Bull. Korean Math. Soc. **36** (1999), No. 4, pp. 717–722

ON THE DUALITY OF THE SPACE X AND THE ALGEBRA $C_p(X)$

Sung Ki Park

ABSTRACT. The set of continuous maps of a space X to real usual space R equipped with the topology of pointwise convergence will be denoted by $C_p(X)$. In this paper, we prove that: $C_p(X)$ is hereditarily separable and hereditary Lindelöf if and only if X^n is hereditarily separable and hereditary Lindelöf.

1. Introduction

We will be interested in the interrelation between a space X and the space $C_p(X)$ formed by all continuous real valued functions on a space X with the topology of pointwise convergence, some progress has been made recently by N. Nagata [2], N. Noble [3] and A. V. Arkhangel'skii [1].

We will study the problems associated with separability and Lindelöfness. And we generalize and extend the theorem of N. Noble [3], i.e., Proposition 1.

The main result of this paper is shown : $C_p(X)$ is hereditarily separable and hereditary Lindelöf if and only if X^n is hereditarily separable and hereditary Lindelöf.

2. Countability between X and $C_p(X)$

DEFINITION. All hypothesized spaces are assumed to be completely regular Hausdorff. The set of continuous functions from X to Z is

Received December 10, 1998.

1991 Mathematics Subject Classification: 54C35.

Key words and phrases: density character, the topology of pointwise convergence, hereditarily separable, hereditary Lindelöf.

This was supported by Chosun University Research Fund., 1998.

denoted by $C(X,Z)$ or, when $Z = R$, by $C(X)$. The density character, δX, of a space X is the least cardinality of a dense subset of X and the weight, wX, of X is the least cardinality of an open basis of X. We define the weak weigh, wwX, of X to be the least of the cardinals wY for Y a continuous one-to-one image of X.

PROPOSITION 1. *Let X be any infinite space and let $C(X)$ have any topology between the pointwise topology and the compact-open topology. Then $\delta C(X) = wwX$ (See [3]).*

PROPOSITION 2. *Let $X = Y \times R$, i.e., the space X is the product of a space Y and the real line R. Then the space $(C_p(X))^{\aleph_0}$ is linearly homemorphic to the space $C_p(X)$.*

Proof. Let Z be the discrete space of integers, $Z \subset R$, and $X_0 = Y \times Z \subset X$.

For an $r \in R$ we denote by $[r]$ its entier, i.e., the largest integer not exceeding r. Further, let $\langle r \rangle = r - [r]$ be the fractional part of r. For $x = (y,r) \in X$ we put $x^- = (y,[r]) \in X_0$ and $x^+ = (y,[r]+1) \in X_0$.

Let us construct the continuous extension operator $\phi : C_p(X_0) \to C_p(X)$, i.e., the continuous linear map $\phi : C_p(X_0) \to C_p(X)$ such that for each function $f \in C(X_0)$ the restriction of the function $\phi(f) \in C(X)$ to X_0 coincides with f. So, let $f \in C_p(X_0)$ and $x = (y,r) \in X$. Put

$$\phi(f)(y,r) = \phi(f)(x) = \langle r \rangle \cdot f(x^+) + (1 - \langle r \rangle) f(x^-).$$

If $r \in Z$, then $\langle r \rangle = 0, x^- = x$, and $\phi(f)(x) = f(x)$, i.e., the restriction of $\phi(f)$ to X_0 coincides with f. It is easy to verify the continuity of $\phi(f)$.

Finally, the linearity of the map $\phi : C(X_0) \to C(X)$ is obvious.

Since the value of $\phi(f)$ at an arbitrary point $x \in X$ depends only on the values of f at the two points x^- and x^+, the map ϕ is continuous with respect to the topology of pointwise convergence on $C(X)$ and on $C(X_0)$.

Thus, $\phi : C_p(X_0) \to C_p(X)$ is a continuous extension operator.

Therefore the space $C_p(X)$ is linearly homeomorphic to the space $C_p(X_0) \times L$, where $L = \{g \in C_p(X) : g(X_0) = \{0\}\}$. In face, the map $\psi : C_p(X_0) \times L \to C_p(X)$ defined by the rule $\psi((f,g)) = \phi(f) + g \in$

$C_p(X)$ for all $(f,g) \in C_p(X) \times L$, is clearly a continuous linear map. It is easily verified that ψ is also bijective, and that $\psi(C_p(X) \times L) = C_p(X)$ (if $h \in C_p(X), f = h|x_0$, and $g = h - \phi(f)$, then $g \in L, f \in C_p(X_0)$, and $\psi((f,g)) = h$).

Since X_0 is the free topological sum of $\aleph_0$ copies of Y, the spaces $C_p(X)$ and $(C_p(Y))^{\aleph_0}$ are linearly homeomorphic.

For $n \in Z$ we put

$$L_n = \{g \in C_p(Y \times [n, n+1]) : g(Y \times \{n\}) = g(Y \times \{n+1\}) = \{0\}\}.$$

Taking for each $n \in Z$ an arbitrary function in L_n, we obtain a function in L. Hence it is clear that L is linearly homeomorphic to the product $\prod\{L_n : n \in Z\}$. However, clearly all L_n are linearly homeomorphic to L_0. Hence L is linearly homeomorphic to $(L_0)^{\aleph_0}$. If we denote the relation of linear homeomorphism by $\overset{lh}{\sim}$, we can now write:

$$C_p(X) \overset{lh}{\sim} C_p(X_0) \times L \overset{lh}{\sim} (C_p(Y))^{\aleph_0} \times L_0^{\aleph_0} \overset{lh}{\sim}$$

$$(C_p(Y) \times L_0)^{\aleph_0} \overset{lh}{\sim} ((C_p(Y) \times L_0)^{\aleph_0})^{\aleph_0} \overset{lh}{\sim} (C_p(X))^{\aleph_0}. \qquad \square$$

LEMMA 1. *If $C_p(X)$ is a hereditary Lindelöf space, then X is a hereditary Lindelöf space.*

Proof. Suppose that $\mathcal{F} \subseteq C_p(X)$ is a Lindelöf subspace and separates points from closed sets i.e., if $x \notin F$ and F in a subspace $\tilde{X}$ is closed, then there is $f \in \mathcal{F}$ such that $f(x) = 1$ and $f(F) = 0$.

Let $\mathcal{H} = \{H\}$ be a family of sets open in $\tilde{X}$. For each point $x \in \tilde{\mathcal{H}}$ where $\tilde{\mathcal{H}} = \cup\{H : H \in \mathcal{H}\}$, we choose $H_x \in \mathcal{H}$ and $f_x \in \mathcal{F}$ such that $f_x(x) = 1$ and $f_x(\tilde{X} - H_x) = 0$. There exists a countable set $B \subseteq \tilde{\mathcal{H}}$ such that $\{f_x : x \in B\}$ is dense in $\{f_x : x \in \tilde{\mathcal{H}}\}$. Then $\tilde{\mathcal{H}} = \cup\{H_x : x \in B\}$. Accordingly, if $x \in \tilde{H}$, then there exists $y \in B$ subject to condition

$$|f_x(x) - f_y(x)| < \frac{1}{2}.$$

Then $f_y(x) \neq 0$ and $x \in H_y$. The open cover $\mathcal{H}$ has a countable subcover $\mathcal{H}_0 = \{H_x : x \in B\}$. And $\mathcal{H}_0$ is a countable base for $\tilde{X}$.

Accordingly, $\tilde{X}$ is the second countable space and hereditary Lindelöf. Therefore $\tilde{X}$ is a hereditary Lindelöf space. □

An open set $U = \prod_{i=1}^{n}\{u_i : \text{open in } \tilde{X}\} \subseteq X^n$ is called an elementary set if $u_i \cap u_j = \phi$ for $i \neq j$.

LEMMA 2. *If $C_p(X)$ is hereditarily separable, then each elementary set U is separable.*

Proof. Let $U = \prod_{i=1}^{n} u_i$ be an elementary set, ξ a family of open sets contained in U. For each point $x = (x_1, \cdots, x_n) \in \tilde{\xi}$, where $\tilde{\xi} = \cup\{u : u \in \xi\}$ we choose a neighborhood $u_x = \prod_{i=1}^{n} u_{x_i}$, inscribed in ξ and construct a function $f_x \in C_p(X)$ such that $f_x(x) = 1$ and $f_x(X - u_x) = 0$. There exist a countable set $B \subseteq \tilde{\xi}$ such that $\{f_x : x \in B\}$ is dense in $\{f_x : x \in \tilde{\xi}\}$. Then $\tilde{\xi} = \cup\{u_x : x \in B\}$.

Let $x \in \tilde{\xi}$. There exists $y \in B$ such that

$$|f_x(x_i) - f_y(x_i)| < \frac{1}{2}, \quad i = 1, 2, \cdots, n.$$

Then $f_y(x_i) \neq 0$ and $x_i \in u_y$, by the property of the elementary set U. Therefore, $x \in u_y$. That is, the countable set B is a dense subset of U. □

THEOREM 1. (separability) *$C_p(X)$ is hereditarily separable if and only if X^n is hereditarily separable.*

Proof. (necessity) Let $C_p(X)$ be hereditarily separable.

We will now prove that X^n is hereditarily separable by the mathematical induction.

Step 1. In the case $n = 1$, it is obvious by proposition 1. Suppose that X^{n-1} is hereditarily separable. Let $A \subseteq X^n$, $A_n = \{x = (x_1, \cdots, x_n) \in A : x_i \neq x_j \text{ for } i \neq j\}$

The set $A - A_n$ lies in the union of a finite set of hereditarily separable space homeomorphic to X^{n-1}, so it is separable.

We will prove the separability of the set A_n. Let $\beta = \{V_n : n \in N\}$ be a countable base of R. For each function $f \in C_p(X)$ and each n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n) \in N^n$, we define a set $w(f, \alpha) = \prod_{i=1}^{n} f^{-1}(V_{\alpha_i})$,

$V_{\alpha_i} \in \beta$. The set $w(f,\alpha)$ is an elementary set if and only if $V_{\alpha_i} \cap V_{\alpha_j} = \phi$ for $i \neq j$.

Step 2. Let $\{w(f_t,\alpha) : t \in T\}$ be an arbitrary family of nonempty set. Then there exists a countable $T_0 \subset T$ such that

$$\cup\{w(f_t,\alpha) : t \in T_0\} = \cup\{w(f_t,\alpha) : t \in T\}$$

by Lemma 2. Indeed, the family $\{f_t : t \in T\}$ contains a countable dense subfamily $\{f_t : t \in T_0\}$ in it. If $x = (x_1, \cdots, x_n) \in w(f_t,\alpha)$, then there exists $g \in T_0$ such that $f_g(x_i) \in V_{\alpha_i}$, $i = 1, 2, \cdots, n$, where $\alpha = (\alpha_1, \cdots, \alpha)$. Therefore, $x_i \in f_g^{-1}(V_{\alpha_i})$ and $x \in w(f_g,\alpha)$. So A_n is separable, and X^n is hereditarily separable.

(sufficiency) On the other hand, let X^n be hereditarily separable, $n \in N$. Let $\mathcal{F} \subseteq C_p(X)$. For each m-tuple $\alpha \in N^m$, $\alpha = (\alpha_1, \cdots, \alpha_m)$, let $\pi(\alpha) = \{w(f,\alpha) : f \in \mathcal{F}\}$. There exists a countable set $\mathcal{F}(\alpha) \subseteq \mathcal{F}$ such that $\tilde{\pi}(\alpha) = \cup\{w(f,\alpha) : f \in \mathcal{F}(\alpha)\}$. Let $\mathcal{F}_0 = \cup \mathcal{F}(\alpha) : \alpha \in N^m\}$. Then $\mathcal{F}_0$ is countable dense in $\mathcal{F}$. □

The proof of theorem 2 is similar to the proof of theorem 1.

THEOREM 2. (Lindelöfness) *$C_p(X)$ is a hereditary Lindelöf space if and only if X^n is a hereditary Lindelöf space.*

Proof. (necessity) In the proof of theorem 1, choosing an elementary neighborhood for each point $x \in A$, we extract from the obtained family a countable cover of the set A. This is possible by step 2 in the proof of theorem 1. From Lemma 1, A is a Lindelöf space.

(sufficiency) Let X^n be a hereditary Lindelöf space, $n \in N$. Let $\mathcal{F} \subseteq C_p(X)$. For each n-tuple $\alpha \in N^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, let $\pi(\alpha) = \{w(f,\alpha) : f \in \mathcal{F}\}$ and $\tilde{\pi}(\alpha) = \cup\{w(f,\alpha) : f \in \mathcal{F}\}$. There exists a countable set $\mathcal{F}(\alpha) \subseteq \mathcal{F}$ such that $\tilde{\pi}(\alpha) = \cup\{w(f,\alpha) : f \in \mathcal{F}(\alpha)\}$.

Let $\mathcal{F}_0 = \cup\{\mathcal{F}(\alpha) : \alpha \in N^n\}$. Then $\mathcal{F}_0$ is countable and dense in $\mathcal{F}$.

Let $f \in \mathcal{F}$ and $V = \{f : f(x_k) \in V_{\alpha_k},\ \ k = 1, \cdots, l\}$ be a neighborhood of f.

Let $x = (x_1, \cdots, x_l)$ and $\alpha = (\alpha_1, \cdots, \alpha_l)$. Since $f(x_k) \in V_{\alpha_k}$, $x \in w(f,\alpha) \in \pi(\alpha)$. There exists $g \in \mathcal{F}(\alpha)$ such that $x \in w(g,\alpha)$. Then $g(x_k) \in V_{\alpha_k}$ and $g \in V$. □

By theorem 1,2, we obtain the following symmetric result.

COROLLARY. *The followings are equivalent:*
a) $C_p(X)$ *is hereditarily separable and hereditary Lindelöf.*
b) $C_p(X^n)$ *is hereditarily separable and hereditary Lindelöf.*
c) $C_p(nX)$ *is hereditarily separable and hereditary Lindelöf.*
d) $(C_p(X))^n$ *is hereditarily separable and hereditary Lindelöf.*
e) $(C_p(X))^{\aleph_0}$ *is hereditarily separable and hereditary Lindelöf.*

Proof. Suppose that $C_p(X)$ is hereditarily separable and hereditary Lindelöf, then $(X^n)^m = X^{nm}$ is hereditarily separable and hereditary Lindelöf for any $m = 1, 2, \cdots$. Therefore, $C_p(X^n)$ is hereditarily separable and hereditary Lindelöf. Since $(C_p(X))^n$ is homeomorphic to $C_p(X^n)$, $(C_p(X))^n$ is hereditarily separable and hereditary Lindelöf if $C_p(X)$ is so. Since this is true for each $n \in N$, $(C_p(X))^{\aleph_0}$ is hereditarily separable and hereditary Lindelöf if $C_p(X)$ is so. □

References

[1] Arkhangel'skii A. V., *Function spaces in the topology of pointwise convergence and compact set*, Russian Math. Surveys **39** (1984), no. 5, 9–56 (Translated from Russian).

[2] Nagata, J., *On lattices of functions on topological spaces and of functions on uniform space*, Osaka Math. **2** (1949), no. 2, 166–181.

[3] Noble, N., *The density character of function spaces*, Proc. Am. Math Soc. **1** (1974), 228–223.

[4] Okunev, O. G., *The spaces of functions in the topology of pointwise convergence*, Moscow Univ. Math. bull. **40** (1985), no. 4, 84–87 (Translated from Russian).

[5] Tkacuk, V. V., *The spaces $C_p(X)$ decomposition into a countable union of bounded subspaces and completeness properties*, Topol. Apple. (1986), 241–354.

DEPARTMENT OF MATHEMATICS EDUCATION, CHOSUN UNIVERSITY, KWANGJU 501-759, KOREA
E-mail: santamoni@channeli.net