)\in Q\}$.*

The following theorem plays an important role in the development of a simple formula for conditional Yeh-Wiener integrals of generalized conditioning functions.

THEOREM 3.3. *Let F be an Yeh-Wiener integrable function. Then*

$$(3.4)\qquad \begin{aligned} &E[F(x)|\gamma_{ij}(x)=\xi_{ij},\ i=1,2,\cdots,\ j=1,2,\cdots] \\ &= E[F(x-x_\infty+\vec{\xi}_\infty)], \text{ and} \\ &E[F(x)|\gamma_{ij}(x)=\xi_{ij},\ i=1,\cdots,m,\ j=1,\cdots,n] \\ &= E[F(x-x_{(mn)}+\vec{\xi}_{(mn)})]. \end{aligned}$$

Proof. Using the results of Theorem 3.1 and Corollary 3.2, we have

$$\begin{aligned} &E[F(x)|\gamma_{ij}(x)=\xi_{ij},\ i=1,2,\cdots,\ j=1,2,\cdots] \\ &= E[F(x-x_\infty+x_\infty)|\gamma_{ij}(x)=\xi_{ij},\ i=1,2,\cdots,\ j=1,2,\cdots] \\ &= E_y\{F(y-y_\infty+\vec{\xi}_\infty\} = E[F(x-x_\infty+\vec{\xi}_\infty)]. \end{aligned}$$

Also the second formula of (3.4) follows by the same reasoning. □

COROLLARY 3.4. *Let F be an Yeh-Wiener integrable function. Then*

$$E[F(x)|\gamma_{ij}(x)=\xi_{ij},\ i=1,2,\cdots,\ j=1,2,\cdots] = F(\vec{\xi}_\infty).$$

The following corollary is an immediate consequence of the second formula in Theorem 3.3.

COROLLARY 3.5. *If F is an Yeh-Wiener integrable function, then for every Borel measurable set B in $\mathbb{R}^{mn}$,*

$$(3.5)\quad \int_{X_{mn}{}^{-1}(B)} F(x)m_y(dx) = \int_B E[F(x-x_{(mn)}+\vec{\xi}_{(mn)})]P_{X_{mn}}(d\vec{\xi}).$$

REMARK 3.6. For each partition $\tau\equiv\tau_{mn}=\{(s_1,t_1),\cdots,(s_m,t_n)\}$ with $0=s_0<s_1<\cdots<s_m=S$ and $0=t_0<t_1<\cdots<t_n=T$, let $X_\tau : C_2(Q)\to\mathbb{R}^{mn}$ be defined by $X_\tau(x)=(x(s_1,t_1),\cdots,x(s_m,t_n))$. In [8] they considered vector-valued conditional Yeh-Wiener integrals of the type $E[F(x)|X_\tau(x)=\vec{\xi}]$ for Yeh-Wiener integrable function F.

We next find the relationship between the ordinary conditioning function X_τ and the generalized conditioning function X_{mn}. If we define $\alpha_{ij}(s,t) = \frac{I_{[s_{i-1},s_i]\times[t_{j-1},t_j]}(s,t)}{\sqrt{(s_i-s_{i-1})\times(t_j-t_{j-1})}}$, then

$$E[F(x)|X_\tau(x)=\vec{\xi}] = E\Bigg[F(x)|X_{mn}(x) = \frac{\xi_{ij}-\xi_{i-1\,j}-\xi_{i\,j-1}+\xi_{i-1\,j-1}}{\sqrt{(s_i-s_{i-1})\times(t_j-t_{j-1})}},\ i=1,\cdots,m,\ j=1,\cdots,n\Bigg]$$

where $\xi_{0j}=\xi_{i0}=0$ for $i=1,\cdots,m$ and $j=1,\cdots,n$.

4. Evaluation Formulas of Conditional Yeh-Wiener Integrals

In this section we establish various evaluation formulas of conditional Yeh-Wiener integrals for generalized conditioning functions.

Using the same notation as in Section 3, for $h \in L_2(Q)$, let

$$h_{(mn)}(s,t) = P_{(mn)}h(s,t) = \sum_{i=1}^{m}\sum_{j=1}^{n}(h,\alpha_{ij})\alpha_{ij}(s,t), \tag{4.1}$$

and

$$h_{(\infty)}(s,t) = P_\infty h(s,t) = \sum_{k=2}^{\infty}\sum_{i+j=k}(h,\alpha_{ij})\alpha_{ij}(s,t).$$

The following propositions give interesting relationships involving h, $h_{(mn)}$, x and $x_{(mn)}$ that are very useful in computing conditional and ordinary expectations of functions including the stochastic integral $\int_Q h(s,t)\,\tilde{d}x_{(mn)}(s,t)$.

PROPOSITION 4.1. *Let $h \in L_2(Q)$. Then*

$$\int_Q h(s,t)h_{(mn)}(s,t)\,dsdt = \int_Q {h_{(mn)}}^2(s,t)\,dsdt, \tag{4.2}$$

and also the above formula holds when $mn=\infty$.

PROPOSITION 4.2. *Let $h \in L_2(Q)$. Then for each $x \in C_2(Q)$*

$$\int_Q h(s,t)\,\tilde{d}x_{(mn)}(s,t) = \int_Q h_{(mn)}(s,t)\,\tilde{d}x(s,t) \tag{4.3}$$
$$= \int_Q h_{(mn)}(s,t)\,\tilde{d}x_{(mn)}(s,t),$$

and the above formula also holds for $mn = \infty$ if we consider $\int_Q h(s,t)$ $\tilde{d}x_\infty(s,t) = \sum_{k=2}^{\infty} \sum_{i+j=k} \gamma_{ij}(x)(h, \alpha_{ij})$.

Proof. It follows from the fact that the above integrals have the same value $\sum_{i=1}^{m} \sum_{j=1}^{n} \gamma_{ij}(x)(h, \alpha_{ij})$. □

THEOREM 4.3. *Let $h \in L_2(Q)$, and assume that*

$$F(x) = f\left[\int_Q h(s,t)\,\tilde{d}x(s,t)\right]$$

is in $L_1(C_2(Q), m_y)$.
(i) *If $\{h, \alpha_{11}, \cdots, \alpha_{mn}\}$ is a linearly dependent set of functions in $L_2(Q)$, say $h(s,t) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\alpha_{ij}(s,t)$ on Q, then*

$$E\left[f\left[\int_Q h(s,t)\,\tilde{d}x(s,t)\right] \Big| X_{mn}(x) = \vec{\xi}\right] = f\left(\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\xi_{ij}\right). \tag{4.4}$$

(ii) *If $\{h, \alpha_{11}, \cdots, \alpha_{mn}\}$ is a linearly independent set of functions in $L_2(Q)$, then*

$$E\left[f\left[\int_Q h(s,t)\,\tilde{d}x(s,t)\right] \Big| X_{mn}(x) = \vec{\xi}\right] \tag{4.5}$$
$$= \left[2\pi\left(\|h\|^2 - \|h_{(mn)}\|^2\right)\right]^{-\frac{1}{2}}$$
$$\times \int_{\mathbb{R}} f(u) \exp\left\{-\frac{\left(u - \int_Q h(s,t)\,d\vec{\xi}_{(mn)}(s,t)\right)^2}{2\|h - h_{(mn)}\|^2}\right\} du.$$

Proof. (i). Using Theorem 3.3 and Proposition 3.2, we have

$$E\left[f\left[\int_Q h(s,t)\,\tilde{d}x(s,t)\right]|X_{mn}(x)=\vec{\xi}\right]$$
$$=E\left[f\left[\int_Q h(s,t)\,\{\tilde{d}(x-x_{(mn)}+\xi_{(mn)})\}(s,t)\right]\right]$$
$$=E\left[f\left[\int_Q\Big(h(s,t)-h_{(mn)}(s,t)\Big)\,\tilde{d}x(s,t)+\int_Q h(s,t)\,d\vec{\xi}_{(mn)}(s,t)\right]\right]$$
$$=f\left[\int_Q h(s,t)\left\{d\left(\sum_{i=1}^m\sum_{j=1}^n\xi_{ij}\int_0^s\int_0^t\alpha_{ij}(u,v)\,dudv\right)\right\}(s,t)\right]$$
$$=f\left(\sum_{i=1}^m\sum_{j=1}^n\xi_{ij}c_{ij}\right).$$

(ii). In this case, we use (3.4),(4.3) and a well known Yeh-Wiener integration formula to obtain

$$E\left[f\left[\int_Q h(s,t)\,\tilde{d}x(s,t)\right]|X_{mn}(x)=\vec{\xi}\right]$$
$$=E\left[f\left[\int_Q(h(s,t)-h_{(mn)}(s,t))\,\tilde{d}x(s,t)+\int_Q h(s,t)\,d\vec{\xi}_{(mn)}(s,t)\right]\right]$$
$$=(2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}}f\left(\|h-h_{(mn)}\|v+\int_Q h(s,t)\,d\vec{\xi}_{(mn)}(s,t)\right)\exp\left\{-\frac{v^2}{2}\right\}dv$$
$$=\left[2\pi\left(\|h-h_{(mn)}\|^2\right)\right]^{-\frac{1}{2}}$$
$$\times\int_{\mathbb{R}}f(u)\exp\left\{-\frac{\left(u-\int_Q h(s,t)d\vec{\xi}_{(mn)}(s,t)\right)^2}{2\|h-h_{(mn)}\|^2}\right\}du.\qquad\square$$

The following two corollaries are the special cases of Theorem 4.3 when $h\equiv\alpha_{ij}$ for some i and j or when h is orthogonal to all the α_{ij}'s.

COROLLARY 4.4. *Let h, F and f be as in* Theorem 4.3. *Then*

$$E\left[f\left[\int_Q\alpha_{ij}(s,t)\,\tilde{d}x(s,t)\right]|X_{mn}(x)=\vec{\xi}\right]=f(\xi_{ij}),\tag{4.6}$$

while if $\{h, \alpha_{11}, \cdots, \alpha_{mn}\}$ *is an orthogonal set of functions in* $L_2(Q)$,

$$\begin{aligned}
(4.7) \qquad & E\left[f\left[\int_Q h(s,t)\,\tilde{d}x(s,t)\right]\Big|X_{mn}(x)=\vec{\xi}\right] \\
&= (2\pi\|h\|^2)^{-\frac{1}{2}}\int_{\mathbb{R}} f(u)\exp\left\{-\frac{u^2}{2\|h\|^2}\right\}du \\
&= E\left[f\left[\int_Q h(s,t)\,\tilde{d}x(s,t)\right]\right].
\end{aligned}$$

Proceeding as above, we obtain the following formula which is a generalization of (4.7).

COROLLARY 4.5. *If* $\{\phi_{11}, \cdots, \phi_{kl}, \alpha_{11}, \cdots, \alpha_{mn}\}$ *is an orthonormal set of functions in* $L_2(Q)$ *and*

$$F(x) = f\left[\int_Q \phi_{11}(s,t)\,\tilde{d}x(s,t), \cdots, \int_Q \phi_{kl}(s,t)\,\tilde{d}x(s,t)\right]$$

is an Yeh-Wiener integrable function, then

$$\begin{aligned}
&E\left[f\left[\int_Q \phi_{11}(s,t)\,\tilde{d}x(s,t), \cdots, \int_Q \phi_{kl}(s,t)\,\tilde{d}x(s,t)\right]\Big|X_{mn}(x)=\vec{\xi}\right] \\
&= E\left[f\left[\int_Q \phi(s,t)\,\tilde{d}x(s,t), \cdots, \int_Q \phi_{kl}(s,t)\,\tilde{d}x(s,t)\right]\right] \\
&= (2\pi)^{-\frac{kl}{2}}\int_{\mathbb{R}^{kl}} f(u_{11}, \cdots, u_{kl})\exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{{u_{ij}}^2}{2}\right\}d\vec{u}
\end{aligned}$$

where $\vec{u} = (u_{11}, \cdots, u_{kl})$.

Our next corollary follows from the fact that $\int_Q (h(s,t) - h_{(mn)}(s,t))\, d\vec{\xi}_{(mn)}(s,t) = 0$ and $(h - h_{(mn)})_{(mn)}(s,t) = 0$.

COROLLARY 4.6. *Let h, F and f be as in* Theorem 4.2. *Then*

$$E\left[f\left[\int_Q h(s,t)\,\tilde{d}(x(s,t)-x_{(mn)}(s,t))\right]\Big|X_{mn}(x)=\vec{\xi}\right]$$
$$=E\left[f\left[\int_Q (h(s,t)-h_{(mn)}(s,t))\,\tilde{d}x(s,t)\right]\right]$$
$$=\left[2\pi\|h-h_{(mn)}\|^2\right]^{-\frac{1}{2}}\int_{\mathbb{R}} f(u)\exp\left\{-\frac{u^2}{2\|h-h_{(mn)}\|^2}\right\}du.$$

Many interesting examples of conditional Yeh-Wiener integrals are obtained as special cases of the following theorem.

THEOREM 4.7. *Let $g \in L_2(Q)$. Then*

$$E\left[\exp\left\{\int_Q g(s,t)x(s,t)\,dsdt\right\}\Big|X_{mn}(x)=\vec{\xi}\right]$$
$$=\exp\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}(g,\beta_{ij})+\frac{1}{2}\int_Q\left[\int_t^T\int_s^S g(u,v)\,dudv\right]^2 dsdt\right.$$
$$\left.-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}(g,\beta_{ij})^2\right\}.$$

Proof. Using the integration by parts formula, it follows that

$$\int_Q g(s,t)x(s,t)\,dsdt=\int_Q\left[\int_t^T\int_s^S g(u,v)\,dudv\right]\tilde{d}x(s,t)$$

and that

$$\int_Q\left[\int_t^T\int_s^S g(u,v)\,dudv\right]\alpha_{ij}(s,t)\,dsdt$$
$$=\int_Q g(s,t)\beta_{ij}(s,t)\,dsdt=(g,\beta_{ij}).$$

Hence using Theorem 3.3, we obtain

$$
\begin{aligned}
&E\left[\exp\left\{\int_Q g(s,t)x(s,t)\,dsdt\right\}|X_{mn}(x)=\vec{\xi}\right]\\
&=E\left[\exp\left[\int_Q\left[\int_t^T\int_s^S g(u,v)\,dudv\right]\{\tilde{d}(x-x_{(mn)}-\vec{\xi}_{(mn)})\}(s,t)\right.\right.\\
&=\exp\left\{\sum_{i=1}^m\sum_{j=1}^n \xi_{ij}\left[\int_Q\left[\int_t^T\int_s^S g(u,v)\,dudv\right]\alpha_{ij}(s,t)\,dsdt\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\exp\int_Q\left[\int_t^T\int_s^S g(u,v)\,dudv\ \tilde{d}x(s,t)\right.\right.\\
&\qquad-\sum_{i=1}^m\sum_{j=1}^n\gamma_{ij}(x)\left[\int_Q\left[\int_t^T\int_s^S g(u,v)\,dudv\right]\alpha_{ij}(s,t)\,dsdt\right]\Bigg]\Bigg]\\
&=\exp\left\{\sum_{i=1}^m\sum_{j=1}^n\xi_{ij}(g,\beta_{ij})+\frac{1}{2}\int_Q\left[\int_t^T\int_s^S g(u,v)\,dudv\right]^2 dsdt\right.\\
&\qquad\left.-\frac{1}{2}\sum_{i=1}^m\sum_{j=1}^n(g,\beta_{ij})^2\right\}.
\end{aligned}
$$

□

The following corollary is a special case of Theorem 4.7.

COROLLARY 4.8. *Let* $g(s,t)=1$ *and* $\alpha_{ij}(u,v)=\frac{I_{[s_{i-1},s_i]\times[t_{j-1},t_j]}(u,v)}{\sqrt{(s_i-s_{i-1})(t_j-t_{j-1})}}$. *Then*

$$
\begin{aligned}
&E\left[\exp\left\{\int_Q x(s,t)\,dsdt\right\}|X_{mn}(x)=\vec{\xi}\right]\\
&=\exp\left\{\sum_{i=1}^m\sum_{j=1}^n\xi_{ij}\sqrt{(s_i-s_{i-1})(t_j-t_{j-1})}\right.\\
&\qquad\left(S-\frac{1}{2}s_i-\frac{1}{2}s_{i-1}\right)\left(T-\frac{1}{2}t_j-t_{j-1}\right)+\frac{1}{18}S^3T^3\\
&\qquad\left.-\frac{1}{32}\sum_{i=1}^m\sum_{j=1}^n(s_i-s_{i-1})(t_j-t_{j-1})(s_i+s_{i-1})^2(t_j+t_{j-1})^2\right\}.
\end{aligned}
$$

Using Theorem 4.7 and Corollary 4.8 we obtain various formulas for conditional Yeh-Wiener integrals.

COROLLARY 4.9. *If $m = 1, n = 1$ and $\alpha_{11}(s,t) = \frac{1}{\sqrt{ST}}$, then*

$$E\left[\exp\left\{\int_Q g(s,t)x(s,t)dsdt\right\}|x(S,T)=\xi\right]$$
$$=\exp\left\{\frac{\xi}{ST}\int_Q st\,g(s,t)\,dsdt+\frac{1}{2}\int_Q\left[\int_t^T\int_s^S g(u,v)\,dudv\right]^2 dsdt\right.$$
$$\left.-\frac{1}{2ST}\left[\int_Q st\,g(s,t)\,dsdt\right]^2\right\},$$

$$E\left[\exp\left\{\int_Q st\,x(s,t)\,dsdt\right\}|x(S,T)=\xi\right]$$
$$=\exp\left\{\frac{\xi}{9}S^2T^2+\frac{11}{4050}S^5T^5\right\},$$

and

$$E\left[\exp\left\{\int_Q x(s,t)\,dsdt\right\}|x(S,T)=\xi\right]$$
$$=\exp\left\{\frac{\xi}{4}ST+\frac{7}{288}S^3T^3\right\}.$$

ACKNOWLEDGEMENT. This work was supported in part by Yonsei University Research Fund of 1997, Non Directed Research Fund, Korea Research Foundation and BSRIP, Ministry of Education, 1998.

References

[1] R. H. Cameron and W. T. Martin, *Transformation of Wiener Integrals under Translations*, Ann. Math. **45** (1944), 386-396.

[2] K. S. Chang and J. M. Ahn, *Translation Theorem for Conditional Yeh-Wiener Integrals*, J. Korean Math. Soc. **20** (1983), 163-172.

[3] K. S. Chang, J. M. Ahn and J. S. Chang, *An Evaluation of the Conditional Yeh-Wiener Integral*, Pacific J. Math. **124** (1986), 107-117.
[4] K. S. Chang and J. S. Chang, *Evaluation of Some Conditional Wiener Integrals*, Bull. Korean Math. Soc. **21** (1984), 99-106.
[5] D. M. Chung and J. M. Ahn, *Conditional Yeh-Wiener Integrals*, J. Korean Math. Soc. **20** (1983), 209-221.
[6] J. L. Doob, *Stochastic Processes*, Wiley, New York (1953).
[7] C. Park and D. L. Skoug, *A Simple Formula for Conditional Wiener Integrals with Applications*, Pacific J. Math. **135** (1988), 381-394.
[8] ———, *Conditional Yeh-Wiener Integrals with Vector-valued Conditioning Functions*, Proc. Amer. Math. Soc. **105** (1989), 450-461.
[9] ———, *Conditional Wiener Integrals II*, Pacific J. Math. **167** (1995), 293-312.
[10] H. G. Tucker, *A Graduate Course in Probability*, Academic Press, New York (1967).
[11] J. Yeh, *Wiener Measure in a Space of Functions of Two Variables*, Trans. Amer. Math. Soc. **95** (1960), 433-450.
[12] ———, *Cameron-Martin Translation Theorems in the Wiener Space of Functions of Two Variables*, Trans. Amer. Math. Soc. **107** (1963), 409-420.
[13] ———, *Stochastic Processes and the Wiener Integral*, Marcel Dekker, New York (1973).
[14] ———, *Inversion of Conditional Expectations*, Pacific J. Math. **52** (1974), 631-640.
[15] ———, *Inversion of Conditional Wiener Integrals*, Pacific J. Math. **59** (1975), 623-638.
[16] ———, *Transformations of Conditional Wiener Integrals under Translation and Cameron-Martin Translation Theorem*, Tohoku Math. J. **30** (1978), 505-515.

Jae Moon Ahn, Department of Mathematics Education, Kon-Kuk University, Seoul 143-701, Korea
E-mail: jmahn@kkucc.konkuk.ac.kr

Kun Soo Chang, Department of Mathematics, Yonsei University, Seoul 120-749, Korea
E-mail: kunchang@yonsei.ac.kr

Sunkyung Kim, Department of Mathematics Education, Andong National University, Andong 760-749, Korea

Il Yoo, Department of Mathematics, Yonsei University, Kangwondo 220-710, Korea
E-mail: iyoo@dragon.yonsei.ac.kr