Bull. Korean Math. Soc. **36** (1999), No. 4, pp. 747–754

HOMOLOGY OF THE DOUBLE LOOP SPACE OF THE HOMOGENEOUS SPACE $Sp(n)/U(n)$

YOUNGGI CHOI

ABSTRACT. We compute the mod p homology of the double loop space of $Sp(n)/U(n)$ by the Serre spectral sequence and the Eilenberg–Moore spectral sequence with the homology operations.

1. Introduction

Let $U(n)$ be the unitary group and $Sp(n)$ the sympletic group. Let $Map^*(S^k, M) = \Omega^k M$ be the k–fold loop space of a space M, the space of all the base point preserving continuous maps from S^k to M. A map $\phi : S^2 \to Sp(n)/U(n)$ between Riemannian manifolds is said to be harmonic if it is the critical point of the energy functional on $\Omega^2 Sp(n)/U(n)$ defined by $E(\phi) = \frac{1}{2}\int_{S^2} |d\phi(x)|^2 dx$. This means that harmonic maps are two dimensional analogue of geodesics, such as minimal surfaces. So it is meaningful to study the homology of the double loop space of $Sp(n)/U(n)$ from the view point of the Morse theory.

On the other hands, consider the space $Hol^*(S^2, Sp(n)/U(n))$ of all the base preserving holomorphic maps from the Riemannian sphere $S^2 = C \bigcup \infty$ to the homogeneous space $Sp(n)/U(n)$. Then forgetting the complex structure, we have the natural inclusion

$$Hol^*_k(S^2, Sp(n)/U(n)) \to \Omega^2_k Sp(n)/U(n)$$

where $k \in \pi_0(\Omega^2 Sp(n)/U(n)) = Z$. By exploiting the inclusion map, we can obtain the homological information of the space $Hol^*_k(S^2, Sp(n)/U(n))$ from the homology of $\Omega^2_k Sp(n)/U(n)$.

Received March 2, 1999.
1991 Mathematics Subject Classification: 55P35, 55R20.
Key words and phrases: loop space, homogeneous space, Serre spectral sequence, Dyer–Lashof operation.
The present research has been supported by the Research Fund of Seoul National University in 1998.

In these aspects we study the homology of the double loop space of $Sp(n)/U(n)$. Main tools of the computation are the Serre spectral sequence and the Eilenberg–Moore spectral sequence with homology operations. Throughout this paper the subscript of an element always means the degree of the element and p always stands for the odd prime.

2. Mod 2 Case

We compute the homology of the double loop space of $Sp(n)/U(n)$ with $\mathbb{F}_2$ coefficients. For $(n+1)$–fold loop spaces, there are homology operations

$$Q_i : H_q(\Omega^{n+1}X;\mathbb{F}_2) \longrightarrow H_{2q+i}(\Omega^{n+1}X;\mathbb{F}_2)$$

defined for $0 \le i \le n$ which is natural for $(n+1)$–fold loop spaces.

The cohomology of $Sp(n)/U(n)$ with Z coefficients is well-known [1], [3]. It is torsion free and even–dimensional.

THEOREM 2.1. *$H^*(Sp(n)/U(n));Z)$ is*

$$Z[c_2,\dots,c_{2n}]/\left(\sum_{i+j=2k}(-1)^i c_{2i}c_{2j}, k\ge 1\right).$$

In mod 2 coefficients, we get $c_{2i}^2 = 0, 1 \le i \le n$ from the relation. Hence we have the following.

COROLLARY 2.2. *$H^*(Sp(n)/U(n));\mathbb{F}_2)$ is $E(c_2,\dots,c_{2n})$, where $E(c_i)$ is the exterior algebra on c_i.*

Since $\pi_2(Sp(n)/U(n)) = Z$, $\pi_0(\Omega^2 Sp(n)/U(n)) = Z$. So each component of the space $\Omega^2 Sp(n)/U(n)$ can be labeled by the integer $k \in Z$ and we denote the k component of the space by $\Omega^2_k Sp(n)/U(n)$. Since all components are homotopy equivalent to each other, it is enough to compute the homology of any component to get the homology of $\Omega^2 Sp(n)/U(n)$.

THEOREM 2.3. *$H_*(\Omega_0^2 Sp(n)/U(n));\mathbb{F}_2)$ is*

$$\mathbb{F}_2[Q_1^a x_0 : a \geq 0] \otimes \mathbb{F}_2[Q_1^a x_{2i} : 1 \leq i \leq n-1, a \geq 0].$$

Here $\mathbb{F}_2[Q_1^a x_0 : a \geq 0] = \mathbb{F}_2[Q_1^a[1] * [-2^a] : a \geq 0]$ where $[1]$ is the image of the generator in $\tilde{H}_0(S^0;\mathbb{F}_2)$ for the map $S^0 \to \Omega^2 Sp(n)/U(n)$ and $*$ is the loop sum product.

Proof. We have the following morphism of fibrations:

$$\begin{array}{ccccc} \Omega^2 Sp(n) & \longrightarrow & \Omega_0^2 Sp(n)/U(n) & \longrightarrow & \Omega_0 U(n) \\ {\scriptstyle i}\downarrow & & \downarrow & & \downarrow{\scriptstyle j} \\ \Omega^2 Sp & \longrightarrow & \Omega_0^2 Sp/U & \longrightarrow & \Omega_0 U \end{array}$$

It is well-known that

$H_*(\Omega_0 U(n);\mathbb{F}_2) = \mathbb{F}_2[y_{2i} : 1 \leq i \leq n-1]$, $n \geq 2$,
$H_*(\Omega_0 U;\mathbb{F}_2) = \mathbb{F}_2[y_{2i} : i \geq 1]$,
$H_*(\Omega^2 Sp(n);\mathbb{F}_2) = \mathbb{F}_2[Q_1^a x_{4i+1} : 0 \leq i \leq n-1]$,
$H_*(\Omega^2 Sp;\mathbb{F}_2) = \mathbb{F}_2[x_{2i+1} : i \geq 0]$,
$H_*(\Omega_0^2 Sp/U;\mathbb{F}_2) = \mathbb{F}_2[Q_1^a x_{2i-2} : a \geq 0, i \geq 1] = \mathbb{F}_2[x_i : i \geq 1]$.

In order to calculate differentials of the Serre spectral sequence of the top row fibration, we first exploit differentials of the infinite dimensional case and use the naturality. Consider the Serre spectral sequence for the bottom row fibration converging to $H_*(\Omega_0^2 Sp/U;\mathbb{F}_2)$ with

$$E_2 = H_*(\Omega_0 U;\mathbb{F}_2) \otimes H_*(\Omega^2 Sp;\mathbb{F}_2).$$

Since the homology of the total space is just tensor products of those of the base and fiber space, we can deduce that the spectral sequence collapses at E_2. Since i_* and j_* are monomorphism, we can conclude that the spectral sequence for the upper fibration also collapses at the E_2 from the naturality. So it follows that the E_∞-term for the $H_*(\Omega_0^2 Sp(n)/U(n);\mathbb{F}_2)$ is

$$\mathbb{F}_2[y_{2i} : 1 \leq i \leq n-1] \otimes \mathbb{F}_2[Q_1^a x_{4i+1} : 0 \leq i \leq n-1].$$

However, we have that

$$H_*(\Omega_0^2 Sp/U;\mathbb{F}_2) = \mathbb{F}_2[Q_1^a x_{2i-2} : a \geq 0, i \geq 1] = \mathbb{F}_2[x_i : i \geq 1].$$

Therefore there is a choice of the generators such that

$$H_*(\Omega_0^2 Sp(n)/U(n);\mathbb{F}_2) = \mathbb{F}_2[Q_1^a x_{2i} : a \geq 0, 0 \leq i \leq n-1]. \qquad \square$$

3. Mod p Case

Now turn to the mod p case. To make basic data, we recall the followings:

$$\begin{aligned}&H_*(Sp/U;\mathbb{F}_p)=\mathbb{F}_p[x_{4i+2}:i\geq 0]\\&H_*(\Omega Sp/U;\mathbb{F}_p)=E(x_{4i+1}:i\geq 0)\\&H_*(U;\mathbb{F}_p)=E(x_{2i+1}:i\geq 0)\\&H_*(\Omega Sp;\mathbb{F}_p)=\mathbb{F}_p[x_{4i+2}:i\geq 0]\end{aligned}$$

First of all, we study the the Serre spectral sequence for the following fibration:

$$\Omega Sp \longrightarrow \Omega Sp/U \longrightarrow U$$

We have that $E_2=H_*(U;\mathbb{F}_p)\otimes H_*(\Omega Sp;\mathbb{F}_p)$. From above data we can obtain following transgressions:

$$d_{4i+3}(x_{4i+3})=x_{4i+2}.$$

The remaining generators $\{x_{4i+1}:i\geq 0\}$ survive permanently so that $H_*(\Omega Sp/U;\mathbb{F}_p)=E(x_{4i+1}:i\geq 0)$. We consider the following map of fibrations:

$$\begin{array}{ccccc}\Omega Sp(n+1) & \longrightarrow & \Omega Sp(n+1)/U(n+1) & \longrightarrow & U(n+1)\\ \downarrow & & \downarrow & & \downarrow\\ \Omega Sp & \longrightarrow & \Omega Sp/U & \longrightarrow & U\end{array}$$

We also have that

$$\begin{aligned}&H_*(U(n+1);\mathbb{F}_p)=E(x_{2i+1}:0\leq i\leq n),\\&H_*(\Omega Sp(n+1);\mathbb{F}_p)=\mathbb{F}_p[x_{4i+2}:0\leq i\leq n].\end{aligned}$$

Now we compute the homology of the loop space of $Sp(n+1)/U(n+1)$. We divide the computation by two cases on n. First we compute the Serre spectral sequence for the following fibration:

$$\Omega Sp(2n+1) \longrightarrow \Omega Sp(2n+1)/U(2n+1) \longrightarrow U(2n+1)$$

By the naturality of differentials, we get following transgressions from the infinite dimensional case:

$$d_{4i+3}(x_{4i+3}) = x_{4i+2}, 0 \leq i \leq n-1.$$

So we have that

$$H_*(\Omega Sp(2n+1)/U(2n+1);\mathbb{F}_p) = E(x_{4i+1} : 0 \leq i \leq n) \\ \otimes \mathbb{F}_p[x_{4i+2} : n \leq i \leq 2n].$$

Similarly we can compute the $2n$ case, so that

$$H_*(\Omega Sp(2n)/U(2n);\mathbb{F}_p) = E(x_{4i+1} : 0 \leq i \leq n-1) \\ \otimes \mathbb{F}_p[x_{4i+2} : n \leq i \leq 2n-1].$$

Hence we have the following theorem.

THEOREM 3.1.

$$H_*(\Omega Sp(n+1)/U(n+1);\mathbb{F}_p) = E(x_{4i+1} : 0 \leq 2i \leq n) \\ \otimes \mathbb{F}_p[x_{4i+2} : n < 2i+1 \leq 2n+1].$$

To develop further we recall the following result in [4].

THEOREM 3.2.

$$H_*(\Omega^2 SU(n+1);\mathbb{F}_p) = E(x_{2jp^i-1} : 0 < j \leq n,\ j \neq 0 \mod p,\ i \geq 0) \\ \otimes \mathbb{F}_p[y_{2jp^i-2} : 0 < j \leq n,\ j \neq 0 \mod p,\ p^i j > n]$$

where t is the smallest integer such that $p^t j > n$.

Note that in paper of [4] there are some minor mistakes in the degrees of some generators. The above statement is corrected form.

We have the following fibration

$$Sp(n+1) \longrightarrow SU(2n+2) \longrightarrow SU(2n+2)/Sp(n+1).$$

For odd primes, there is mod p cross sections $s : SU(2n+2)/Sp(n+1) \to SU(2n+2)$ [2] so that we get the following decomposition

$$SU(2n+2) \simeq_{(p)} SU(2n+2)/Sp(n+1) \times Sp(n+1)$$

where $\simeq$ is the homotopy equivalence localized at p. Hence looping twice, we get that

$$\Omega^2 SU(2n+2) \simeq_{(p)} \Omega^2 SU(2n+2)/Sp(n+1) \times \Omega^2 Sp(n+1).$$

This implies that the mod p homology of $\Omega^2 Sp(n+1))$ is the parts of direct summands of the mod p homology $\Omega^2 SU(2n+2))$. So in the same way as [4], we can obtain the following.

THEOREM 3.3. *$H_*(\Omega^2 Sp(n+1); \mathbb{F}_p)$ is*

$$E(x_{2jp^i-1} : j : odd,\ 0 < j \le 2n+1,\ j \neq 0 \mod p,\ i \ge 0)$$
$$\otimes\, \mathbb{F}_p[y_{2jp^i-2} : j : odd,\ 0 < j \le 2n+1,\ j \neq 0 \mod p,\ p^i j > 2n+1]$$

where t is the smallest integer such that $p^t j > 2n+1$.

Now we work on the homology of our target space.

THEOREM 3.4.

$$\begin{aligned}
&H_*(\Omega_0^2 Sp(n+1)/U(n+1); \mathbb{F}_p) = \mathbb{F}_p[x_{4i} : 0 < 2i \le n]\\
&\otimes E(x_{2jp^i-1} : j : odd, n < j \le 2n+1, j \neq 0 \mod p, i \ge 0)\\
&\otimes \mathbb{F}_p[y_{2jp^i-2} : j : odd, n < j \le 2n+1, j \neq 0 \mod p, i \ge 1]\\
&\otimes E(x_{2jp^{k+i}-1} : j : odd, 0 < j \le n, j \neq 0 \mod p, n < jp^k \le 2n+1, i \ge 0)\\
&\otimes \mathbb{F}_p[y_{2jp^{k+i}-2} : j : odd, 0 < j \le n, j \neq 0 \mod p, n < jp^k \le 2n+1, i \ge 1).
\end{aligned}$$

Proof. Before the main argument, it is necessary to consider the infinite dimensional case. The followings are well-known:

$$\begin{aligned}
H_*(\Omega_0^2 Sp/U; \mathbb{F}_p) &= \mathbb{F}_p[x_{4i} : i \ge 1]\\
H_*(\Omega_0 U; \mathbb{F}_p) &= \mathbb{F}_p[x_{2i} : i \ge 1]\\
H_*(\Omega^2 Sp; \mathbb{F}_p) &= E(x_{4i+1} : i \ge 0)
\end{aligned}$$

In the Serre spectral sequence converging to $H_*(\Omega_0^2 Sp/U; \mathbb{F}_p)$ for the following fibration

$$\Omega^2 Sp \longrightarrow \Omega_0^2 Sp/U \longrightarrow \Omega_0 U,$$

we can make an analysis of the behavior of differentials from the above homology information. Now every differential from each generator of the form x_{4i} for $i \ge 1$ is trivial so that they become permanent cycles, and differentials from other generators are detected as follows. If some x_{2k} happens to transgress to x_{2k-1}, then differentials behave after the following pattern:

$$\begin{aligned}
&d(x_{2k}^{p^i}) = x_{2kp^i-1},\ i \ge 0,\\
&d(x_{2kp}) = x_{2k-1} x_{2k}^{p-1},\\
&d(x_{2kp^2}) = x_{2k-1} x_{2k}^{p-1} x_{2kp}^{p-1},\\
&\quad\cdot\\
&\quad\cdot\\
&d(x_{2kp^\ell}) = x_{2k-1} x_{2k}^{p-1} \dots x_{2kp^{\ell-1}}^{p-1},\ \ell \ge 1.
\end{aligned}$$

Now to get $H_*(\Omega_0^2 Sp(n+1)/U(n+1);\mathbb{F}_p)$, we take into consideration of the Serre spectral sequence for the fibration

$$\Omega^2 Sp(n+1) \longrightarrow \Omega_0^2 Sp(n+1)/U(n+1) \longrightarrow \Omega_0 U(n+1).$$

From the naturality, we get that each generator of the form x_{4i} becomes a permanent cycle for $0 \le 2i \le n$. Since the degrees of all generators in $H_*(\Omega U(n+1);\mathbb{F}_p)$ are less than or equal to $2n$, the following parts,

$$E(x_{2jp^i-1} : j : odd,\ n < j \le 2n+1,\ j \not\equiv 0 \mod p,\ i \ge 0)$$
$$\otimes \mathbb{F}_p[y_{2jp^i-2} : j : odd,\ n < j \le 2n+1,\ j \not\equiv 0 \mod p,\ i \ge 1],$$

are surviving permanently from the dimensional reason.

Using differentials of the infinite dimensional case and the information of the E_2 term of the Eilenberg–Moore spectral sequence for the path–loop fibration converging to $H_*(\Omega^2 Sp(n+1)/U(n+1))$, we can compute the other differentials as follows. For an odd j with the condition that $0 < j \le n$ and $j \not\equiv 0 \mod p$, we divide two cases:

(1) there is no $k \ge 1$ such that $n < jp^k \le 2n+1$;

(2) there is some $k \ge 1$ such that $n < jp^k \le 2n+1$.

For the case (1), the following parts of $H_*(\Omega^2 Sp(n+1);\mathbb{F}_p)$,

$$E(x_{2jp^i-1} : i \ge 0) \otimes \mathbb{F}_p[y_{2jp^i-2} : p^i j > 2n+1],$$

are targets of the differentials, so that they do not survive.
For the case (2), even though $E(x_{2jp^i-1} : i \ge 0)$ can not survive,

$$\mathbb{F}_p[y_{2jp^i-2} : p^i j > 2n+1]$$

survive permanently. Moreover in that case

$$x_{2jp^i-1}(x_{2j}^{p^i})^{p-1}(x_{2jp}^{p^i})^{p-1}\cdots(x_{2jp^{k-1}}^{p^i})^{p-1}$$

survives for any $i \ge 0$, so that we represent it by $x_{2jp^{k+i}-1}$. Hence we get the result. $\square$

In fact, each result in Theorem 2.3 and Theorem 3.4 is consistent with the result of the E_2 term of the Eilenberg–Moore spectral sequence for the path–loop fibration converging to the homology of the double loop space.

References

[1] H. Cartan, *Démonstration homologique des théorèmes de périodicitè de Bott II*, Séminaire Cartan et Moore 1959/1960, Exposé 17 Ecole Norm. Sup. Paris.

[2] B. Harris, *On the homotopy groups of the classical groups*, Ann. Math. 74 (1961), 407-413.

[3] M. Mimura and H. Toda, *Topology of Lie groups, I and II*, Trans. Math. Monographs vol. 91, Amer. Math. Soc., 1991.

[4] D. C. Ravenel, *The homology and Morava K-theories of* $\Omega^2 SU(n)$, Forum Math. 5 (1993), 1-21.

DEPARTMENT OF MATHEMATICS EDUCATION, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA

E-mail: yochoi@plaza.snu.ac.kr