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COMPATIBLE PAIRS OF ORTHOGONAL POLYNOMIALS

D. H. KIM, KIL H. KWON, D. W. LEE, AND F. MARCELLÁN

ABSTRACT. We find necessary and sufficient conditions for an orthogonal polynomial system to be compatible with another orthogonal polynomial system. As applications, we find new characterizations of semi-classical and classical orthogonal polynomials

1. Introduction

In [4] Bonan et al. raised and solved the following problem: Characterize distribution functions $d\alpha(x)$ and $d\beta(x)$ for which there are integers $r \geq 1$, $s \geq 0$, and $t \geq 0$, and a rational function $R(x) = S(x)/Q(x)(\neq 0)$ such that

$$R(x)Q_n^{(r)}(x) = \sum_{i=n-r-t}^{n-r+s} c_{n,i}P_i(x), \quad n \geq 0, \tag{1.1}$$

where $c_{n,i}$ are real numbers with $c_{n,i} = 0$ for $i < 0$, $c_{n,n-r-t} \neq 0$ and $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ are real orthogonal polynomial systems relative to $d\alpha(x)$ and $d\beta(x)$ respectively. Due to the three-term recurrence relations satisfied by any orthogonal polynomial system, in the relation (1.1) the denominator $Q(x)$ of $R(x)$ plays no significant role.

When $R(x) = S(x)$, $r = 1$, and $\{P_n(x)\}_{n=0}^{\infty} = \{Q_n(x)\}_{n=0}^{\infty}$, the relation (1.1) is the so-called structure relation characterizing semi-classical orthogonal polynomials, which was first introduced by Maroni [12,13] in answering to questions raised by Askey (see Al-Salam and Chihara [2, p. 69]).

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Here, we will consider the same problem in a more general setting by allowing $d\alpha(x)$ and $d\beta(x)$ to be signed measures, that is, $\alpha(x)$ and $\beta(x)$ are functions of bounded variation.

In other words, let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be orthogonal polynomial systems relative to quasi-definite moment functionals σ and τ respectively and ask: When are there a polynomial $S(x) \neq 0$ and integers $r \geq 0$, $s \geq 0$, and t such that

$$(1.2) \qquad S(x)Q_n^{(r)}(x) = \sum_{i=n-r-t}^{n-r+s} a_{n,i}^{(r)} P_i(x), \quad n \geq 0,$$

where $a_{n,i}^{(r)}$ are complex numbers with $a_{n,i}^{(r)} = 0$ for $i < 0$. Then, t must be non-negative as we shall see later in Theorem 2.2.

A moment functional σ (i.e., a linear functional on $\mathcal{P}$, the space of all polynomials with complex coefficients) is said to be quasi-definite if its moments $\sigma_n := \langle \sigma, x^n \rangle$, $n \geq 0$, satisfy the Hamburger condition:

$$\Delta_n(\sigma) := \det [\sigma_{i+j}]_{i,j=0}^{n} \neq 0, \quad n \geq 0.$$

Then, σ is quasi-definite if and only if there is an orthogonal polynomial system (OPS) $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ (cf. [5]), that is, $\deg(P_n) = n$, $n \geq 0$ and

$$\langle \sigma, P_m P_n \rangle = K_n \delta_{mn}, \quad m \text{ and } n \geq 0,$$

where $K_n \neq 0$. We say that an OPS $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order $r(\geq 0)$ and depth $\leq t$ (respectively, depth t) with another OPS $\{P_n(x)\}_{n=0}^{\infty}$ if the relation (1.2) holds for some polynomial $S(x)(\neq 0)$ of degree s (respectively, $a_{n,n-r-t}^{(r)} \neq 0$ for some $n \geq r+t$).

Compatibility of order 1 was studied in [10] as an inverse problem for orthogonal polynomials.

The primary goal of this work is to solve the following inverse problem: Characterize quasi-definite moment functionals σ and τ for which the corresponding OPS's are compatible. In Section 2 we find necessary and sufficient conditions for an OPS $\{Q_n(x)\}_{n=0}^{\infty}$ to be compatible with another OPS $\{P_n(x)\}_{n=0}^{\infty}$. Then it turns out that as long as $r \geq 1$, r plays no significant role in the compatibility condition (1.2). To be precise, if $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order $r \geq 1$ with $\{P_n(x)\}_{n=0}^{\infty}$, then $\{Q_n(x)\}_{n=0}^{\infty}$ (respectively, $\{P_n(x)\}_{n=0}^{\infty}$) is compatible of any order ≥ 0 with $\{P_n(x)\}_{n=0}^{\infty}$ (respectively, $\{Q_n(x)\}_{n=0}^{\infty}$). In Section 3 we find, as applications of results in Section 2 new characterizations of semi-classical and classical OPS's.

2. Main Results

For a moment functional σ and a polynomial $\phi(x)$ we let σ' and $\phi\sigma$ be the moment functionals defined respectively by

$$\langle \sigma', \psi \rangle = -\langle \sigma, \psi' \rangle$$

and

$$\langle \phi\sigma, \psi \rangle = \langle \sigma, \phi\psi \rangle$$

for any $\psi \in \mathcal{P}$. Then it is straightforward to prove

$$(\phi\sigma)' = \phi'\sigma + \phi\sigma'.$$

LEMMA 2.1. *Let τ be a quasi-definite moment functional and $\{Q_n(x)\}_{n=0}^{\infty}$ a monic OPS relative to τ. Then*

(i) *for any polynomial $\phi(x)$, $\phi(x)\tau = 0$ if and only if $\phi(x) = 0$;*
(ii) *for any other moment functional σ and any integer $k \geq 0$, $\langle \sigma, Q_n \rangle = 0$ for $n \geq k+1$ if and only if $\sigma = \pi_k(x)\tau$ for some polynomial $\pi_k(x)$ of degree $\leq k$.*

Proof. See [7, Lemma 2.2] and [13, Proposition 2.2]. □

In fact, in Lemma 2.1 (ii) we have

$$\pi_k(x) = \sum_{j=0}^{k} \frac{\langle \sigma, Q_j \rangle Q_j(x)}{\langle \tau, Q_j^2 \rangle} = \langle \sigma_y, K_k(x,y) \rangle \tag{2.1}$$

where $K_k(x,y) := \sum_{j=0}^{k} \frac{Q_j(x)Q_j(y)}{\langle \tau, Q_j^2 \rangle}$ is the k-th kernel polynomial for $\{Q_n(x)\}_{n=0}^{\infty}$ and σ_y means the action of σ over the variable y for polynomials in two variables (x, y).

Following Maroni [12,13] a moment functional σ is said to be semi-classical if σ is quasi-definite and satisfies a Pearson type functional equation

$$(\alpha\sigma)' - \beta\sigma = 0 \tag{2.2}$$

for some polynomials $\alpha(x)$ and $\beta(x)$ with $|\alpha(x)| + |\beta(x)| \neq 0$. It is then easy to see that $\alpha(x) \neq 0$ and $\deg(\beta) \geq 1$. For a semi-classical moment functional σ we call

$$s := \min\ \max(\deg(\alpha) - 2, \deg(\beta) - 1)$$

the class number of σ, where the minimum is taken over all pairs $(\alpha, \beta) \neq (0,0)$ of polynomials satisfying (2.2). An OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to a semi-classical moment functional σ (of class s) is said to be a semi-classical OPS (SCOPS) (of class s). It is well known (cf. [7,13]) that an OPS $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS if and only if $\{P_n(x)\}_{n=0}^{\infty}$ is a SCOPS of class 0.

We are now ready to state and prove our main result consisting in the characterization of compatibility. In the following we always let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ be the monic OPS's relative to quasi-definite moment functionals σ and τ respectively. We also let

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_n P_{n-1}(x), \quad n \geq 0 \quad (P_{-1}(x) = 0) \tag{2.3}$$

and

$$Q_{n+1}(x) = (x - \beta_n)Q_n(x) - \gamma_n Q_{n-1}(x), \quad n \geq 0 \quad (Q_{-1}(x) = 0) \tag{2.4}$$

be the three-term recurrence relations for $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$, where b_n and β_n, $n \geq 0$, are complex numbers and c_n and γ_n, $n \geq 1$, are non-zero complex numbers.

THEOREM 2.2. *$\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order $r(\geq 0)$ and depth $\leq t$ with $\{P_n(x)\}_{n=0}^{\infty}$ if and only if there are non-zero polynomials $S(x)$ and $T_j(x)$, $0 \leq j \leq r$, such that $\deg(T_j) \leq t + 2r - j$ and*

$$(S(x)\sigma)^{(j)} = T_j(x)\tau, \quad 0 \leq j \leq r. \tag{2.5}$$

In this case we have $j \leq \deg(T_j) \leq t + 2r - j$ (so that $t \geq 0$),

$$T_j(x) = (-1)^j \langle \sigma_y, S(y) K_{t+2r-j}^{(0,j)}(x,y) \rangle, \quad 0 \leq j \leq r \tag{2.6}$$

and

$$S(x)Q_n^{(j)}(x) = \sum_{i=n+j-2r-t}^{n-j+s} a_{n,i}^{(j)} P_i(x), \quad n \geq 0, \text{ and } 0 \leq j \leq r, \tag{2.7}$$

where $a_{n,i}^{(j)} = 0$ for $i < 0$ and $s := \deg(S)$.

Hence, for any $j = 0, 1, \cdots, r$, $\{Q_n(x)\}_{n=0}^\infty$ is also compatible of order j and depth $\le t + 2(r-j)$ for any $j = 0, 1, \cdots, r$ with $\{P_n(x)\}_{n=0}^\infty$.

Moreover, if $r = 0$, then σ is semi-classical if and only if τ is semi-classical and if $r \ge 1$, then both σ and τ must be semi-classical.

Proof. Assume that $\{Q_n(x)\}_{n=0}^\infty$ is compatible of order $r(\ge 0)$ and depth $\le t$ with $\{P_n(x)\}_{n=0}^\infty$, that is, the relation (1.2) holds for some polynomial $S(x)$ of degree s and a positive integer t. Then

$$\begin{aligned}\langle (S(x)\sigma)^{(r)}, Q_n(x)\rangle &= (-1)^r\langle \sigma, S(x)Q_n^{(r)}(x)\rangle \\ &= (-1)^r\langle \sigma, \sum_{n-r-t}^{n-r+s} a_{n,i}^{(r)} P_i(x)\rangle = 0\end{aligned}$$

if $n \ge r+t+1$. Hence, by Lemma 2.1 (ii), $(S(x)\sigma)^{(r)} = T_r(x)\tau$ for some polynomial $T_r(x)$ of degree $\le t+r$. If $T_r(x) = 0$ then $(S(x)\sigma)^{(r)} = 0$ so that $S(x)\sigma = 0$ and $S(x) = 0$ by Lemma 2.1 (i), which is a contradiction. Hence, $T_r(x) \ne 0$.

We now assume $r \ge 1$. Differentiating r-times the three-term recurrence relation (2.4) we obtain

$$Q_{n+1}^{(r)}(x) = xQ_n^{(r)}(x) + rQ_n^{(r-1)}(x) - \beta_n Q_n^{(r)}(x) - \gamma_n Q_{n-1}^{(r)}(x)$$

so that

$$\begin{aligned}&S(x)Q_n^{(r-1)}(x) \\ &= \frac{1}{r}[S(x)Q_{n+1}^{(r)}(x) + \beta_n S(x)Q_n^{(r)}(x) + \gamma_n S(x)Q_{n-1}^{(r)}(x) - xS(x)Q_n^{(r)}(x)] \\ &= \frac{1}{r}\left[\sum_{i=n+1-r-t}^{n+1-r+s} a_{n+1,i}^{(r)}P_i(x) + \beta_n \sum_{i=n-r-t}^{n-r+s} a_{n,i}^{(r)}P_i(x) + \gamma_n \sum_{i=n-1-r-t}^{n-1-r+s} a_{n-1,i}^{(r)}P_i(x)\right. \\ &\quad \left. - \sum_{i=n-r-t}^{n-r+s} a_{n,i}^{(r)}(P_{i+1}(x) + b_i P_i(x) + c_i P_{i-1}(x))\right] \\ &= \sum_{i=n-r-1-t}^{n-r+1+s} a_{n,i}^{(r-1)}P_i(x)\end{aligned}$$

by (1.2) and (2.3). Hence, (2.7) holds for $j = r-1$ so that $\{Q_n(x)\}_{n=0}^\infty$ is compatible of order $r-1$ and depth $\le t+2$ with $\{P_n(x)\}_{n=0}^\infty$. Repeating the same process we can obtain (2.7) for $0 \le j \le r$ so that $\{Q_n(x)\}_{n=0}^\infty$

is compatible of order j and depth $\leq t+2(r-j)$ for $0 \leq j \leq r$ with $\{P_n(x)\}_{n=0}^{\infty}$.

Now, from (2.7) we have for $0 \leq j \leq r$

$$\begin{aligned}\langle (S(x)\sigma)^{(j)}, Q_n(x)\rangle &= (-1)^j\langle \sigma, S(x)Q_n^{(j)}(x)\rangle \\ &= (-1)^j\langle \sigma, \sum_{i=n+j-2r-t}^{n-j+s} a_{n,i}^{(j)}P_i(x)\rangle = 0\end{aligned}$$

if $n \geq t+2r-j+1$. Hence, by Lemma 2.1, we have (2.5) for some polynomial $T_j(x)$ ($\neq 0$) of degree $\leq t+2r-j$. Moreover, from (2.1) we obtain

$$T_j(x) = \langle (S(y)\sigma)_y^{(j)}, K_{t+2r-j}(x,y)\rangle = (-1)^j\langle \sigma_y, S(y)K_{t+2r-j}^{(0,j)}(x,y)\rangle$$

where $K_n^{(i,j)}(x,y) := \sum_{k=0}^{n} \frac{Q_k^{(i)}(x)Q_k^{(j)}(y)}{\langle \tau, Q_k^2\rangle}$, which gives (2.6).

Assume that $0 \leq \deg(T_j(x)) = k < j$ for some $j = 0, 1, \cdots, r$. Then

$$0 = \langle S(x)\sigma, Q_k^{(j)}(x)\rangle = (-1)^j\langle (S(x)\sigma)^{(j)}, Q_k(x)\rangle = (-1)^j\langle \tau, T_j(x)Q_k(x)\rangle,$$

which is impossible since $\langle \tau, T_j(x)Q_k(x)\rangle \neq 0$. Hence, $j \leq \deg(T_j(x))$.

Conversely, assume that (2.5) holds. Write $S(x)Q_n^{(r)}(x)$ as

$$S(x)Q_n^{(r)}(x) = \sum_{i=0}^{n-r+s} a_{n,i}^{(r)}P_i(x), \quad n \geq 0.$$

Then

$$\begin{aligned}a_{n,i}^{(r)}\langle \sigma, P_i^2(x)\rangle &= \langle \sigma, S(x)Q_n^{(r)}(x)P_i(x)\rangle = (-1)^r\langle (P_i(x)S(x)\sigma)^{(r)}, Q_n(x)\rangle \\ &= (-1)^r\sum_{j=0}^{r}\binom{r}{j}\langle P_i^{(r-j)}(x)(S(x)\sigma)^{(j)}, Q_n(x)\rangle \\ &= (-1)^r\sum_{j=0}^{r}\binom{r}{j}\langle \tau, Q_n(x)P_i^{(r-j)}(x)T_j(x)\rangle = 0\end{aligned}$$

if $i < n-r-t$ so that $a_{n,i}^{(r)} = 0$ if $i < n-r-t$. Hence, (1.2) holds, that is, $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order r and depth $\leq t$ with $\{P_n(x)\}_{n=0}^{\infty}$.

Assume now that $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 with $\{P_n(x)\}_{n=0}^{\infty}$. Then, $S(x)\sigma = T(x)\tau$ for some non-zero polynomials $S(x)$ and $T(x)$. If one of σ and τ, say, σ is semi-classical satisfying (2.2), then τ satisfies

$$(S(x)\alpha(x)T(x)\tau)' = (2S'(x)\alpha(x)T(x) + S(x)\beta(x)T(x))\tau$$

so that τ is also semi-classical since $S(x)\alpha(x)T(x) \neq 0$. Finally, assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order $r \geq 1$ and depth $\leq t$ with $\{P_n(x)\}_{n=0}^{\infty}$. Then, (2.5) holds. In particular, we have

$$S(x)\sigma = T_0(x)\tau \quad \text{and} \quad (S(x)\sigma)' = T_1(x)\tau$$

so that

$$(T_0(x)\tau)' = T_1(x)\tau$$

and so

$$(T_0(x)S(x)\sigma)' = (T_0'(x) + T_1(x))S(x)\sigma.$$

Hence, both σ and τ must be semi-classical since $T_0(x) \neq 0$ and $T_0(x)S(x) \neq 0$. □

In particular, $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 with $\{P_n(x)\}_{n=0}^{\infty}$ if and only if $\{P_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 with $\{Q_n(x)\}_{n=0}^{\infty}$. Later, we will see that the compatibility of any order is a reflexive property for $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$.

We may also express $T_j(x)$ in terms of $a_{n,0}^{(j)}$'s: Write $T_j(x)$ as

$$T_j(x) = \sum_{k=0}^{t+2r-j} c_{j,k} Q_k(x).$$

Then

$$\begin{aligned} c_{j,k}\langle \tau, Q_k^2(x)\rangle &= \langle T_j(x)\tau, Q_k(x)\rangle = (-1)^j \langle \sigma, S(x)Q_k^{(j)}(x)\rangle \\ &= \begin{cases} 0 & \text{if } 0 \leq k < j \\ (-1)^j \langle \sigma, \sum_{i=k+j-2r-t}^{k-j+s} a_{k,i}^{(j)} P_i(x)\rangle & \\ = (-1)^j a_{k,0}^{(j)} \langle \sigma, P_0(x)\rangle & \text{if } j \leq k \leq t+2r-j \end{cases} \end{aligned}$$

by (2.7) so that

$$T_j(x) = (-1)^j \langle \sigma, P_0(x)\rangle \sum_{k=j}^{t+2r-j} \frac{a_{k,0}^{(j)}}{\langle \tau, Q_k^2\rangle} Q_k(x).$$

Hence, $\deg(T_j(x)) = t + 2r - j$ if and only if $a_{t+2r-j,0}^{(j)} \neq 0$. In particular, if $t = 0$, then $\deg(T_r(x)) = r$ so that $a_{r,0}^{(r)} \neq 0$. Moreover for $j = 0$, either

$a^{(0)}_{n,n-2r-t} \neq 0$ for all $n \geq 2r+t$ (if $\deg(T_0(x)) = t+2r$) or $a^{(0)}_{n,n-2r-t} = 0$ for all $n \geq 2r+t$ (if $\deg(T_0(x)) < t+2r$) since we have:

PROPOSITION 2.3. *For any polynomial $S(x) \neq 0$ of degree $s\,(\geq 0)$, write $S(x)Q_n(x)$ as*

$$S(x)Q_n(x) = \sum_{i=0}^{n+s} a_{n,i}P_i(x), \quad n \geq 0.$$

Then, either $a_{n,0} \neq 0$ for infinitely many n's or

$$S(x)Q_n(x) = \sum_{i=n-t}^{n+s} a_{n,i}P_i(x), \quad n \geq 0. \tag{2.8}$$

where $a_{n,i} = 0$ for $i < 0$ and $a_{t,0} \neq 0$ for some integer $t \geq 0$. In the latter case, $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 and depth t with $\{P_n(x)\}_{n=0}^{\infty}$ and $a_{n,n-t} \neq 0$, $n \geq t$.

Proof. Assume that $a_{n,0} \neq 0$ for only finitely many n's. Let t (≥ 0) be the largest integer such that $a_{t,0} \neq 0$. Then

$$S(x)Q_n(x) = \sum_{i=1}^{n+s} a_{n,i}P_i(x), \quad n \geq t+1 \tag{2.9}$$

so that $\langle S(x)\sigma, Q_n(x)\rangle = 0$, $n \geq t+1$. Hence, by Lemma 2.1,

$$S(x)\sigma = T(x)\tau \tag{2.10}$$

for some polynomial $T(x)(\neq 0)$ of degree $\leq t$. Then from (2.9) and (2.10) we obtain

$$\begin{aligned} a_{n,i}\langle \sigma, P_i^2(x)\rangle &= \langle S(x)Q_n(x)\sigma, P_i(x)\rangle \\ &= \langle \tau, Q_n(x)P_i(x)T(x)\rangle = 0 \end{aligned}$$

if $i < n-t$ so that $a_{n,i} = 0$ if $i < n-t$. Hence (2.8) holds since (2.8) for $0 \leq n \leq t$ holds trivially. Similarly, we have from (2.8) and (2.10)

$$a_{n,n-t} = \frac{\langle \tau, Q_n(x)P_{n-t}(x)T(x)\rangle}{\langle \sigma, P_{n-t}^2(x)\rangle}, \quad n \geq t.$$

Since $a_{t,0} \neq 0$, $\langle \tau, Q_t(x)P_0(x)T(x)\rangle = \langle \tau, Q_t(x)T(x)\rangle \neq 0$ and so $\deg(T(x)) = t$. Then $\langle \tau, Q_n(x)P_{n-t}(x)T(x)\rangle \neq 0$, $n \geq t$ so that $a_{n,n-t} \neq 0$, $n \geq t$. $\square$

Theorem 2.2 shows, in particular, that if $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order $r(\geq 1)$ with $\{P_n(x)\}_{n=0}^{\infty}$, then $\{Q_n(x)\}_{n=0}^{\infty}$ must be compatible of order j for any $j = 0, 1, \cdots, r$ with $\{P_n(x)\}_{n=0}^{\infty}$. This fact is essentially proved in [4; See Theorem 1.1 and Lemma 2.1], where they showed that the relation (1.1) implies the relation (2.7) for positive-definite moment functionals σ and τ. Note that in this case, t and s for (1.1) are different from t and s for (2.7) in general.

On the other hand, Marcellán et al. [10] showed that if $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible with $\{P_n(x)\}_{n=0}^{\infty}$ of order 1, then $S\sigma = T_0\tau$ for some non-zero polynomials $S(x)$ and $T_0(x)$ and σ and τ must be semi-classical (see Proposition 1.1 in [10]).

In general, compatibility of order 0 does not imply compatibility of higher order. However, we have:

THEOREM 2.4. *For any integer $r \geq 1$ the following statements are equivalent.*

(i) $\{Q_n(x)\}_{n=0}^{\infty}$ *(or $\{P_n(x)\}_{n=0}^{\infty}$) is an SCOPS and is compatible of order 0 (and depth $\leq t$) with $\{P_n(x)\}_{n=0}^{\infty}$, that is, there are non-negative integers s and t and a polynomial $S(x)$ of degree s such that*

$$S(x)Q_n(x) = \sum_{i=n-t}^{n+s} a_{n,i}P_i(x), \quad n \geq 0 \quad (a_{n,i} = 0 \text{ for } i < 0). \tag{2.11}$$

(ii) $\{Q_n(x)\}_{n=0}^{\infty}$ *is compatible of order r (and depth $\leq t^*$) with $\{P_n(x)\}_{n=0}^{\infty}$, that is, there are non-negative integers s^* and t^* and a polynomial $S^*(x)$ of degree s^* such that*

$$S^*(x)Q_n^{(r)}(x) = \sum_{i=n-r-t^*}^{n-r+s^*} a_{n,i}^{(r)}P_i(x), \quad n \geq 0 \quad (a_{n,i}^{(r)} = 0 \text{ for } i < 0). \tag{2.12}$$

Moreover, in this case, $\{P_n(x)\}_{n=0}^{\infty}$ is also an SCOPS and

$$S(x)[\alpha^*(x)Q_n''(x) + \beta^*(x)Q_n'(x)] = \sum_{i=n-v}^{n+s+s^*} a_{n,i}^*P_i(x), \tag{2.13}$$
$$n \geq 0 \quad (a_{n,i}^* = 0 \quad \text{for } i < 0)$$

if $(\alpha^*(x)\tau)' = \beta^*(x)\tau$ and $|\alpha^*(x)| + |\beta^*(x)| \neq 0$, where v is a non-negative integer and $s^* := \max(\deg(\alpha^*(x)) - 2, \deg(\beta^*(x)) - 1)$.

Proof. Assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order r and depth $\leq t^*$ with $\{P_n(x)\}_{n=0}^{\infty}$. Then by Theorem 2.2, $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 and depth $\leq t^*+2r$ with $\{P_n(x)\}_{n=0}^{\infty}$ and both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ are SCOPS's. Conversely, assume that τ is a semi-classical moment functional satisfying $(\alpha^*(x)\tau)' = \beta^*(x)\tau$ with $|\alpha^*(x)|+|\beta^*(x)| \neq 0$. Then by [Theorem 3.1, 13], $\{Q_n^{(r)}(x)\}_{n=r}^{\infty}$ is quasi-orthogonal relative to $\alpha^*(x)^r\tau$ of order $\leq rs^*$, that is,

$$\langle \alpha^*(x)^r\tau, x^k Q_n^{(r)}(x)\rangle = 0, \quad 0 \leq k < n-r-rs^*. \tag{2.14}$$

On the other hand, (2.11) implies

$$S(x)\sigma = T(x)\tau \tag{2.15}$$

for some polynomial $T(x)$ with $0 \leq \deg(T(x)) \leq t$. Write $S(x)\alpha^*(x)^rQ_n^{(r)}(x)$ as

$$S(x)\alpha^*(x)^rQ_n^{(r)}(x) = \sum_{i=0}^{n-r+s+ur} a_{n,i}^{(r)}P_i(x), \quad n \geq 0,$$

where $u := \deg(\alpha^*(x))$. Then from (2.14) and (2.15) we obtain

$$\begin{aligned} a_{n,i}^{(r)}\langle\sigma, P_i^2(x)\rangle &= \langle\sigma, S(x)(\alpha^*(x))^rQ_n^{(r)}(x)P_i(x)\rangle \\ &= \langle(\alpha^*(x))^r\tau, Q_n^{(r)}(x)P_i(x)T(x)\rangle = 0 \end{aligned}$$

if $i < n-r-rs^*-\tilde{t}$, where $\tilde{t} = \deg(T(x))$ so that $a_{n,i}^{(r)} = 0$ if $i < n-r-rs^*-\tilde{t}$. Hence we have (2.12) with $S^*(x) = S(x)\alpha^*(x)^r$, $s^* = s+ur$, and $t^* = rs^*+\tilde{t}$.

Finally to show (2.13) note first that $\deg(S(x)[\alpha^*(x)Q_n''(x)+\beta^*(x)Q_n'(x)]) \leq n+s+s^*$ so that

$$S(x)[\alpha^*(x)Q_n''(x)+\beta^*(x)Q_n'(x)] = \sum_{i=0}^{n+s+s^*} a_{n,i}^*P_i(x), \quad n \geq 0.$$

Then, by (2.14) with $r=1$ and (2.15) we deduce

$$\begin{aligned} a_{n,i}^*\langle\sigma, P_i^2(x)\rangle &= \langle\sigma, S[\alpha^*(x)Q_n''(x)+\beta^*(x)Q_n'(x)]P_i(x)\rangle \\ &= -\langle(T(x)P_i(x)\alpha^*(x)\tau)' + T(x)P_i(x)\beta^*(x)\tau, Q_n'(x)\rangle \\ &= -\langle\alpha^*(x)\tau, Q_n'(x)(T(x)P_i(x))'\rangle = 0 \end{aligned}$$

if $i < n-s^*-\tilde{t}$ so that $a_{n,i}^* = 0$ if $i < n-s^*-\tilde{t}$. Hence we have (2.13) with $v = s^*+\tilde{t} \geq 0$. □

REMARK 2.1. We can now see from Theorem 2.2 and Theorem 2.4: Assume $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order $r\,(\geq 0)$ with $\{P_n(x)\}_{n=0}^{\infty}$. If $r=0$ we also assume that either $\{P_n(x)\}_{n=0}^{\infty}$ or $\{Q_n(x)\}_{n=0}^{\infty}$ is an SCOPS. Then $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ are compatible each other of any order ≥ 0 and both $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ must be SCOPS's. Thus, for $r \geq 1$, r plays no significant role in the compatibility condition (1.2).

3. Applications

As applications of Theorem 2.2 and Theorem 2.4 we give some new characterizations of SCOPS's and Classical OPS's.

THEOREM 3.1. *For an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ and an integer $r \geq 1$ the following statements are equivalent.*

(i) *$\{P_n(x)\}_{n=0}^{\infty}$ is an SCOPS.*

(ii) *$\{P_n(x)\}_{n=0}^{\infty}$ is compatible of order r with $\{P_n(x)\}_{n=0}^{\infty}$, that is, there are a polynomial $S(x)$ of degree $s \geq 0$ and an integer $t \geq 0$ such that*

$$S(x)P_n^{(r)}(x) = \sum_{i=n-r-t}^{n-r+s} a_{n,i}^{(r)} P_i(x), \quad n \geq 0 \ \ (a_{n,i}^{(r)} = 0 \text{ for } i < 0). \tag{3.1}$$

(iii) *(cf. [1]) There are polynomials $\alpha(x) \neq 0$ and $\beta(x)$ and integers r and s with $0 \leq r \leq s$ such that*

(3.2)

$$\alpha(x)P_n''(x) + \beta(x)P_n'(x) = \sum_{i=n-r}^{n+s} a_{n,i} P_i(x), \quad n \geq 0 \ \ (a_{n,i} = 0 \text{ for } i < 0).$$

Proof. (i) $\Rightarrow$ (ii) and (iii): It comes from Theorem 2.4 since $\{P_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 with $\{P_n(x)\}_{n=0}^{\infty}$.

(ii) $\Rightarrow$ (i): It comes from Theorem 2.2.

(iii) $\Rightarrow$ (i): For any integer $k \geq 0$, we have by (3.2)

$$\begin{aligned}\langle [(x^k\alpha(x)\sigma)' - x^k\beta(x)\sigma]', P_n(x)\rangle &= \langle \sigma, x^k(\alpha(x)P_n''(x) + \beta(x)P_n'(x))\rangle \\ &= \sum_{i=n-r}^{n+s} a_{n,i}\langle \sigma, x^k P_i(x)\rangle = 0\end{aligned}$$

if $n > r + k$ so that

$$(3.3) \qquad [(x^k\alpha(x)\sigma)' - x^k\beta(x)\sigma]' = \pi_{r+k}(x)\sigma$$

for some polynomial $\pi_{r+k}(x)$ of degree $\leq r + k$ by Lemma 2.1. In particular, for $k = 1$,

$$\begin{aligned}\pi_{r+1}(x)\sigma = [(x\alpha(x)\sigma)' - x\beta(x)\sigma]' &= [\alpha(x)\sigma + x\{(\alpha(x)\sigma)' - \beta(x)\sigma\}]' \\ &= 2(\alpha(x)\sigma)' + \beta(x)\sigma + x\pi_r(x)\sigma\end{aligned}$$

by (3.3) for $k = 0$ so that

$$2(\alpha(x)\sigma)' = [\pi_{r+1}(x) - x\pi_r(x) - \beta(x)]\sigma.$$

Hence, σ must be semi-classical since $\alpha(x) \neq 0$. □

Characterization of SCOPS's by the relation (3.1) with $r = 1$ was first proved by Maroni [12,13], who called it a structure relation of the SCOPS $\{P_n(x)\}_{n=0}^{\infty}$. We may call (3.1) a structure relation of order $r (\geq 1)$. Characterization of SCOPS's by the relation (3.2) was first proved in [1] assuming $r = s$, where they used the dual basis of a polynomial sequence.

Al-Salam and Chihara [2] (see also [7]) characterized classical OPS'a via a structure relation of order 1: an OPS $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS if and only if there is a polynomial $S(x) \neq 0$ of degree at most 2 such that

$$(3.4) \qquad S(x)P_n'(x) = r_nP_{n+1}(x) + s_nP_n(x) + t_nP_{n-1}(x), \quad n \geq 1,$$

where r_n, s_n, and t_n are real numbers with $t_n \neq 0$, that is, $\{P_n(x)\}_{n=0}^{\infty}$ is compatible of order 1 and depth 0 with $\{P_n(x)\}_{n=0}^{\infty}$. We can now extend Al-Salam and Chihara's characterization of classical OPS's for which we need the following extended Hahn's characterization of classical OPS's (see Theorem 3.3 in [8]): An OPS $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS if and only if for some integer $r \geq 1$, $\{P_n^{(r)}(x)\}_{n=r}^{\infty}$ is quasi-orthogonal of order 0, that is, there is a non-zero moment functional μ such that

$$\langle \mu, P_m^{(r)}P_n^{(r)}\rangle = 0 \quad \text{for } m \neq n.$$

THEOREM 3.2. *If $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order $r \geq 1$ and depth 0 with $\{P_n(x)\}_{n=0}^{\infty}$ then $\{Q_n(x)\}_{n=0}^{\infty}$ must be a classical OPS.*

Proof. Assume that (1.2) holds for some polynomial $S(x)$ of degree $s\,(\geq 0)$ and $t = 0$. Then

$$\begin{aligned}\langle S(x)\sigma, Q_m^{(r)}(x)Q_n^{(r)}(x)\rangle &= \langle \sigma, Q_m^{(r)}(x)\sum_{i=n-r}^{n-r+s} a_{n,i}^{(r)}P_i(x)\rangle \\ &= \sum_{i=n-r}^{n-r+s} a_{n,i}^{(r)}\langle \sigma, Q_m^{(r)}(x)P_i(x)\rangle = 0\end{aligned}$$

if $n > m$. Hence $\{Q_n^{(r)}(x)\}_{n=r}^{\infty}$ is quasi-orthogonal of order 0 relative to $S(x)\sigma$ so that $\{Q_n(x)\}_{n=0}^{\infty}$ must be a classical OPS. □

COROLLARY 3.3. *Let $r \geq 1$ be any integer and $\{P_n(x)\}_{n=0}^{\infty}$ an OPS. Then $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS if and only if there is a polynomial $S(x)$ of degree $s \geq 0$ such that*

$$(3.5)\qquad S(x)P_n^{(r)}(x) = \sum_{i=n-r}^{n-r+s} a_{n,i}P_i(x), \quad n \geq 0 \ \ (a_{n,i} = 0 \ \text{ for } i < 0),$$

that is, $\{P_n(x)\}_{n=0}^{\infty}$ is compatible of order $r \geq 1$ and depth 0 with $\{P_n(x)\}_{n=0}^{\infty}$.

Proof. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a classical OPS relative to σ satisfying $(\alpha(x)\sigma)' = \beta(x)\sigma$ with $\max(\deg(\alpha(x)) - 2, \deg(\beta(x)) - 1) = 0$. Then we have (3.5) with $S(x) = \alpha(x)^r$ (see the proof of (i) $\Rightarrow$ (ii) in Theorem 2.4). The converse result comes from Theorem 3.2. □

The relation (3.5) (which is exactly (3.4) for $r = 1$) gives a characterization of classical OPS's via higher order structure relations.

EXAMPLE 3.1. Let σ be the moment functional defined by

$$\langle \sigma, \phi\rangle = \int_0^{\infty} \phi(x)x^{\alpha}e^{-x}dx, \quad \phi \in \mathcal{P}.$$

Then σ is positive-definite for $\alpha > -1$ and the corresponding monic OPS is the Laguerre OPS $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$:

$$L_n^{(\alpha)}(x) = (-1)^n n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}, \quad n \geq 0$$

and

$$\langle \sigma, L_n^{(\alpha)}(x)^2 \rangle = n!\Gamma(n+\alpha+1), \quad n \geq 0.$$

We now consider another moment functional τ satisfying

$$x\tau = x\sigma. \tag{3.6}$$

Then

$$\tau = \sigma + (\tau_0 - \sigma_0)\delta(x)$$

so that τ is quasi-definite if and only if (cf. [9, Corollary 3.2] or [11, Theorem 2.1])

$$1 + (\tau_0 - \sigma_0)\frac{(\alpha+1)_n}{\Gamma(\alpha+2)(n-1)!} \neq 0, \quad n \geq 1,$$

i.e.,

$$\begin{aligned} \tau_0 \neq \sigma_0 - \frac{\Gamma(\alpha+2)(n-1)!}{(\alpha+1)_n} &= \Gamma(\alpha+1) - \frac{\Gamma(\alpha+2)(n-1)!}{(\alpha+1)_n} \\ &= \Gamma(\alpha+1)\left[1 - \frac{(\alpha+1)(n-1)!}{(\alpha+1)_n}\right] \\ &= \Gamma(\alpha+1)\left[1 - \frac{(n-1)!}{(\alpha+2)_{n-1}}\right], \quad n \geq 1. \end{aligned}$$

From now on we will assume $\tau_0 \neq \sigma_0$, $\Gamma(\alpha+1)[1 - \frac{n!}{(\alpha+2)_n}]$, $n \geq 1$ so that τ is quasi-definite. Then the monic OPS $\{Q_n(x)\}_{n=0}^{\infty}$ relative to τ is

$$Q_n(x) = L_n^{(\alpha)}(x) + \frac{(\sigma_0 - \tau_0)}{d_{n-1}} L_n^{(\alpha)}(0) K_{n-1}(x,0), \quad n \geq 0,$$

where $d_{-1} = K_{-1}(x,y) = 1$ and

$$\begin{aligned} d_n &= 1 + (\tau_0 - \sigma_0)K_n(0,0) \\ &= 1 + (\tau_0 - \sigma_0)\frac{(\alpha+2)_{n-1}}{\Gamma(\alpha+1)(n-1)!}. \end{aligned}$$

Then by (3.6) and Theorem 2.2 $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 and depth 1 with $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$:

$$xQ_n(x) = \sum_{i=n-1}^{n+1} a_{n,i} L_i^{(\alpha)}(x), \quad n \geq 0, \tag{3.7}$$

where $a_{n,i} = 0$ for $i < 0$ and $a_{n,n-1} \neq 0$ for $n \geq 1$. On the other hand, since $K_{n-1}(x,0) = \frac{(-1)^{(n-1)}}{\Gamma(\alpha+1)(n-1)!} L_{n-1}^{(\alpha+1)}(x)$ and

(3.8)
$$\begin{aligned} L_n^{(\alpha)}(x) &= L_n^{(\alpha+1)}(x) + nL_{n-1}^{(\alpha+1)}(x), \\ Q_n(x) &= L_n^{(\alpha+1)}(x) + L_{n-1}^{(\alpha+1)}(x)\left[n + \frac{\sigma_0 - \tau_0}{d_{n-1}} L_n^{(\alpha)}(0) \frac{(-1)^{n-1}}{\Gamma(\alpha+1)(n-1)!}\right], \end{aligned}$$

i.e., $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 and depth 1 with $\{L_n^{(\alpha+1)}(x)\}_{n=0}^{\infty}$. Note that the basic difference with the compatibility condition (3.7) is that there is no polynomial factor in (3.8). Since

$$\begin{aligned} (x\sigma)' &= (-x+\alpha+1)\sigma \\ (x\tau)' &= (-x+\alpha+1)\sigma, \end{aligned} \tag{3.9}$$

we have, by (3.9) and Theorem 2.2, $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is compatible of order 1 and depth 0 with $\{Q_n(x)\}_{n=0}^{\infty}$:

$$x[L_n^{(\alpha)}]' = \sum_{i=n-1}^{n} b_{n,i} Q_i(x), \quad n \geq 0$$

where $b_{n,i} = 0$ for $i < 0$ and $b_{n,n-1} \neq 0$.

EXAMPLE 3.2. Let σ be the Bessel moment functional, i.e.,

$$(x^2\sigma)' = [(\alpha+2)x + 2]\sigma$$

with $\alpha \neq -n$ and $n \geq 2$. σ is quasi-definite and the corresponding monic OPS is the Bessel OPS $\{B_n^{(\alpha)}(x)\}_{n=0}^{\infty}$:

$$B_n^{(\alpha)}(x) = \frac{2^n}{(\alpha+n+1)_n} \sum_{k=0}^{n} \binom{n}{k} (n+\alpha+1)_k \left(\frac{x}{2}\right)^k$$

and

$$\langle \sigma, B_n^{(\alpha)}(x)^2 \rangle = \frac{(-1)^{n+1} 2^{2n+\alpha+1} \Gamma(n+\alpha+1) n!}{(2n+\alpha+1)\Gamma(2n+\alpha+2)}.$$

We now consider another moment functional τ satisfying

$$x^2\tau = x^2\sigma.$$

Then

$$\tau = \sigma + (\tau_0 - \sigma_0)\delta(x) + (\sigma_1 - \tau_1)\delta'(x) .$$

According to Proposition 1 in [3] a necessary and sufficient condition for the quasi-definiteness of τ is

$$0 \neq \begin{vmatrix} 1 + (\tau_0 - \sigma_0)K_n(0,0) + (\sigma_1 - \tau_1)K_n^{(0,1)}(0,0) & (\sigma_1 - \tau_1)K_n(0,0) \\ (\tau_0 - \sigma_0)K_n^{(0,1)}(0,0) + (\sigma_1 - \tau_1)K_n^{(1,1)}(0,0) & 1 + (\sigma_1 - \tau_1)K_n^{(0,1)}(0,0) \end{vmatrix}, \quad n \geq 0 .$$

With this hypothesis, if $\{Q_n(x)\}_{n=0}^{\infty}$ is the monic OPS relative to τ then

$$Q_n(x) = B_n^{(\alpha)}(x) + a_n K_{n-1}(x,0) + b_n K_{n-1}^{(0,1)}(x,0) \tag{3.10}$$

and from (19) in [3],

$$x^2 Q_n(x) = \sum_{j=n-2}^{n+2} a_{n,j} B_j^{(\alpha)}(x), \quad n \geq 0 .$$

That is, $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 and depth 2 with $\{B_n^{(\alpha)}(x)\}_{n=0}^{\infty}$, where $a_{n,j} = 0$ for $j < 0$ and $a_{n,n-2} \neq 0$ for $n \geq 2$.

On the other hand, from (46) in [3] we obtain

$$Q_n(x) = B_n^{(\alpha+2)}(x) + c_n B_{n-1}^{(\alpha+2)}(x) + e_n B_{n-2}^{(\alpha+2)}(x) .$$

That is, $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 and depth 2 with $\{B_n^{(\alpha+2)}(x)\}_{n=0}^{\infty}$. Since

$$\begin{aligned} (x^2\sigma)' &= [(\alpha+2)x + 2]\sigma \\ (x^2\tau)' &= [(\alpha+2)x + 2]\sigma , \end{aligned}$$

$\{B_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is compatible of order 1 and depth 0 with $\{Q_n(x)\}_{n=0}^{\infty}$:

$$x^2 [B_n^{(\alpha)}(x)]' = \sum_{i=n-1}^{n+1} b_{n,i} Q_i(x)$$

with $b_{n,n-1} \neq 0$ for $n \geq 1$.

EXAMPLE 3.3. Let σ be the moment functional defined by

$$\langle \sigma, \phi \rangle = \int_{-1}^{1} \phi(x)(1-x)^{\alpha}(1+x)^{\beta} dx, \quad \phi \in \mathcal{P}.$$

Then σ is positive-definite for $\alpha, \beta > -1$ and the corresponding monic OPS is the Jacobi OPS $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{\binom{2n+\alpha+\beta}{n}} \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k}$$

and

$$\langle \sigma, P_n^{(\alpha,\beta)}(x)^2 \rangle = \frac{2^{\alpha+\beta+2n+1} n! \Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)(2n+\alpha+\beta+1)(n+\alpha+\beta+1)_n^2}, \quad n \geq 0.$$

We now consider another moment functional τ satisfying

$$(1-x^2)\tau = (1-x^2)\sigma.$$

Then

$$\tau = \sigma + \overbrace{\frac{1}{2}[(\tau_0+\tau_1)-(\sigma_0+\sigma_1)]}^{A_1}\,\delta(x-1) + \overbrace{\frac{1}{2}[(\tau_0-\tau_1)+(\sigma_0-\sigma_1)]}^{A_2}\,\delta(x+1)$$

so that τ is quasi-definite if and only if (cf. [9, Theorem 3.1])

$$0 \neq \begin{vmatrix} 1+A_1K_{n-1}(1,1) & A_2K_{n-1}(1,-1) \\ A_1K_{n-1}(1,-1) & 1+A_2K_{n-1}(-1,-1) \end{vmatrix}.$$

Under this hypothesis, let $\{Q_n(x)\}_{n=0}^{\infty}$ be the monic OPS relative to τ. Note that if $A_1 \geq 0$ and $A_2 \geq 0$ τ is a positive-definite moment functional. The corresponding sequence of orthogonal polynomials was studied by T. H. Koornwinder [6]. It is easy to prove

$$Q_n(x) = P_n^{(\alpha,\beta)}(x) + a_nK_{n-1}(x,1) + b_nK_{n-1}(x,-1)$$

and

$$(1-x^2)Q_n(x) = \sum_{j=n-2}^{n+2} a_{n,j}P_j^{(\alpha,\beta)}(x), \quad n \geq 0.$$

This means that $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 and depth 2 with $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ where $a_{n,j} = 0$ for $j < 0$ and $a_{n,n-2} \neq 0$ for $n \geq 2$. On the other hand,

$$Q_n(x) = P_n^{(\alpha+1,\beta+1)}(x) + c_n P_{n-1}^{(\alpha+1,\beta+1)}(x) + e_n P_{n-2}^{(\alpha+1,\beta+1)}(x)$$

so that $\{Q_n(x)\}_{n=0}^{\infty}$ is compatible of order 0 and depth 2 with $\{P_n^{(\alpha+1,\beta+1)}(x)\}_{n=0}^{\infty}$. Since

$$\begin{aligned} [(1-x^2)\sigma]' &= [-(\alpha+\beta+2)x+\beta-\alpha]\sigma \\ [(1-x^2)\tau]' &= [-(\alpha+\beta+2)x+\beta-\alpha]\sigma \ , \end{aligned}$$

$\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ is compatible of order 1 and depth 0 with $\{Q_n(x)\}_{n=0}^{\infty}$:

$$(1-x^2)[P_n^{(\alpha,\beta)}(x)]' = \sum_{i=n-1}^{n+1} b_{n,i} Q_i(x)$$

where $b_{n,n-1} \neq 0$ for $n \geq 1$.

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D. H. Kim and Kil H. Kwon, Department of Mathematics, KAIST, Taejon 305-701, Korea
E-mail: khkwon@jacobi.kaist.ac.kr
E-mail: dhkim@camis.kaist.ac.kr

D. W. Lee, Topology and Geometry Research Center, Kyungpook National University, Taegu 702-701, Korea
E-mail: dwlee@gauss.kyungpook.ac.kr

F. Marcellán, Dept. de Matemáticas, Univ. Carlos III, C. Butarque 15, 28911 Leganés, Spain
E-mail: pacomarc@ing.uc3m.es