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CONTINUOUS SELECTIONS UNDER WEAKER SEPARATION AXIOMS AND REFLEXIVE BANACH SPACES

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Abstract. The paper is devoted to generalizations of continuity of set-valued mappings and some properties of hypertopologies on the collection of some subsets of a topological space. It is also dedicated to continuous selection theorems without relatively higher separation axioms. More precisely, we give characterizations of λ-collectionwise normality using continuous functions as in Michael's papers.

1. Introduction

Let X and Y be topological spaces, and 2^Y be the family of nonempty subsets of Y. A mapping $\Phi : X \to 2^Y$ is called a *set-valued mapping.* A *selection* for $\Phi : X \to 2^Y$ is a continuous map $f : X \to Y$ such that $f(x) \in \Phi(x)$ for every $x \in X$. A set-valued mapping $\Phi : X \to 2^Y$ is called *lower semi-continuous* (respectively, *upper semi-continuous*) or l.s.c. (respectively, u.s.c.) if for every open subset V of Y,

$$\Phi^{-1}(V) = \{x \in X : \Phi(x) \cap V \neq \emptyset\}$$

$$(\text{respectively}, \Phi^{\#}(V) = \{x \in X : \Phi(x) \subset V\})$$

is an open subset of X.

Let

$$\mathcal{F}(Y) = \{S \in 2^Y : S \text{ is closed}\}.$$

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In particular, if Y is a Banach space, then we let

$$\mathcal{F}_c(Y) = \{S \in 2^Y : S \text{ is closed and convex}\},$$

$$\mathcal{C}(Y) = \{S \in \mathcal{F}(Y) : S \text{ is compact or } S = Y\}.$$

This paper is dedicated to the theory of continuous selections of set-valued mappings which is a classical area of mathematics as well as an area which has been intensively developing in recent decades. The fundamental results in this theory stemmed from the mid 1950's by E. Michael [15,16,17]. Most of the classical Michael's selection theorems establish that existence of continuous selections for l.s.c. set-valued mapping $\Phi : X \to \mathcal{F}(Y)$ is equivalent to some higher separation axioms (i.e., paracompactness, collectionwise normality, normality, etc.) of X. One of the theorems is the following.

THEOREM 1.1 ([16, Theorem 3.2″]). *For a T_1 space X the following are equivalent:*

(a) *X is paracompact.*

(b) *If Y is a Banach space, then every l.s.c. $\Phi : X \to \mathcal{F}_c(Y)$ admits a selection.*

This theorem characterizes paracompactness using continuous functions rather than covering properties. Since the continuous functions are relatively easier to handle covering properties, the above Michael's theorem is important and applicable to certain proofs.

Let (Y, d) be a metric space and let for $S \in 2^Y$ and $\epsilon > 0$, $B^d_\epsilon(S)$ denote $\{y \in Y : d(y, S) < \epsilon\}$. A mapping $\Phi : X \to 2^Y$ is d-l.s.c. (respectively, d-u.s.c.) if, given $\epsilon > 0$, every $x \in X$ admits a neighborhood U such that for every $u \in U$,

$$\Phi(x) \subset B^d_\epsilon(\Phi(u)) \quad (\text{respectively, } \Phi(u) \subset B^d_\epsilon(\Phi(x))).$$

The following new important notions of continuity of set-valued mappings arise via combinations of the above versions of l.s.c. and u.s.c. Namely, a set-valued mapping Φ is *continuous* if it is both l.s.c. and u.s.c.; Φ is *d-continuous* if it is d-l.s.c. and d-u.s.c.; and finally Φ is *d-proximal continuous* (see [12]) if it is both l.s.c. and d-u.s.c.

A continuous Φ is not necessarily d-continuous and vice versa (see, e.g., [12, Proposition 2.5]), while every continuous or d-continuous Φ is d-proximal continuous (see Proposition 2.2 and 2.4). But, there are d-proximal continuous mappings Φ which are neither continuous nor d-continuous (see [12]). In view of that, we shall henceforth restrict our attention only to d-proximal continuity. This property, however, depends on the metric d on the range Y. To overcome this, following [12], we shall say that $\Phi : X \to 2^Y$ is *proximal continuous*, where Y is metrizable, if there exists a compatible metric d on Y such that Φ is d-proximal continuous.

An extension problem is one of the important branch in general topology. As we will see, selection theories are very closely related to extension problems (see Theorem 4.3 and 4.4). We consider a following general extension problem in general topology.

General Question ([18]): Let X and Y be two topological spaces with $A \subset X$ closed, and let $f : A \to Y$ be continuous. Under what conditions on X and Y, does f have a continuous extension over X (or at least over some open $U \supset A$)?

A classical and basic answer to the above question is the following.

THEOREM 1.2 (Tietze's extension Theorem). *Let X be a normal space. Then for every closed $A \subset X$ and every continuous function $f : A \to \mathbb{R}$, there exists a continuous extension of f over X.*

Collectionwise normality is another strengthening of normality, but it is weaker than paracompactness. We can also give a characterization of collectionwise normality using continuous functions as in Michael's papers (see [16], [17]). One of the purpose of this paper is to give characterizations of λ-collectionwise normality using continuous functions as in Michael's papers. The paper is also devoted to generalizations of continuity of set-valued mappings and some properties of hypertopologies on the collection of some subsets of a topological space.

This paper is organized as follows: Section 1 is the introduction. Section 2 consists of preliminaries which involve generalizations of set-valued continuous mappings. Section 3 is devoted to hypertopologies. Section 4 is dedicated to extensions and selections.

Throughout this paper, by a space we always mean a topological space. As far as topological concepts are concerned, we follow [7].

2. Generalizations of Set-valued Continuous Mappings

Let X be a space, (Y, d) be a metric space, and let τ be a topology on $\mathcal{F}(Y)$. We say [9] that a set-valued mapping $\Phi : X \to \mathcal{F}(Y)$ is τ-continuous if Φ is continuous as a single-valued mapping from X to the space $(\mathcal{F}(Y), \tau)$. So far, the two best known topologies on $\mathcal{F}(Y)$ are the Hausdorff metric topology and Vietories topology.

For $A, B \in \mathcal{F}(Y)$, we define $H(d) : \mathcal{F}(Y) \times \mathcal{F}(Y) \to \mathbb{R}$ by

$$H(d)(A, B) = \inf\{\epsilon > 0 : A \subset B^d_\epsilon(B) \text{ and } B \subset B^d_\epsilon(A)\}.$$

Then $H(d)$ is a metric on $\mathcal{F}(Y)$ which is called the Hausdorff metric, and the topology $\tau_{H(d)}$ on $\mathcal{F}(Y)$ generated by $H(d)$ is called the Hausdorff metric topology.

The Vietories topology τ_V on $\mathcal{F}(Y)$ is a topology generated by all collections of the form

$$\langle \mathcal{V} \rangle = \{S \in \mathcal{F}(Y) : S \cap V \neq \emptyset, V \in \mathcal{V}, \text{ and } S \subset \bigcup \mathcal{V}\},$$

where $\mathcal{V}$ runs over the finite families of open subsets of Y.

The d-proximal topology $\tau_{\delta(d)}$ on $\mathcal{F}(Y)$ is a topology generated by all collections of the form

$$\langle\langle \mathcal{V} \rangle\rangle = \{S \in \langle \mathcal{V} \rangle : d(S, Y \backslash \bigcup \mathcal{V}) > 0\},$$

where $\mathcal{V}$ is again a finite family of open subsets of Y.

We note that $\tau_{\delta(d)} \subset \tau_V \cap \tau_{H(d)}$.

PROPOSITION 2.1 ([10]). *Let X be a topological space, (Y, d) be a metric space, and let $\Phi : X \to \mathcal{F}(Y)$. Then*

(a) *Φ is d-continuous if and only if it is $\tau_{H(d)}$-continuous.*

(b) *Φ is continuous if and only if it is τ_V-continuous.*

(c) *Φ is d-proximal if and only if it is $\tau_{\delta(d)}$-continuous.*

The following proposition shows us that d-proximal continuity is a generalization of continuity of a set-valued mapping.

PROPOSITION 2.2. *Let X be a topological space and (Y, d) be a metric space. Then every continuous $\Phi : X \to \mathcal{F}(Y)$ is d-proximal continuous.*

Proof. Assume that $\Phi : X \to \mathcal{F}(Y)$ is continuous. Then Φ is l.s.c. and u.s.c. So it suffices to claim that if Φ is u.s.c., then it is d-u.s.c. Let $\epsilon > 0$ and $x \in X$. Then $B_\epsilon^d(\Phi(x))$ is open in Y. Since Φ is u.s.c., $\Phi^\#(B_\epsilon^d(\Phi(x))) = \{z \in X : \Phi(z) \subset B_\epsilon^d(\Phi(x))\}$ is open in X. Clearly, $x \in \Phi^\#(B_\epsilon^d(\Phi(x)))$. Therefore there is a neighborhood V of x such that $x \in V \subset \Phi^\#(B_\epsilon^d(\Phi(x)))$. Let $z \in V$. Then $z \in \Phi^\#(B_\epsilon^d(\Phi(x)))$, i.e., $\Phi(z) \subset B_\epsilon^d(\Phi(x))$. Thus Φ is d-u.s.c. □

We recall that a metric space (Y, d) is called a UC space (or an Atsuji space) if each real-valued continuous function $f : X \to \mathbb{R}$ is uniformly continuous.

THEOREM 2.3. *If (Y, d) is a UC space (or an Atsuji space), then a map $\Phi : X \to \mathcal{F}(Y)$ is continuous if and only if it is d-proximal continuous.*

Proof. It follows from Proposition 2.2 above and Proposition 2.3 in [10]. □

The following proposition shows us that d-proximal continuity is also a generalization of d-continuity of a set-valued mapping.

PROPOSITION 2.4. *Let X be a topological space and (Y, d) be a metric space. Then every d-continuous $\Phi : X \to \mathcal{F}(Y)$ is d-proximal continuous.*

Proof. Assume $\Phi : X \to \mathcal{F}(Y)$ is d-continuous. Then Φ is both d-l.s.c. and d-u.s.c. To show it is d-proximal continuous, it suffices to claim that if Φ is d-l.s.c., then it is l.s.c. Let Φ be d-l.s.c. and let $U \subset Y$ be open. We want to show that $\Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is open in X. Let $x \in \Phi^{-1}(U)$ and $\epsilon > 0$. Since Φ is d-l.s.c., there exists a neighborhood V of x such that if $z \in V$, then $\Phi(x) \subset B_\epsilon^d(\Phi(z))$. It is not difficult to show that $V \subset \Phi^{-1}(U)$. Therefore $\Phi^{-1}(U)$ is open in X. Hence Φ is l.s.c. and thus it is d-proximal continuous. □

THEOREM 2.5. *If (Y, d) is a totally bounded metric space, then a map $\Phi : X \to \mathcal{F}(Y)$ is d-continuous if and only if it is d-proximal continuous.*

Proof. It follows from Proposition 2.4 above and Proposition 2.4 in [10]. □

It should be mentioned here that a d-continuous set-valued mapping Φ is not necessarily d-proximal continuous and vice versa.

PROPOSITION 2.6 ([10, Proposition 2.5]). *The following two properties of a metric space (Y, d) are equivalent:*

(a) (Y, d) *is compact.*

(b) *Every $\Phi : X \to \mathcal{F}(Y)$ is d-continuous if and only if it is continuous.*

COROLLARY 2.7. *If (Y, d) is a compact metric space, then every d-proximal continuous $\Phi : X \to \mathcal{F}(Y)$ is continuous.*

Proof. We first recall that a metric space is compact if and only if it is totally bounded and complete. Let $\Phi : X \to \mathcal{F}(Y)$ be d-proximal. Then by Theorem 2.5, it is d-continuous and thus it is continuous by Proposition 2.6. □

It should be also mentioned that a d-continuous set-valued mapping Φ is not necessarily continuous and vice versa.

3. Hypertopologies

By a *hyperspace* of a T_1 topological space $(X, \mathcal{T})$ we mean the set $\mathcal{F}(X)$ of the nonempty closed subsets of X, endowed with a topology τ such that the function $i : (X, \mathcal{T}) \to (\mathcal{F}(X), \tau)$ defined as $i(x) = \{x\}$ is a homeomorphism onto its image. Since the beginning of this century some *hyperspace topologies*, also called *hypertopologies*, have been introduced and developed; in particular, the Hausdorff metric and Vietories topologies. These two topologies are very fine, at least in view of some applications. It is a remarkable fact that the most important hyperspace topologies arise as topologies induced by families of geometric set functionals. We give a particular attention to the interplay between hyperspaces and geometrical functional analysis. Although hyperspace topologies and related set convergence notions have been investigated since the beginning of the century, the approach we take to the subject reflects decisive modern contributions by mathematicians whose primary research interests lie outside general topology. The revival of the subject stems from work of Robert Wijsman[22] in the middle of 1960's, and its development over the next fifteen years was to a large extent in the hands of U. Mosco, R. Wets, H. Attouch, and their associates. This increasing interest is owing to usefulness of these in different fields of applications such as probability, statistics or variational problems for instance.

Throughout this section, we do an effort in understanding structures, common features, and general patterns of hypertopologies in order to find a common description for them. The papers [5], [21] or more recently [14], are partially or completely devoted to this goal, offering various possibilities of generalization.

Let us first describe notations we are going to deal with. For a topological space X and $E \subset X$, write

$$E^- = \{A \in \mathcal{F}(X) : A \cap E \neq \emptyset\},$$
$$E^+ = \{A \in \mathcal{F}(X) : A \subset E\};$$

further if $(X, \mathcal{U})$ is a uniform space, put

$$E^{++} = \{A \in \mathcal{F}(X) : \exists U \in \mathcal{U} \text{ with } U[A] \subset E\},$$

where $U[A] = \{x \in X : \exists a \in A \; with \; (x,a) \in U\}$. There are three types of hypertopologies: the *hit-and-miss*, the *proximal hit-and-miss*, and the *weak topologies generated by gap and excess functionals* on $\mathcal{F}(X)$, respectively.

Hit-and-miss topology: The abstract hit-and-miss topology on $\mathcal{F}(X)$ has as a subbase all sets of the from V^-, where V is open in X, plus all sets of the form $(B^c)^+$, where $B^c = X \backslash B$ and B ranges over a fixed nonempty subfamily $\triangle \subset \mathcal{F}(X)$. The well-known prototypes of hit-and-miss topologies are the *Vietories topology*, with $\triangle =$

$\mathcal{F}(X)$([4], [15]) and the *Fell topology*, with $\triangle$=nonempty closed compact subsets of X([4],[8]).

Proximal hit-and-miss topology: If $(X, \mathcal{U})$ is a uniform space and $(B^c)^+$ is replaced by $(B^c)^{++}$ in the above definition, we get the proximal hit-and-miss topology (or hit-and-far topology). For example, among its useful prototypes we can find the *proximal topology*, with $\triangle = \mathcal{F}(X)$([6]) or the *ball-proximal topology*, with $\triangle$= closed proper balls in a metric space X([4]).

Weak topologies generated by gap and excess functionals: In a metric space (X, d), we define the *distance* functional

$$d(x, A) = \inf\{d(x, a) : a \in A\}(x \in X, \emptyset \neq A \subset X),$$

the *gap* functional

$$D(A, B) = \inf\{d(a, B) : a \in A\}(A, B \subset X),$$

and the *excess* functional

$$e(A, B) = \sup\{d(a, B) : a \in A\}(A, B \subset X).$$

Then the so-called *weak hypertopologies* (or *initial topologies*) on $\mathcal{F}(X)$ are defined as the weak topologies generated by gap (in particular, distance) and excess functionals, where one of the set arguments of $D(A, B)$ and $e(A, B)$, respectively ranges over given subfamilies of $\mathcal{F}(X)$. As a prototype of weak topologies we should mention the *Wijsman topology*, denoted by $\tau_{W(d)}$, which is the weak topology generated by the distance functionals viewed as functionals of set argument.

THEOREM 3.1 ([6]). *Let X be a metrizable space, and let D denote the set of compatible metrics for X. Then the Vietories topology on $\mathcal{F}(X)$ is the weak topology determined by the family of distance functionals $\{d(x, \cdot) : x \in X, d \in D\}$.*

The following theorem is basically due to [4, 3.2.3] and [10, 2.4]:

THEOREM 3.2. *Let (Y, d) be a metric space. Then the following are equivalent:*

(1) *(Y, d) is totally bounded.*

(2) $\tau_{H(d)} = \tau_{W(d)}$ *on* $\mathcal{F}(Y)$.
(3) $(\mathcal{F}(Y), \tau_{H(d)})$ *is second countable.*
(4) *Every* $\Phi : X \to \mathcal{F}(Y)$ *is d-continuous if and only if it is d-proximal continuous.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is proved in [4, 3.2.3]. (1) $\Leftrightarrow$ (4) is proved by [10, 2.4] and Theorem 2.5. □

It is well-known [4] that if (Y, d) is a metric space, then the Wijsman topology $\tau_{W(d)}$ contains the Fell topology τ_F.

THEOREM 3.3. *Let* (Y, d) *be a metric space. Then the following are equivalent:*
(1) (Y, d) *is compact.*
(2) $\tau_{H(d)} = \tau_F$ *on* $\mathcal{F}(Y)$.
(3) *Every* $\Phi : X \to \mathcal{F}(Y)$ *is d-continuous if and only if it is continuous.*

Throughout the remainder of this section, we consider basic cardinal functions on hyperspaces which will be used in Section 4. A cardinal function is a function ϕ from the class of all topological spaces (or some precisely defined subclass) into the class of all infinite cardinal numbers such that $\phi(X) = \phi(Y)$ whenever topological spaces X and Y are homeomorphic. There are various topological invariant cardinal numbers that are assigned to each topological space X, e.g., the cardinality $|X|$, weight $w(X)$, character $\chi(X)$, density $d(X)$, and Lindelöff degree $L(X)$. Cardinal functions enable us to study general topology more systematically. All cardinal functions defined in the rest of this paper will assumed to have infinite values.

For any space $(X, \mathcal{T})$,

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\}$$

is called the *weight* of X. We can easily see that if $w(X) \leq \aleph_0$, then X is second countable.

Let $(X, \mathcal{T})$ be a space and $x \in X$. The *character* of $x \in X$, denoted by $\chi(x, (X, \mathcal{T}))$, or $\chi(x)$, is $min\{|\mathcal{B}(X)| : \mathcal{B}(X)$ is a local base at $x\}$.

The *character* of $(X, \mathcal{T})$, denoted by $\chi(X)$, is $\sup\{\chi(x, (X, \mathcal{T})) : x \in X\}$.

Note that if $\chi((X, \mathcal{T})) \leq \aleph_0$, then $(X, \mathcal{T})$ is first countable. We can easily see that $\chi(X) \leq \omega(X)$ for any space X, which implies that every second countable space is first countable.

THEOREM 3.4. *Let X be a Hausdorff space. Then $\chi(X) = \chi(\mathcal{F}(X))$, where $\mathcal{F}(X)$ has the Vietories topology τ_V.*

Proof. We first show that $\chi(\mathcal{F}(X)) \leq \chi(X)$. Let $\chi(X) = \kappa$ and $E = \{x_1, x_2, \dots, x_n\} \in \mathcal{F}(X)$. For each $i = 1, 2, \dots, n$ let $\mathcal{V}_i$ be a neighborhood base at x_i with $|\mathcal{V}_i| \leq \kappa$. Let $\mathcal{W}$ be the collection of all open sets of the form $\langle \mathcal{V} \rangle \cap \mathcal{F}(X)$, say $\langle \mathcal{V} \rangle = \langle V_1, V_2, \dots, V_n \rangle$, where each $V_i \in \mathcal{V}_i$. Then $|\mathcal{W}| \leq \kappa$. We claim that $\mathcal{W}$ is a neighborhood base at E. Let $\langle U_1, U_2, \dots, U_m \rangle \cap \mathcal{F}(X)$ be a neighborhood of E, where each U_j is open in X. Let $\mathcal{U} = \{U_1, U_2, \dots, U_m\}$. For each $x_i \in E$, $x_i \in U_j$ for some $U_j \in \mathcal{U}$. Since $\mathcal{V}_i$ is a neighborhood base at x_i, we can choose an element $V_i \in \mathcal{V}_i$ such that $V_i \subset \bigcap\{U_j \in \mathcal{U} : x_i \in U_j\}$. Then $E \in \langle V_1, V_2, \dots, V_n \rangle \cap \mathcal{F}(X) \subset \langle U_1, U_2, \dots, U_m \rangle \cap \mathcal{F}(X)$. This proves our claim. Hence $\mathcal{W}$ is a neighborhood base at E with $|\mathcal{W}| \leq \kappa$ and thus $\chi(\mathcal{F}(X)) \leq \kappa$. The inequality $\chi(X) \leq \chi(\mathcal{F}(X))$ easily follows from the fact that X is homeomorphic to the subspace $\mathcal{F}_1(X) \subset \mathcal{F}(X)$, where $\mathcal{F}_1(X)$ is the family of all one-point subsets of X. Therefore $\chi(X) = \chi(\mathcal{F}(X))$. □

COROLLARY 3.5. *Let X be a Hausdorff space. Then X is first-countable if and only if $\mathcal{F}(X)$ is first-countable.*

4. Extensions and Selections

Suppose that A is a subset of a topological space X. Let $C(X)$ be the collection of all real-valued continuous functions on X. Let $C^*(X)$ be the subset of $C(X)$ consisting of all bounded functions in $C(X)$. We say that A is C-embedded (respectively, C^*-embedded) in X if every function in $C(A)$ (respectively, $C^*(A)$) can be extended to a function in $C(X)$ (respectively, $C^*(X)$).

A subset A of a space X is said to be P^λ*-embedded*, where λ is an infinite cardinal, if every continuous pseudometric on A with $w(A) \leq \lambda$

can be extended to a continuous pseudometric on X with $w(X) \leq \lambda$. The subset A is *P-embedded* if every continuous pseudometric on A can be extended to a continuous pseudometric on X. The notion of *P^λ-embedded* was introduced by Arens[2] under the name of "λ-normally embedded". It is well-known ([1], Theorem 14.5) that a subset A of a space X is *P^λ-embedded* if and only if for every locally finite cozero-set cover $\mathcal{W}$ of A of cardinality $|\mathcal{W}| \leq \lambda$ there exists a locally finite cozero-set cover $\mathcal{U}$ of X such that $\mathcal{W}$ is refined by $\mathcal{U}|_A = \{U \cap A : U \in \mathcal{U}\}$, i.e., every locally finite cozero set cover $\mathcal{W}$ of A of cardinality $|\mathcal{W}| \leq \lambda$ has a refinement that can be extended to a locally finite cozero set cover of X.

We can generalize P^λ-embedding as follows ([10]): Let X be a space and $A \subset X$. We shall say that a map $g : A \to Y$ is *A-regular* if for every locally finite cozero set cover $\mathcal{V}$ of Y there exists a locally finite cozero set cover $\mathcal{U}$ of X such that $g(\mathcal{U}|_A)$ refines $\mathcal{V}$.

PROPOSITION 4.1. *Let X be a space, $A \subset X$, and let Y be a second countable space. Then every continuous map $g : A \to Y$ is A-regular whenever A is $P^{\aleph_0}$-embedded in X (or equivalently, A is C-embedded).*

Proof. We first note from [1, Theorem 16.3] that A is C-embedded if and only if it is $P^{\aleph_0}$-embedded. Suppose A is $P^{\aleph_0}$-embedded. Let $g : A \to Y$ be continuous and $\mathcal{V}$ be a locally finite cozero-set cover of Y with $|\mathcal{V}| \leq \aleph_0$(since Y is second countable). Let $\mathcal{W} = \{g^{-1}(V) : V \in \mathcal{V}\}$. Then $\mathcal{W}$ is a locally finite cozero set cover of A of cardinality $|\mathcal{W}| \leq \aleph_0$. Since A is $P^{\aleph_0}$-embedded, there exists a locally finite cozero set cover $\mathcal{U}$ of X such that $\mathcal{W}$ is refined by $\mathcal{U}|_A = \{U \cap A : U \in \mathcal{U}\}$. Then $g(\mathcal{U}|_A) = \{g(U \cap A) : U \in \mathcal{U}\}$ refines $\mathcal{V}$. Hence g is A-regular. □

REMARK. In fact, we may generalize Proposition 4.1 as follows (see [10]): If X is a space, $A \subset X$, and Y is a space, then every continuous map $g : A \to Y$ is A-regular whenever if A is $P^{w(Y)}$-embedded in X or A is C^*-embedded in X and $g(A) \subset Y$ is compact.

If $\lambda > \aleph_0$ we can ask for a characterization of P^λ-embedding in terms of extending covers. It seems unlikely that every locally finite cozero-set cover of A of cardinality at most λ can be extended to a locally finite cozero-set cover of X if and only if A is P^λ-embedded in X. So the following is a natural problem.

PROBLEM 4.2. Characterize A-regularity in terms of extending covers.

An normality can be characterized by C-embedding (Theorem 1.2), collectionwise normality can be characterized in terms of P-embedding ([1]). More generally, we have the following:

First, we recall that a space X is called *λ-collectionwise normal* if X is a T_1-space and every discrete collection $\mathcal{D}$ of closed subsets of X with $|\mathcal{D}| \leq \lambda$ can be separated by an open discrete collection $\mathcal{E} = \{E_D : D \in \mathcal{D}\}$ (i.e., $D \subset E_D$ for every $D \in \mathcal{D}$).

THEOREM 4.3. *The following properties of a T_1-space X are equivalent:*

(a) *X is λ-collectionwise normal.*

(b) *Every closed subset A of X is P^λ-embedded.*

(c) *Every continuous mapping $f : A \to Y$, where A is a closed subset of X and Y is a closed convex subset of a Banach space with $w(Y) \leq \lambda$ has a continuous extension on X.*

(d) *Every continuous mapping $f : A \to H(\lambda)$, where A is a closed subset of X and $H(\lambda)$ is the generalized Hilbert space of weight λ.*

Proof. For $(a) \Leftrightarrow (b)$, see Theorem 15.6 in [1]. For $(a) \Leftrightarrow (c) \Leftrightarrow (d)$, see Theorem 4.7 in [19]. □

COROLLARY 4.4. *The following properties of a T_1-space X are equivalent:*

(a) *X is collectionwise normal.*

(b) *Every closed subset of X is P-embedded.*

(c) *For every Banach space Y and for every l.s.c. $\Phi : X \to \mathcal{C}(Y)$, there is a selection.*

Proof. For $(a) \Leftrightarrow (b)$ is an immediate consequence of $(a) \Leftrightarrow (b)$ in Theorem 4.3. For $(a) \Leftrightarrow (c)$, see Theorem 3.2′ in [16]. □

In the rest of this section, we include recent results by Gutev and Nedev related to the more weakening of separation axioms on the domain of a l.s.c. set-valued mapping in comparison with Theorems 1.1, 4.3, and 4.4.

THEOREM 4.5 ([10]). *Let X be a topological space, Y a Banach space and $\Phi : X \rightarrow \mathcal{F}_c(Y)$ be d-proximal continuous, where d is the metric generated by a norm on Y. Then Φ admits a single-valued continuous selection.*

It is interesting that we can weaken the restriction on the continuity of Φ if Y is a reflexive Banach space in Theorem 4.5. More precisely, $\Phi : X \rightarrow \mathcal{F}_c(Y)$ is called *weakly continuous* if it is l.s.c. and the set $\Phi^{\#}(Y \backslash K)$ is open in X for every weakly compact $K \subset Y$. We can easily show that every d-proximal continuous $\Phi : X \rightarrow \mathcal{F}_c(Y)$ is weakly continuous.

THEOREM 4.6 ([13]). *Let X be a topological space, Y be a reflexive Banach space, and let $\Phi : X \rightarrow \mathcal{F}_c(Y)$ be weakly continuous. Then Φ admits a single-valued continuous selection.*

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