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MIXED VOLUMES OF A CONVEX BODY AND ITS POLAR DUAL

Y. D. CHAI* AND YOUNG SOO LEE

ABSTRACT. In this paper, we obtain some geometric inequalities for mixed volumes of a convex body and its polar dual. We also develop a lower bound of the product of quermassintegral of a convex body and its polar dual and give a lower bound for the product of the dual quermassintegral of any index of centrally symmetric convex body and that of its polar dual.

1. Introduction

Polar dual convex bodies are mainly important in Minkowski geometry [2]. Firey [3] showed that the mixed area of a plane convex body and its polar dual is at least π. Sangwine-Yager [6] obtained integral lower bound for certain mixed volumes by using a method of generalized outer parallel sets of a compact set. As a consequence, she generalized Firey's result to the higher dimensions. Ghandehari [4] found a lower bound of $W_{n-1}(K)W_{n-1}(K^*)$ for all convex bodies K. However, the problem of finding the lower bound of the product $W_i(K)W_i(K^*)$ for all convex bodies, for each i, is not solved completely yet. See Bambah [1], Firey [3], Lutwak [5], and Ghandehari [4] for partial results.

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In this paper, we prove that for a convex body K and its polar dua K^*,

$$W_1(K)W_1(K^*) \geq \left(\frac{r_i}{r_e}\right)^{n-1} \omega_n^2$$

where $\frac{r_i}{r_e} = \max_{p\in K}\{\frac{r_i(p)}{r_e(p)} | r_i(p)$ is the radius of the largest ball contained in K and centered at p and $r_e(p)$ is the radius of the smallest ball containing K and centered at $p\}$.

We also give a lower bound for the product of the dual quermassintegral of any index of centrally symmetric convex body and that of its polar dual, that is, for a convex body K and its polar dual K^*

$$\tilde{W}_i(K)\tilde{W}_i(K^*) \geq \left(\frac{r_i}{r_e}\right)^{n-i} \omega_n^2, \qquad i = 0, 1, \cdots, n.$$

2. Preliminaries

By a *convex body* in $E^n, n \geq 2$, we mean a compact convex subset of E^n with nonempty interior. A set E is said to be *centered* if $-x \in E$ whenever $x \in E$, and *centrally symmetric* if there is a vector c such that the translate $E - c$ of E by $-c$ is centered. Let B be the closed unit ball in E^n. The inradius, and outradius of a convex body K with respect to B are defined to be the largest scalar for which a homothet of B is contained in K, and the smallest scalar for which a homothet of B contains K, respectively. We denote inradius of K by $r_{in}(K)$ and outradius of K by $r_{out}(K)$, respectively. For each direction $u \in S^{n-1}$ where S^{n-1} is the unit sphere centered at the origin in E^n, we define the support function $h(K, u)$ on S^{n-1} of the convex body K by

$$h(K, u) = \sup\{u \cdot x | x \in K\}$$

and the radial function $\rho(K, u)$ on S^{n-1} of the convex body K is

$$\rho(K, u) = \sup\{\lambda > 0 | \lambda u \in K\}.$$

The *polar dual* of a convex body K, denoted by K^*, is another convex body defined by

$$K^* = \{y | x \cdot y \leq 1 \text{ for all } x \in K\}.$$

The polar dual has the following well known property:

$$h(K^*, u) = 1/\rho(K, u) \text{ and } \rho(K^*, u) = 1/h(K, u).$$

The outer parallel set of K at the distance $\lambda > 0$, K_λ, is given by

$$K_\lambda = K + \lambda B.$$

Then the volume $V(K_\lambda)$ is a polynomial in λ whose coefficients $W_i(K)$ are geometric invariants of K:

$$V(K + \lambda B) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\lambda^i. \tag{1}$$

The functionals $W_i(K)$, $i = 0, \cdots, n$, are called the ith *quermassintegrals* of K. The followings are true:

$$W_0(K) = V(K); \quad nW_1(K) = S(K); \quad W_n(K) = \omega_n,$$

where $V(K)$ and $S(K)$ are the volume and surface area of K, respectively and ω_n is the volume of the unit ball B in E^n. If $K_1, \cdots, K_r$ are convex bodies in E^n and $\lambda_1, \cdots, \lambda_r$ range over the positive real numbers, then the volume of $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial, of degree n , in $\lambda_1, \cdots, \lambda_r$. That is,

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum V(K_{i_1}, \cdots, K_{i_n})\lambda_{i_1} \cdots \lambda_{i_n}, \tag{2}$$

where $i_1, \cdots, i_n$ range independently over $1, \cdots, r$. The coefficients $V(K_{i_1}, \cdots, K_{i_n})$ depending on $K_1, \cdots, K_n$ are symmetric in their variables. This coefficient is called *mixed volumes* of $K_{i_1}, \cdots, K_{i_n}$. It follows from (1) and (2) that

$$W_i(K) = V(\underbrace{K, \cdots, K}_{n-i}, \underbrace{B, \cdots, B}_{i}). \tag{3}$$

Now we need to introduce another interpretation of quermassintegrals in terms of surface area measure. Let K be a convex body in E^n. For each $\beta \in \mathcal{B}(E^n)$, a Borel measurable set in E^n, $\sigma(K, \beta)$ is the set of all $u \in S^{n-1}$ such that u is an outer normal to K at the boundary point of $K \cap \beta$. For $u \in \mathcal{B}(S^{n-1})$, a Borel set in S^{n-1}, $\sigma^{-1}(K, u)$ is the set of boundary points of K at which there exists an outer normal in u. Then for all convex body in E^n, it is possible to show the existence of the measure denoted by $S_{n-1}(K,u)$ on $\mathcal{B}(S^{n-1})$ which is the $(n-1)$-dimensional Hausdorff measure of $\sigma^{-1}(K,u)$. Then it is well known that

$$(4)\quad V(K_1,\cdots,K_{n-1},K_n)=\frac{1}{n}\int_{S^{n-1}}h(K_n,u)dS(K_1,\cdots,K_{n-1},u)$$

and

$$(5)\; S_{n-1}(\sum_{i=1}^{r}\lambda_iK_i,u)=\sum_{i_1=1}^{r}\cdots\sum_{i_{n-1}=1}^{r}S(K_{i_1},\cdots,K_{i_{n-1}},\cdot)\lambda_{i_1}\cdots\lambda_{i_{n-1}}.$$

Introducing the notation:

$$S_i(K,u)=S(\underbrace{K,\cdots,K}_{i},\underbrace{B,\cdots,B}_{n-i-1},u),\qquad i=0,1,\cdots,n-1,$$

representations for the quermassintegrals are

$$\begin{aligned} W_i(K) &= \frac{1}{n}\int_{S^{n-1}}h(K,u)dS_{n-i-1}(K,u), \qquad i=0,1,\cdots,n-1,\\ (6)\qquad &= \frac{1}{n}\int_{S^{n-1}}dS_{n-i}(K,u), \qquad i=1,\cdots,n.\end{aligned}$$

Let f be a nonnegative measurable function on S^{n-1} for which the integrals in (7) are finite. At each boundary point of K at which u is an outer normal add $\theta f(u)u$, $0<\theta\le t$, $t>0$. The resultant (probably nonconvex) set will be called a generalized outer parallel set of K at distance t denoted by $K_{tf(u)}$. It follows from a result of [9] that

$$(7)\quad V(K_{tf(u)})=V(K)+\frac{1}{n}\sum_{i=1}^{n}\binom{n}{i}t^i\int_{S^{n-1}}f(u)^i dS_{n-1}(K,u),$$

where the S_i are the area measures.

Sangwine-Yager [6] used (7) to obtain an integral formula as a lower bound of a mixed volume: If K and L are convex bodies in E^n and the origin is in the interior of L, then

$$(8)\qquad V(K,L,\cdots,L)\ge\frac{1}{n}\int_{S^{n-1}}\rho(L,u)^{n-1}dS_1(K,u).$$

Let K_j be a convex body in E^n with $o \in K_j$, $1 \leq j \leq n$. Then we define the *dual mixed volumes* $\tilde{V}(K_1, \cdots, K_n)$ by

$$\tilde{V}(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n, u) du, \tag{9}$$

where du signifies the area element on S^{n-1}. Let

$$\tilde{V}_i(K_1, K_2) = \tilde{V}(\underbrace{K_1, \cdots, K_1}_{n-i}, \underbrace{K_2, \cdots, K_2}_{i}).$$

The *dual quermassintegrals* are the special dual mixed volumes defined by

$$\tilde{W}_i(K) = \tilde{V}_i(K, B).$$

Note that $\tilde{V}_0(K, B) = V(K)$ is the volume of K, while $\tilde{V}_n(K) = \omega_n$ does not depend on K.

3. Main Results

In the following theorem, we develop a lower bound for $W_1(K)W_1(K^*)$ for a convex body K in E^n.

THEOREM 1. *Let K be a convex body in E^n. Then the quermassintegrals $W_1(K)$ and $W_1(K^*)$ satisfy*

$$W_1(K)W_1(K^*) \geq \left(\frac{r_i}{r_e}\right)^{n-1} \omega_n^2$$

where $\frac{r_i}{r_e} = \max\limits_{p \in K}\{\frac{r_i(p)}{r_e(p)} | r_i(p)$ is the radius of the largest ball contained in K and centered at p and $r_e(p)$ is the radius of the smallest ball containing K and centered at $p\}$.

Proof. By (8), we have

$$V(B, K, \cdots, K) \geq \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-1} du$$

and

$$V(B, K^*, \cdots, K^*) \geq \frac{1}{n} \int_{S^{n-1}} \rho(K^*, u)^{n-1} du$$

where we note that $S_1(B, u) = du$.

Multiply both sides of the above two inequalities and use $\rho(K^*, u) = 1/h(K, u)$ and the Cauchy-Schwartz inequality to obtain

$$\begin{aligned} & n^2 V(B, K, \cdots, K) V(B, K^*, \cdots, K^*) \\ & \geq \left(\int_{S^{n-1}} \rho(K, u)^{n-1} du\right) \left(\int_{S^{n-1}} \frac{1}{h(K, u)^{n-1}} du\right) \\ & \geq \left(\int_{S^{n-1}} \frac{\sqrt{\rho(K, u)^{n-1}}}{\sqrt{h(K, u)^{n-1}}} du\right)^2 \\ & \geq \left(\frac{r_i}{r_e}\right)^{n-1} \left(\int_{S^{n-1}} du\right)^2 \\ & = \left(\frac{r_i}{r_e}\right)^{n-1} O_{n-1}^2 \end{aligned}$$

where O_{n-1} is the $(n-1)$-dimensional volume of the unit sphere S^{n-1}.

The last inequality follows since $h(K, u) \leq r_e$ and $\rho(K, u) \geq r_i$ and the equality follows from (6). □

As a special case of Theorem 1 we obtain the following result.

COROLLARY 1. *Let K be a centrally symmetric convex body in E^n. Then*

$$W_1(K) W_1(K^*) \geq \left(\frac{r_{in}}{r_{out}}\right)^{n-1} \omega_n^2$$

where r_{in} and r_{out} are the inradius and outradius of K, respectively.

Proof. It is obvious, because of $\frac{r_i}{r_e} = \frac{r_{in}}{r_{out}}$ for centrally symmetric convex body. □

The *Santaló point* of K is often defined as the unique point in the interior of K with respect to which the volume of the polar dual is a minimum.

Ghandehari [4] gave a upper bound for the product of the dual quermassintegral of any index and that of its polar dual:

Assume K is a convex body in R^n and K^* is the polar dual of K with respect to Santaló point. Then

$$\tilde{W}_i(K) \tilde{W}_i(K^*) \leq \omega_n^2.$$

Now we obtain a lower bound for the product of the dual quermass-integral of any index of a convex body and that of its polar dual.

THEOREM 2. *Let K be a convex body in E^n. Then the dual quermassintegrals $\tilde{W}_i(K)$ and $\tilde{W}_i(K^*)$ satisfy*

$$\tilde{W}_i(K)\tilde{W}_i(K^*) \geq \left(\frac{r_i}{r_e}\right)^{n-i} \omega_n^2 \qquad i = 0, 1, \cdots, n.$$

Proof. By (9), we have

$$\tilde{V}_i(K, B) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} du$$

and

$$\tilde{V}_i(K^*, B) = \frac{1}{n} \int_{S^{n-1}} \rho(K^*, u)^{n-i} du.$$

Multiply both sides of the above two inequalities and use $\rho(K^*, u) = 1/h(K, u)$ and the Cauchy-Schwartz inequality to obtain

$$\begin{aligned} n^2 \tilde{V}_i(K, B)\tilde{V}_i(K^*, B) &= \left(\int_{S^{n-1}} \rho(K, u)^{n-i} du\right)\left(\int_{S^{n-1}} \rho(K^*, u)^{n-i} du\right) \\ &\geq \left(\int_{S^{n-1}} \sqrt{\left(\frac{\rho(K, u)}{h(K, u)}\right)^{n-i}} du\right)^2 \\ &\geq \left(\frac{r_i}{r_e}\right)^{n-i} \left(\int_{S^{n-1}} du\right)^2 \\ &= \left(\frac{r_i}{r_e}\right)^{n-i} O_{n-1}^2. \end{aligned}$$

The last inequality follows from $\frac{\rho(K,u)}{h(K,u)} \geq \frac{r_i}{r_e}$. Using $O_{n-1} = n\omega_n$, we obtain the desired result. □

COROLLARY 2. *Let K be a centrally symmetric convex body with center o in E^n and K^* polar dual of K. Then*

$$\tilde{W}_i(K)\tilde{W}_i(K^*) \geq \left(\frac{r_{in}}{r_{out}}\right)^{n-i} \omega_n^2 \qquad i = 0, 1, \cdots, n$$

where r_{in}, r_{out} are the inradius, outradius of K, respectively.

Proof. It is obvious, since $\frac{r_i}{r_e} = \frac{r_{in}}{r_{out}}$ for a centrally symmetric convex body. □

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Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea

E-mail: ydchai@yurim.skku.ac.kr

E-mail: yslee@math.skku.ac.kr