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HOMOTOPY FIXED POINT SET OF THE HOMOTOPY FIBRE

Hyang - Sook Lee

Abstract. Let $\mathcal{X}$ be a p-compact group, $\mathcal{Y} \to \mathcal{X}$ be a p-compact subgroup of $\mathcal{X}$ and $G \to \mathcal{X}$ be a p-compact toral subgroup of $\mathcal{X}$ with $(\mathcal{X}/\mathcal{Y})^{hG} \neq \emptyset$. In this paper we show that the homotopy fixed point set of the homotopy fibre $(\mathcal{X}/\mathcal{Y})^{hG}$ is $\mathbb{F}_p$-finite.

1. Introduction

Homotopy Lie group theory was recently invented by W. G. Dwyer and C. W. Wilkerson [3]. The basic philosophy behind the process was formulated by Rector [9] who suggested studying compact Lie group by looking at its classifying spaces and expressing the classical Lie group theory notions in terms of classifying spaces.

A loop space is a triple $\mathcal{X} = (\mathcal{X}, B\mathcal{X}, e)$, where $\mathcal{X}$ is a topological space, $B\mathcal{X}$ is a connected pointed classifying space of $\mathcal{X}$ and $e : \mathcal{X} \to \Omega B\mathcal{X}$ is a homotopy equivalence from $\mathcal{X}$ to the space $\Omega B\mathcal{X}$ of based loops in $B\mathcal{X}$. Such a loop space is called p-compact group if $\mathcal{X}$ is $\mathbb{F}_p$-finite and $B\mathcal{X}$ is $\mathbb{F}_p$-complete. Here the second condition is equivalent to that $\mathcal{X}$ is $\mathbb{F}_p$-complete and $\pi_0(\mathcal{X})$ is a finite p-group. The p-compact r-torus $BT = K((Z_p)^r, 2)$ is an example of p-compact group. Another example is the p-completion of compact Lie group G, $(C_{\mathbb{F}_p}(G), C_{\mathbb{F}_p}(BG), e)$ with $\pi_0(G)$ is a finite p-group where $e : C_{\mathbb{F}_p}(G) \simeq \Omega C_{\mathbb{F}_p}(BG)$.

Dwyer and Wilkerson proved a lot of properties of p-compact groups which are based on homotopy theoretic generalizations of compact Lie

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groups ([3], [4]). In particular, they showed that every p-compact group has a maximal torus, normalizer of the maximal torus and Weyl groups. More homotopy theories of p-compact groups are developed in [4], [6] and [7].

We use the following notations and terminology; Let p be a fixed prime number, $\mathbb{F}_p$ the field with p-elements, $\mathbf{Z}_p$ the ring of p-adic integers. All unspecified homology and cohomology are assumed with coefficients in $\mathbb{F}_p$. A graded vector space H^* over a field $\mathbb{F}_p$ is of *finite type* if each H^i is finite dimensional over $\mathbb{F}_p$ and is *finite dimensional* if in addition $H^i = 0$ for all but a finite number of i. A space X is $\mathbb{F}_p$*-finite* if H^*X is finite dimensional over a field $\mathbb{F}_p$. Bousfield and Kan [1] construct a functor $C_{\mathbb{F}_p}(\_)$ on the category of spaces, called $\mathbb{F}_p$*-completion functor*, together with a natural map $\epsilon_X : X \to C_{\mathbb{F}_p}(X)$ for any X. A space X is $\mathbb{F}_p$*-good* if $H_*\epsilon_X$ is an isomorphism and $\mathbb{F}_p$*-complete* if ϵ_X is a homotopy equivalence. A map $f : X \to Y$ is an $\mathbb{F}_p$*-equivalence* if it induces an isomorphism on $H^*(\_, \mathbb{F}_p)$. If X and Y are spaces then $Map(X, Y)$ denotes the function space of the maps from X to Y; the component containing a particular map or homotopy class f is $Map(X, Y)_f$. If f is a specific map then $Map(X, Y)_f$ is a pointed space with basepoint f. We assume that any space in this paper has the homotopy type of a CW-complex.

This paper is organized as follows.

As a background we give the basic definitions and properties of p-compact groups in section 2. The third section gives the proof of our following main result.

THEOREM 3.6. *Let $\mathcal{X}$ be a p-compact group. Let $\mathcal{Y} \to \mathcal{X}$ be a p-compact subgroup of $\mathcal{X}$ and $G \to \mathcal{X}$ be a p-compact toral subgroup of $\mathcal{X}$ with $(\mathcal{X}/\mathcal{Y})^{hG}$ is nonempty. Then the homotopy fixed point set of the homotopy fibre satisfies the cohomological finiteness condition.*

2. Preliminaries

In this section we summarize some basic definitions and properties about p-compact groups. We refer to the paper [3], [4] and [8]. Most of constructions are translated from the geometry of classical Lie group theory to the homotopical setting of loop spaces or p-compact groups.

A *homomorphism* $f : \mathcal{Y} \to \mathcal{X}$ of p-compact groups or loop spaces is a pointed map $Bf : B\mathcal{Y} \to B\mathcal{X}$. A homomorphism f is an *equivalence* if Bf is a homotopy equivalence and *trivial* if Bf is null homotopic. A homomorphism of p-compact groups is said to be a *monomorphism* if the homotopy fibre $\mathcal{X}/\mathcal{Y}$ is $\mathbb{F}_p$-finite, and an *epimorphism* if $\Omega\mathcal{X}/\mathcal{Y}$ is a p-compact group. A *subgroup* $\mathcal{Y} \to \mathcal{X}$ of p-compact group $\mathcal{X}$ is a monomorphism of p-compact groups. A sequence $\mathcal{Y} \to \mathcal{X} \to \mathcal{Z}$ of p-compact groups is short exact if the associated sequence $B\mathcal{Y} \to B\mathcal{X} \to B\mathcal{Z}$ is a fibration up to homotopy. In this case f is a monomorphism, g is an epimorphism and $f : \mathcal{Y} \to \mathcal{X}$ is a normal subgroup.

Two homomorphisms $f, g : \mathcal{Y} \to \mathcal{X}$ of p-compact groups are called *conjugate* if Bf and Bg are freely homotopic. For a homomorphism $h : \mathcal{Y} \to \mathcal{X}$ of p-compact groups and for a p-compact toral subgroup $i' : G \to \mathcal{X}$, we say that G is *subconjugate* to $\mathcal{Y}$ if there exists a homomorphism $j' : G \to \mathcal{Y}$ such that $h \circ j'$ and i' are conjugate.

For a group G, the homotopy fixed point set of the action of G on X, denoted X^{hG}, is G-equivariant mapping space $map^G(EG, X)$. If G acts trivially on X, then X^{hG} is $map(BG, X)$. Let X_{hG} be the Borel construction $EG \times_G X$. Then the space X^{hG} is isomorphic to the space of sections of the fibration $X_{hG} \to BG$. A G-map $f : X \to Y$ induces a map $f^{hG} : X^{hG} \to Y^{hG}$; if f is an ordinary (non-equivariant) homotopy equivalence, then f^{hG} is a homotopy equivalence.

REMARK 2.1. Let G be a loop space. A G-space X is defined to be a fibration $X_{hG} \to BG$ with X as the fibre, and an equivariant map $X \to Y$ to be a map of spaces together with an extension to a map $X_{hG} \to Y_{hG}$ of spaces over BG. The homotopy fixed point set of such an action of G on X is defined to be the space of sections of the fibration $X_{hG} \to BG$. A *proxy action* of G on X is a space Y which is homotopy equivalent to X together with an action of G on Y.

LEMMA 2.2. [3, 10.6] *Let $f : X \to Y$ be a map of G-spaces which is an ordinary (non-equivariant) fibration. Assume that Y is connected, and that the homotopy fibre of f is F. Then the map*

$$f^{hG} : X^{hG} \to Y^{hG}$$

induced by f is a fibration. Moreover, for any point $y \in Y^{hG}$ there is a proxy action $\alpha(y)$ of G on F such that the (homotopy) fibre of f^{hG} over y is equivalent to $F^{hG}_{\alpha(y)}$.

A p-compact torus T of rank r is a p-compact group such that BT is an Eilenberg-Mac Lane space of type $K((\mathbf{Z}_p)^r, 2)$. A *p-compact toral group* G is a p-compact group which is an extension of a p-compact torus by a finite p-group. A p-discrete torus $T_\infty = \{x \in T \mid x^{p^n} = 1 \text{ for some } n\}$ of rank r is a discrete group isomorphic to $(\mathbf{Z}/p^\infty)^r$. A *p-discrete toral group* G_∞ is a discrete group which is an extension of a p-discrete torus by a finite p-group. Let $f : G_\infty \to G$ be a homomorphism where G_∞ is a p-discrete toral group and G is a p-compact toral group. Then G_∞ is said to be a *discrete approximation* to G if Bf is an $\mathbb{F}_p$-equivalence.

PROPOSITION 2.3. [3, 6.8] *Let $f : G_\infty \to G$ be a discrete approximation of the p-compact toral group G, and let X be an $\mathbb{F}_p$-complete space with an action of G. Then f induces a homotopy equivalence $X^{hG} \to X^{hG_\infty}$.*

PROPOSITION 2.4. [3, 6.9] *Any p-compact toral group G has a discrete approximation $G_\infty \to G$.*

PROPOSITION 2.5. [3, 6.10] *Any p-discrete toral group G_∞ has a (functorial) closure $G_\infty \to Cl(G_\infty)$.*

REMARK 2.6. If G_∞ is a p-discrete toral group then the homotopy fibre of the map $BG_\infty \to BCl(G_\infty)$ is the same as the homotopy fibre $K((Q_p)^r, 1)$ of the completion map $BA \to C_{F_p}(BA)$ where A is isomorphic to $(Z/p^\infty)^r (r \geq 0)$ such that G_∞/A is a finite p-group. By obstruction theory this implies that if K_∞ is a p-discrete toral group then any homomorphism $K_\infty \to Cl(G_\infty)$ lifts up to conjugacy to a homomorphism $K_\infty \to G_\infty$.

For a homomorphism $f : \mathcal{Y} \to \mathcal{X}$ between p-compact groups, we define the *centralizer* $C_{\mathcal{X}}(f(\mathcal{Y}))$ (or $C_{\mathcal{X}}(Y)$ if f is understood) to be the loop space given by the triple $(\Omega map(B\mathcal{Y}, B\mathcal{X})_{Bf}, map(B\mathcal{Y}, B\mathcal{X})_{Bf}, id)$. The evaluation at the base point $ev : map(B\mathcal{Y}, B\mathcal{X})_{Bf} \to B\mathcal{X}$ establishes a homomorphism $C_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$ of loop spaces. If $\mathcal{Y}$ is a p-compact toral group, then the centralizer $C_{\mathcal{X}}(\mathcal{Y})$ is again a p-compact group and the evaluation $C_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$ is a monomorphism. A subgroup $\mathcal{Y} \to \mathcal{X}$ of a p-compact group $\mathcal{X}$ is called *central* if the monomorphism $C_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$

is an equivalence. For every p-compact group $\mathcal{X}$, the centralizer $C_{\mathcal{X}}(\mathcal{X})$ is a p-compact group and $Z(\mathcal{X}) := C_{\mathcal{X}}(\mathcal{X}) \to \mathcal{X}$ is the center of $\mathcal{X}$. The center $Z(\mathcal{X})$ of $\mathcal{X}$ is the maximal central subgroup of $\mathcal{X}$([3], [6]). A p-compact group $\mathcal{X}$ is *abelian* if the identity homomorphism of $\mathcal{X}$ is central.

3. Homotopy Fixed Point Set of the Homotopy Fibre

Let $i : H \to \mathcal{X}$ and $j : K \to \mathcal{X}$ be p-compact toral subgroups of p-compact group X. If there exist homomorphisms $f : H \to K$ and $g : K \to H$ such that $j \circ f \sim i$ and $i \circ g \sim j$, then we say H is *conjugate* to K in $\mathcal{X}$.

Let $H \to \mathcal{X}$ be a p-compact toral subgroup of a p-compact group $\mathcal{X}$. We define the *Weyl space* $\mathcal{W}_H(\mathcal{X})$ to be the space obtained by replacing $BH \to B\mathcal{X}$ by an equivalent fibration $BH' \to B\mathcal{X}$ and considering the space of self maps of BH' over $B\mathcal{X}$. If we assume that $BH \to B\mathcal{X}$ is already a fibration then $\mathcal{W}_H(\mathcal{X})$ is defined to be the space of self maps of BH over $B\mathcal{X}$. Then the composition of maps gives $\mathcal{W}_H(\mathcal{X})$ the structure of an associative topological monoid.

LEMMA 3.1. *Let $\mathcal{X}$ be a p-compact group and $i : H \to \mathcal{X}$ a p-compact toral subgroup of $\mathcal{X}$. Then any self map of BH over $B\mathcal{X}$ is a homotopy self equivalence of BH.*

Proof. Let $j : H_\infty \to H$ be a discrete approximation to a p-compact toral group H. By Remark 2.6, any map $f : BH \to BH$ over $B\mathcal{X}$ lifts to a map $\tilde{f} : BH_\infty \to BH_\infty$, i.e., to a homomorphism $f_\infty : H_\infty \to H_\infty$ over H. The homomorphism $i \circ j$ must have a trivial algebraic kernel since $H \to \mathcal{X}$ is a monomorphism. Now H_∞ is a p-discrete toral group, and hence it has an extension

$$1 \to T_\infty \to H_\infty \to W \to 1$$

where $T_\infty \approx (Z/p^\infty)^r$, for some r, is a p-discrete torus and W is a finite p-group. We consider the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_\infty & \longrightarrow & H_\infty & \longrightarrow & W & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle g_\infty} & & \Big\downarrow{\scriptstyle f_\infty} & & \Big\downarrow{\scriptstyle h} & & \\ 1 & \longrightarrow & T_\infty & \longrightarrow & H_\infty & \longrightarrow & W & \longrightarrow & 1 \end{array}$$

Any monic self map g_∞ is an isomorphism and right vertical map h is also isomorphic. It follows f_∞ is an isomorphism. Since H_∞ is a discrete approximation to H, $Bj : BH_\infty \to BH$ is $\mathbb{F}_p$-equivalent. Thus it induces f is a self homotopy equivalence since BH is also $\mathbb{F}_p$-complete. □

Lemma 3.1 implies that $\pi_0(\mathcal{W}_H(\mathcal{X}))$ is a group under composition. The group $\pi_0(\mathcal{W}_H(\mathcal{X}))$ is called *Weyl group* of H in $\mathcal{X}$ and we denote it by $W_H(\mathcal{X})$.

If $i : Y \to X$ and $f : G \to X$ are homomorphisms of loop spaces, then taking the homotopy pullback of $Bi : BY \to BX$ along Bf gives an action in the sense of Remark 2.1 of G on X/Y. The homotopy fixed point set of this action is denoted by $(X/Y)^{hf(G)}$, or $(X/Y)^{hG}$ if f is understood.

PROPOSITION 3.2. [3, 6.22] *Let $\mathcal{X}$ be a p-compact group, G a p-compact toral group and $f : G \to \mathcal{X}$ a homomorphism. Then there exists a finite p-group A and a homomorphism $A \to G$ such that the restriction map $C_{\mathcal{X}}(G) \to C_{\mathcal{X}}(A)$ is an equivalence.*

COROLLARY 3.3. *Let $\mathcal{X}$ be a p-compact group, G a p-compact toral group and $f : G \to \mathcal{X}$ a homomorphism. If $\mathcal{Y} \to \mathcal{X}$ be a p-compact subgroup of $\mathcal{X}$ and $(\mathcal{X}/\mathcal{Y})^{hG}$ is nonempty, then there is a finite p-group A such that the restriction $C_{\mathcal{Y}}(G) \to C_{\mathcal{Y}}(A)$ is an equivalence.*

Proof. If $(\mathcal{X}/\mathcal{Y})^{hG} \neq \emptyset$, then there is a section $s : BG \to (\mathcal{X}/\mathcal{Y})_{hG}$ since $(\mathcal{X}/\mathcal{Y})^{hG}$ is the space of sections of the fibration $(\mathcal{X}/\mathcal{Y})_{hG} \to BG$. The homotopy pullback of $B\mathcal{Y} \to B\mathcal{X}$ along Bf gives an action of G on $\mathcal{X}/\mathcal{Y}$. This induces a map $(\mathcal{X}/\mathcal{Y})_{hG} \to B\mathcal{Y}$. Then the composition of this map with s gives a homomorphism $G \to \mathcal{Y}$. Hence we apply the Proposition 3.2 to a p-compact group $\mathcal{Y}$. □

PROPOSITION 3.4. [3, 5.8] *Suppose that $\mathcal{Y}$ and $\mathcal{X}$ are p-compact groups, that A is a finite p-group acting on $B\mathcal{Y}$ and $B\mathcal{X}$ (not necessary in a base point preserving way), and that $f : B\mathcal{Y} \to B\mathcal{X}$ is an equivariant map. For each $x \in (B\mathcal{X})^{hA}$ let $\alpha(x)$ be the proxy action of A on $\mathcal{X}/\mathcal{Y}$. Then for each x, the space $(\mathcal{X}/\mathcal{Y})^{hA}_{\alpha(x)}$ is $\mathbb{F}_p$-complete.*

THEOREM 3.5. [3, 4.6] *Let Z be a space with an action of the finite p-group A. Assume that Z is $\mathbb{F}_p$-finite and that for each subgroup $F \subset A$, Z^{hF} is $\mathbb{F}_p$-complete. Then Z^{hA} is $\mathbb{F}_p$-finite.*

THEOREM 3.6. *Let $\mathcal{X}$ be a p-compact group. Let $i : \mathcal{Y} \to \mathcal{X}$ be a p-compact subgroup of $\mathcal{X}$ and $j : G \to \mathcal{X}$ be a p-compact toral subgroup of $\mathcal{X}$ with $(\mathcal{X}/\mathcal{Y})^{hG}$ is nonempty. Then $(\mathcal{X}/\mathcal{Y})^{hG}$ is $\mathbb{F}_p$-finite.*

Proof. There exists a finite p-group A and a homomorphism $k : A \to G$ such that the restriction $C_{\mathcal{X}}(G) \to C_{\mathcal{X}}(A)$ is an equivalence by Proposition 3.2. Thus $BC_{\mathcal{X}}(G)$ is homotopy equivalent to $BC_{\mathcal{X}}(A)$ for such A. Also $BC_{\mathcal{Y}}(G) \simeq BC_{\mathcal{Y}}(A)$ by Corollary 3.3. Now the homotopy pullback of $Bi : B\mathcal{Y} \to B\mathcal{X}$ along the map $id : B\mathcal{X} \to B\mathcal{X}$ gives an action of $\mathcal{X}$ on $\mathcal{X}/\mathcal{Y}$. This implies that $\alpha : (\mathcal{X}/\mathcal{Y})_{h\mathcal{X}} \to B\mathcal{Y}$ is homotopy equivalent. Hence the maps between mapping spaces $map(BG, (\mathcal{X}/\mathcal{Y})_{h\mathcal{X}})_{\beta\circ B\tilde{j}} \to map(BG, B\mathcal{Y})_{B\tilde{j}}$ ($\simeq BC_{\mathcal{Y}}(G)$) and $map(BA, (\mathcal{X}/\mathcal{Y})_{h\mathcal{X}})_{\beta\circ B\tilde{j}\circ Bk} \to map(BA, B\mathcal{Y})_{B\tilde{j}\circ Bk}$ ($\simeq BC_{\mathcal{Y}}(A)$) are homotopy equivalences, where $\tilde{j} : G \to \mathcal{Y}$ and $\beta : B\mathcal{Y} \to (\mathcal{X}/\mathcal{Y})_{h\mathcal{X}}$ such that $\beta \circ \alpha \simeq id_{(\mathcal{X}/\mathcal{Y})_{h\mathcal{X}}}$ and $\alpha \circ \beta \simeq id_{BY}$. Therefore we consider the following commutative square which the horizontal rows are homotopy equivalent

$$\begin{array}{ccc} map(BG, (\mathcal{X}/\mathcal{Y})_{h\mathcal{X}})_{\beta\circ B\tilde{j}} & \longrightarrow & map(BA, (\mathcal{X}/\mathcal{Y})_{h\mathcal{X}})_{\beta\circ B\tilde{j}\circ Bk} \\ \downarrow & & \downarrow \\ map(BG, B\mathcal{X})_{Bj} & \longrightarrow & map(BA, B\mathcal{X})_{Bj\circ Bk}. \end{array}$$

The homotopy fibres of the vertical maps are $(\mathcal{X}/\mathcal{Y})^{hG}$ and $(\mathcal{X}/\mathcal{Y})^{hA}$ respectively. Therefore $(\mathcal{X}/\mathcal{Y})^{hG}$ is homotopy equivalent to $(\mathcal{X}/\mathcal{Y})^{hA}$. By Proposition 3.4, $(\mathcal{X}/\mathcal{Y})^{hF}$ with proxy action is $\mathbb{F}_p$-complete for each finite subgroup $F \subset A$. Since $\mathcal{X}/\mathcal{Y}$ is $\mathbb{F}_p$-finite, $(\mathcal{X}/\mathcal{Y})^{hA}$ is $\mathbb{F}_p$-finite by Theorem 3.5. Thus we conclude $(\mathcal{X}/\mathcal{Y})^{hG}$ is $\mathbb{F}_p$-finite. □

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Department of Mathematics, Ewha Womans University, Seodaemoon-Ku, Seoul 120-750, Korea
E-mail: hsl@mm.ewha.ac.kr