Bull. Korean Math. Soc. **36** (1999), No. 4, pp. 763–770

MEROMORPHIC UNIVALENT HARMONIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

JAY M. JAHANGIRI AND HERB SILVERMAN

ABSTRACT. The purpose of this paper is to give sufficient coefficient conditions for a class of univalent harmonic functions that map each $|z| = r > 1$ onto a curve that bounds a domain that is starlike with respect to origin. Furthermore, it is shown that these conditions are also necessary when the coefficients are negative. Extreme points for these classes are also determined. Finally, comparable results are given for the convex analog.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset C$ if both u and v are real harmonic in D. In any simply connected domain we write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D. See [1]. There are numerous papers on univalent harmonic functions defined on the domain $U = \{z : |z| < 1\}$.

$$f(z) = h(z) + \overline{g(z)} + A \log|z|, \tag{1}$$

where $h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n}$, $g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n}$, $0 \leq |\beta| < |\alpha|$, and $a = \overline{f}_{\bar{z}}/f_z$ is analytic and satisfies $|a(z)| < 1$ for $z \in \tilde{U}$. Hengartner and Schober [3] used the representation (1) to obtain coefficient bounds and distortion theorems. In this note, we give sufficient coefficient conditions for which functions of the form (1) will be univalent.

Received March 4, 1999.

1991 Mathematics Subject Classification: 30C45, 30C50.

Key words and phrases: harmonic, univalent, starlike, and convex functions.

The work of the first author was supported by KSU Research Council Grant 1999.

Under certain restrictions, we also give necessary and sufficient coefficient conditions for functions to be harmonic and starlike. Finally, we characterize the extreme points for such classes of functions.

2. Main Results

THEOREM 1. *Set* $f(z) = h(z) + \overline{g(z)} + A \log|z|$, *where* $h(z) = \alpha z + \sum_{n=0}^{\infty} a_n z^{-n}$ *and* $g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n}$, $0 \le |\beta| < |\alpha|$. *If*

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \le |\alpha| - |\beta| - |A|,$$

then f *is orientation preserving and univalent in* $\tilde{U}$.

Proof. To show that f is orientation preserving, we need to show that $|a(z)| = |\overline{f}_{\bar{z}}/f_z| < 1,\ z \in \tilde{U}$. We have

$$\begin{aligned} |a(z)| &= \left| \frac{2zg'(z) + \bar{A}}{2zh'(z) + A} \right| \\ &= \left| \frac{2\beta z - 2\sum_{n=1}^{\infty} n b_n z^{-n} + \bar{A}}{2\alpha z - 2\sum_{n=1}^{\infty} n a_n z^{-n} + A} \right| \\ &< \frac{2|\beta| + 2\sum_{n=1}^{\infty} n|b_n| + |A|}{2|\alpha| - 2\sum_{n=1}^{\infty} n|a_n| - |A|} \le 1, \end{aligned}$$

and f is orientation preserving. The univalence follows upon noting that

$$\begin{aligned} &\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2) + A(\log|z_1| - \log|z_2|)} \right| \\ &\ge 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2) + A(\log|z_1| - \log|z_2|)} \right| \\ &= 1 - \left| \frac{\beta(z_1 - z_2) + \sum_{n=1}^{\infty} b_n (z_1^{-n} - z_2^{-n})}{\alpha(z_1 - z_2) + \sum_{n=1}^{\infty} a_n (z_1^{-n} - z_2^{-n}) + A(\log|z_1| - \log|z_2|)} \right| \\ &> 1 - \frac{|\beta| + \sum_{n=1}^{\infty} n|b_n|}{|\alpha| - \sum_{n=1}^{\infty} n|a_n| - |A|} \ge 0. \end{aligned}$$

□

We denote by $\mathcal{H}_o^*$ the subclass of harmonic orientation preserving functions f that are starlike with respect to the origin in $\tilde{U}$ and are of the form

$$f(z) = h(z) + \overline{g(z)}, \tag{2}$$

where .

$$h(z) = \alpha z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \beta z + \sum_{n=1}^{\infty} b_n z^{-n}, \quad 0 \leq |\beta| < |\alpha|, \tag{3}$$

and $a = \overline{f}_{\bar{z}}/f_z$ is analytic and satisfies $|a(z)| < 1$ for $z \in \tilde{U}$.

A necessary and sufficient condition for such f to be starlike in $\tilde{U}$ is (see [2]) that for each z, $|z| = r > 1$, we have $\frac{\partial}{\partial\theta}(arg\ f(re^{i\theta})) > 0$, $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, $r > 1$.

We now give a sufficient coefficient condition for starlikeness of harmonic meromorphic functions.

THEOREM 2. *If f of the form (2) satisfies the inequality*

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq |\alpha| - |\beta|,$$

then f is harmonic univalent in $\tilde{U}$ and $f \in \mathcal{H}_o^*$.

Proof. That f is harmonic univalent and orientation preserving follows from Theorem 1. To show starlikeness, we note for $|z| = r > 1$, that

$$\begin{aligned}\frac{\partial}{\partial\theta}(arg\ f(re^{i\theta})) &= Im\frac{\partial}{\partial\theta}(\log\ f(re^{i\theta})) \\ &= Re\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} := Re\frac{A(z)}{B(z)}.\end{aligned}$$

Now $Re(A(z)/B(z)) \geq 0$ if and only if $|1 + A(z)/B(z)| \geq |1 - A(z)/B(z)|$, or equivalently, $|A(z) + B(z)| - |A(z) - B(z)| \geq 0$. We have

$$
\begin{aligned}
&|A(z)+B(z)|-|A(z)-B(z)| \\
&= |h(z)+zh'(z)+\overline{g(z)-zg'(z)}|-|h(z)-zh'(z)+\overline{g(z)+zg'(z)}| \\
&= \left|2\alpha z+\sum_{n=1}^{\infty}a_nz^{-n}-\sum_{n=1}^{\infty}na_nz^{-n}+\overline{\sum_{n=1}^{\infty}b_nz^{-n}+\sum_{n=1}^{\infty}nb_nz^{-n}}\right| \\
&\quad -\left|2\overline{\beta z}+\sum_{n=1}^{\infty}a_nz^{-n}+\sum_{n=1}^{\infty}na_nz^{-n}+\overline{\sum_{n=1}^{\infty}b_nz^{-n}-\sum_{n=1}^{\infty}nb_nz^{-n}}\right| \\
&= \left|2\alpha z-\sum_{n=1}^{\infty}(n-1)a_nz^{-n}+\overline{\sum_{n=1}^{\infty}(n+1)b_nz^{-n}}\right| \\
&\quad -\left|2\overline{\beta z}+\sum_{n=1}^{\infty}(n+1)a_nz^{-n}-\overline{\sum_{n=1}^{\infty}(n-1)b_nz^{-n}}\right| \\
&\geq 2|\alpha||z|-\sum_{n=1}^{\infty}(n-1)|a_n||z|^{-n}-\sum_{n=1}^{\infty}(n+1)|b_n||z|^{-n} \\
&\quad -2|\beta||z|-\sum_{n=1}^{\infty}(n+1)|a_n||z|^{-n}-\sum_{n=1}^{\infty}(n-1)|b_n||z|^{-n} \\
&= 2|z|\left\{|\alpha|-|\beta|-\sum_{n=1}^{\infty}n(|a_n|+|b_n|)|z|^{-n-1}\right\} \\
&\geq 2|z|\left\{|\alpha|-|\beta|-\left(\sum_{n=1}^{\infty}n(|a_n|+|b_n|)\right)\right\}\geq 0,
\end{aligned}
$$

and the result follows. □

Denote by $\mathcal{TH}_o^*$ the subfamily of $\mathcal{H}_o^*$ consisting of functions f of the form (2) for which the coefficients in (3) are restricted by

$$
\begin{aligned}
&h(z)=-\alpha z-\sum_{n=1}^{\infty}a_nz^{-n}, \\
(4)\qquad & \\
&g(z)=\beta z-\sum_{n=1}^{\infty}b_nz^{-n}\quad (\alpha>\beta\geq 0,\ a_n\geq 0,\ b_n\geq 0).
\end{aligned}
$$

We next show that the sufficient condition for starlikeness is also necessary for functions in $\mathcal{TH}_o^*$.

THEOREM 3. *Let $f(z) = h(z) + \overline{g(z)}$ where h and g are of the form (4). A necessary and sufficient condition for f to be in $\mathcal{TH}_o^*$ is that*

$$\sum_{n=1}^{\infty} n(a_n + b_n) \leq \alpha - \beta. \tag{5}$$

Proof. In view of Theorem 2, we need only show that $f \notin \mathcal{TH}_o^*$ if the coefficient inequality (5) does not hold. In fact, we can do more by showing that f would not even be univalent. Setting $z = r > 1$ and differentiating f with respect to r we obtain, $f'(r) = -\alpha+\beta+\sum_{n=1}^{\infty} na_n r^{-n-1} + \sum_{n=1}^{\infty} nb_n r^{-n-1}$. Since $f'(r) \to -\alpha + \beta + \sum_{n=1}^{\infty} na_n + \sum_{n=1}^{\infty} nb_n > 0$ as $r \to 1^-$ and $f'(r) \to -\alpha + \beta < 0$ as $r \to \infty$, there must exist an r_o, $1 < r_o < \infty$, for which $f'(r_o) = 0$. Hence, $f(r)$ must have a local minimum on the positive real interval $(1, \infty)$ and so cannot even be one-to-one there. □

Consequently, we obtain the following

COROLLARY 1. *$f \in \mathcal{TH}_o^*$ if and only if f is harmonic univalent in $\tilde{U}$.*

We next give a distortion result.

THEOREM 4. *If $f \in \mathcal{TH}_o^*$ and $|z| = r > 1$, then*

$$(\alpha - \beta)r - (\alpha - \beta)r^{-1} \leq |f(z)| \leq (\alpha + \beta)r + (\alpha - \beta)r^{-1}.$$

Proof. For $f \in \mathcal{TH}_o^*$, we see from (4) that

$$|f(z)| \leq (\alpha + \beta)r + \sum_{n=1}^{\infty}(a_n + b_n)r^{-n} \leq (\alpha + \beta)r + \sum_{n=1}^{\infty}(a_n + b_n)r^{-1}$$

and

$$|f(z)| \geq (\alpha - \beta)r - \sum_{n=1}^{\infty}(a_n + b_n)r^{-n} \geq (\alpha - \beta)r - \sum_{n=1}^{\infty}(a_n + b_n)r^{-1}.$$

Now the theorem follows by applying (5) to the above two inequalities. □

3. Extreme Points

The family $\mathcal{TH}_o^*$ is not locally uniformly bounded. For instance, $f_n(z) = -nz$ is in $\mathcal{TH}_o^*$ for every positive integer n, but $f_n(z_o)$ is not uniformly bounded for any z_o in $\tilde{U}$. However, the family is locally uniformly bounded if we fix the coefficient of z. In this section, we examine the extreme points for functions in $\mathcal{TH}_o^*$, for each fixed α, when f is defined by (4). This family is still not compact under the topology of locally uniform convergence. To see this, observe that for $n = 1, 2, ...,$

$$f_n(z) = -\alpha z + \frac{\alpha n}{n+1}\bar{z} \in \mathcal{TH}_o^*$$

but

$$lim_{n\to\infty} f_n(z) = -\alpha z + \alpha\bar{z} \notin \mathcal{TH}_o^*.$$

Nevertheless, we can still use the coefficient bounds of Theorem 3 to determine the extreme points of the closed convex hull of $\mathcal{TH}_o^*$ (*clco* $\mathcal{TH}_o^*$).

THEOREM 5. *Set $h_o(z) = -z$, $g_o(z) = -z + \bar{z}$ and $h_n(z) = -z - z^{-n}/n$, $g_n(z) = -z - \bar{z}^{-n}/n$ $(n = 1, 2, \cdots)$. Then $f \in$ clco $\mathcal{TH}_o^*$ if and only if f can be expressed as $f(z) = \sum_{n=0}^{\infty}(\lambda_n h_n + \gamma_n g_n)$, where $\lambda_n \geq 0$, $\gamma_n \geq 0$, and $\sum_{n=0}^{\infty}(\lambda_n + \gamma_n) = \alpha$.*
In particular, the extreme points of clco $\mathcal{TH}_o^*$ are $\{h_n\}$, $\{g_n\}$, $(n = 0, 1, 2, \cdots)$.

Proof. Note first that

$$\sum_{n=0}^{\infty}(\lambda_n h_n + \gamma_n g_n) = -\sum_{n=0}^{\infty}(\lambda_n + \gamma_n)z + \gamma_o\bar{z} - \sum_{n=1}^{\infty}\frac{\gamma_n}{n}z^{-n} - \sum_{n=1}^{\infty}\frac{\gamma_n}{n}\bar{z}^{-n}$$

is in *clco* $\mathcal{TH}_o^*$.

Conversely, if $f \in$ *clco* $\mathcal{TH}_o^*$, then we may write $f(z) = -\alpha z + \beta\bar{z} - \sum_{n=1}^{\infty} z^{-n} - \sum_{n=1}^{\infty} b_n\bar{z}^{-n}$, where $\alpha > 0$, $\alpha \geq \beta \geq 0$, $a_n \geq 0$, $b_n \geq 0$, and $\sum_{n=1}^{\infty} n(a_n + b_n) \leq \alpha - \beta$. Setting $\lambda_n = na_n$ and $\gamma_n = nb_n$ $(n = 1, 2, \cdots)$, $\gamma_o = \beta$, $\lambda_o = \alpha - \beta - \sum_{n=1}^{\infty} n(\lambda_n + \gamma_n)$, we get $f(z) = \sum_{n=0}^{\infty}(\lambda_n h_n + \gamma_n g_n)$. □

4. The Convex Case

A function f of the form (2) is said to be convex in $\tilde{U}$ if it maps each $|z| = r > 1$ onto a curve that bounds a convex domain. Such functions f are characterized (see [2]) by

$$\frac{\partial}{\partial\theta}\left(arg\left\{\frac{\partial}{\partial\theta}f(re^{i\theta})\right\}\right) \geq 0, \;\; z = re^{i\theta}, \;\; 0 \leq \theta < 2\pi, \;\; r > 1.$$

One can apply this necessary and sufficient condition for convex functions, much like the characterization for starlike functions led to Theorems 2 and 3. The convex meromorphic harmonic functions were also studied by Jun [4]. We state our results as

THEOREM 6. *A sufficient condition for f of the form (2) to be convex in $\tilde{U}$ is that $\sum_{n=1}^{\infty} n^2(|a_n| + |b_n|) \leq |\alpha| - |\beta|$. This condition is also necessary when the coefficients of f are restricted by the conditions in* (4).

REMARK. The extreme points of the closed convex hull of convex harmonic functions in $\tilde{U}$ whose coefficients satisfy (4) may be obtained from Theorem 6, much as the starlike analog in Theorem 5 followed from Theorem 3. The extreme points $\{h_n\}$ and $\{g_n\}$ for this class are found to be those given in Theorem 5, with n replaced by n^2 in the denominator.

References

[1] J. Clunie and T. Sheil-Small, *Harmonic Univalent Functions*, Ann. Acad. Sci. Fenn. Ser. AI Math. **9** (1984), 3-25.

[2] P. L. Duren, *Univalent Functions*, Springer-Verlag, N.Y., 1983.

[3] W. Hengartner and G. Schober, *Univalent Harmonic Functions*, Trans. Amer. Math. Soc. **299** (1987), 1-31.

[4] S. H. Jun, *Planar harmonic mappings and curvature estimates*, J. Korean Math. Soc. **32** (1995), no. 4, 803-814.

[5] T. Sheil-Small, *Constants for planar harmonic mappings*, J. London Math. Soc. **42** (1990), no. 2, 237-248.

[6] H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl. **220** (1998), 283-289.

Jay M. Jahangiri, Kent State University, Burton, OH 44021, USA
E-mail: jay@geauga.kent.edu

Herb Silverman, College of Charleston, Charleston, SC 29424, USA
E-mail: silvermanh@cofc.edu