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POSITIVELY EQUICONTINUOUS FLOWS ARE TOPOLOGICALLY CONJUGATE TO ROTATION FLOWS

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ABSTRACT. In this paper we study the continuity of rotation numbers of liftings of circle maps with degree one. And apply our result to prove that a positively equicontinuous flow of homeomorphisms on the circle S^1 is topologically conjugate to a continuous flow of rotation maps.

1. Introduction

Let X be a compact metric space and $C^0(X,X)$ denote the set of continuous maps from X into itself.

Throughout this paper N, Z, R and C denote the set of all positive integers, integers, reals and complex numbers, respectively. The symbol id_X denotes the identity map on the set X and S^1 stands for the unit circle, i.e.,

$$S^1 = \{z \in C \mid |z| = 1\}.$$

A family of continuous maps $T^t : S^1 \to S^1$, $t \in R$ is said to be a *flow* on S^1 if

$$T^t \circ T^s = T^{t+s} \text{ for } t, s \in R.$$

A flow $\{T^t\}_{t\in R}$ is continuous if the mapping $T : R \times S^1 \to S^1$ defined by $(t, z) \mapsto T^t(z)$ is continuous. Continuous flows of homeomorphisms on S^1 have been studied in [7].

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In 1998, M. Bajger [1] have investigated flows of homeomorphisms $\{T^t\}_{t\in R}$ on the unit circle S^1 satisfying the following hypothesis (Ω) : there exists $w \in R$ such that $(T^w)^n(z) \neq z$ for all $n \in Z \setminus \{0\}, z \in S^1$. Actually he proved that if $\{T^t\}_{t\in R}$ is an s-disjoint flow, then either $\{T^t\}_{t\in R}$ is topologically conjugate to a flow of rotations, or $\{T^t\}_{t\in R}$ is topologically conjugate to a special s-disjoint piecewise linear flow.

On the other hand, in 1993, S. H. Cho, K. J. Min and S. K. Yang [2] determined conditions under which equicontinuity of the family of iterates $\{T^n\}_{n\in N}$ of a continuous function T that maps the circle S^1 into itself does occur. Actually, they showed that when degree of T is 1, $\{T^n\}_{n\in N}$ is equicontinuous if and only if T is topologically conjugate to a rotation map.

In this paper, we will show that a positively equicontinuous flow $\{T^t\}_{t\in R}$ of homeomorphisms on S^1 is topologically conjugate to a continuous flow of rotation maps.

2. Preliminaries and Definitions

The canonical projection map $p : R \to S^1$ defined by for $x \in R$, $p(x) = e^{2\pi i x}$ is said to be a covering map, since it wraps R around S^1 without doubling back (i.e., without critical points). Let $p : R \to S^1$ be the covering map and $z, w \in S^1$ with $z \neq w$. Define the distance d of z and w by $d(z, w) = |x - y|$ for some $x, y \in R$ with $|x - y| \leq \frac{1}{2}$ such that $p(x) = z$ and $p(y) = w$. Then d is a well-defined metric on S^1 which is equivalent to the original one. For the conveniency, we will use this metric d on S^1. The distance between two maps $S, T \in C^0(S^1, S^1)$ defined by $d(S,T) = \sup_{z\in S^1} d(S(z), T(z))$.

Let $T : S^1 \to S^1$ be a continuous map on the circle. We say that a continuous map $F : R \to R$ is a *lifting* of T if $T \circ p = p \circ F$.

If F and F' are liftings of the same map T, then $F = F' + n$ for some integer n. There exists a unique integer m such that $F(x + 1) = F(x) + m$ for all liftings F and all x, which is called the *degree* of T, denoted by $deg(T)$. Define the distance of two liftings F and G by $d(F, G) = \sup\{|F(x) - G(x)| \mid x \in R\}$.

In 1979, Newhouse, Palis and Takens [5] have generalized a rotation number for a homeomorphism of the circle S^1 to a continuous map of degree 1 and defined a rotation set.

DEFINITION. Let $T : S^1 \to S^1$ be a continuous map of $deg(T) = 1$ and let F be a lifting of T. Given $x \in R$, define the *rotation number* of F at x

$$\rho_x(F) = \lim_{n\to\infty} \sup \frac{F^n(x) - x}{n}.$$

And define the rotation set of F

$$\rho(F) = \{\rho_x(F) | x \in R\}.$$

DEFINITION. Let $T : S^1 \to S^1$ be a continuous map of $deg(T) = 1$ and let F be a lifting of T. And let $p : R \to S^1$ be a covering map. Given $z \in S^1$, define the *rotation number* of T at z

$$\rho_z(T) = p(\rho_x(F)),$$

where $p(x) = z$ for some $x \in R$. And define the rotation set of T

$$\rho(T) = \{\rho_z(T) | z \in S^1\}.$$

Notice that if a different lifting is used, this simply has the effect of translating the rotation set by an integer. It is known (see [3],[5]) that $\rho(F)$ is either one point or a closed interval, and if $a/b \in \rho(F)$ and $GCD(a, b) = 1$ then T has a periodic point x of period b with $\rho_x(F) = a/b$. Conversely if T has a periodic point of period b then there exists $l \in Z$ such that $l/b \in \rho(F)$.

3. Main Results

Let $T : X \to X$ be a continuous map. A point $y \in X$ is called an *ω-limit point* of x if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $T^{n_i}(x) \to y$. We denote the set of ω-limit points of x for T by $\omega_T(x)$.

A rotation map on S^1 is a map $T : S^1 \to S^1$ defined by $T(z) = \alpha z$ for some $\alpha \in S^1$. In this case $\alpha = e^{2\pi i\rho}$ where ρ is the rotation number of T, and it is well-known that if $\rho(T) = \rho$ is an irrational number, then $\omega_T(x) = S^1$ for all $x \in S^1$.

THEOREM 1. *Let $T : S^1 \to S^1$ be a rotation map with an irrational rotation number. Suppose that $S : S^1 \to S^1$ is a continuous map with $S \circ T = T \circ S$. Then S is a rotation map.*

Proof. Let $e^{2\pi i\rho}$ be the irrational rotation number of T. Since T is a rotation map, we have $T(z) = \alpha z$ for all $z \in S^1$, where $\alpha = e^{2\pi i\rho}$. By hypothesis, we have $S(T(z)) = S(\alpha z)$, $T(S(z)) = \alpha S(z)$ and

$$\begin{aligned}\frac{S(z)}{z} &= \frac{T(S(z))}{\alpha z} = \frac{S(\alpha z)}{\alpha z} = \frac{S(T(z))}{T(z)} \\ &= \frac{S(T^2(z))}{T^2(z)} = \cdots = \frac{S(T^n(z))}{T^n(z)}.\end{aligned}$$

For a fixed $z \in S^1$, $\omega_T(z) = S^1$ since ρ is irrational. Therefore we have for all $u \in S^1$, there exists $n_i \to \infty$ such that $T^{n_i}(z) \to u$. Thus we have

$$\frac{S(u)}{u} = \lim_{n_i \to \infty} \frac{S(T^{n_i}(z))}{T^{n_i}(z)} = \lim_{n_i \to \infty} \frac{S(z)}{z} = \frac{S(z)}{z},$$

and hence $\frac{S(u)}{u}$ is a constant map. Putting $\frac{S(z)}{z} = \beta$, we have $S(z) = \beta z$. Hence S is a rotation map. □

Remark that if the rotation number of T is rational, then S need not be a rotation map in Theorem 1.

Let $T \in C^0(S^1, S^1)$. Then a family of iterates $\{T^n\}_{n \in N}$ of T is said to be *equicontinuous* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for $x, y \in S^1$, $d(x, y) < \delta$ implies $d(T^n(x), T^n(y)) < \epsilon$ for all $n \in N$.

The following lemma appears in [2].

LEMMA 2. *Let $T, S : S^1 \to S^1$ be topologically conjugate. Then $\{T^n\}_{n \in N}$ is equicontinuous if and only if $\{S^n\}_{n \in N}$ is equicontinuous.*

The following is a slight modification of Theorem 5 in [2].

LEMMA 3. *Let $T \in C^0(S^1, S^1)$ with $deg(T) = 1$. Then $\{T^n\}_{n \in N}$ is equicontinuous if and only if T is topologically conjugate to a rotation map.*

Using the Lemma 2 and Lemma 3, we have the following result.

COROLLARY 4. *Let $T \in C^0(S^1, S^1)$ with an irrational rotation number and $\{T^n\}_{n\in N}$ be equicontinuous. Suppose that $S : S^1 \to S^1$ is a continuous map with $S \circ T = T \circ S$. Then $\{S^n\}_{n\in N}$ is also equicontinuous.*

Proof. By Lemma 3, there exists a homeomorphism $\varphi : S^1 \to S^1$ such that $\varphi^{-1} \circ T \circ \varphi$ is a rotation map with an irrational rotation number. Now since $S \circ T = T \circ S$, we have

$$(\varphi^{-1} \circ S \circ \varphi) \circ (\varphi^{-1} \circ T \circ \varphi) = (\varphi^{-1} \circ T \circ \varphi) \circ (\varphi^{-1} \circ S \circ \varphi).$$

Hence by Theorem 1, $\varphi^{-1} \circ S \circ \varphi$ is a rotation map. Therefore, by Lemma 2, $\{S^n\}_{n\in N}$ is equicontinuous. □

The following lemma is due to Remark 1 in [6].

LEMMA 5. *Let $\{T^t\}_{t\in R}$ be a flow on S^1. If T^t is surjective, then $T^0 = id_{S^1}$ and each T^t is a homeomorphism.*

By Lemma 5, if T^t is surjective, then T^t must be a homeomorphism. Therefore we declare that $deg(T^t) \in \{1, -1\}$. However in this case we have $deg(T^t) = 1$ for all $t \in R$. In fact we have $deg(T^t) = deg(T^{\frac{t}{2}} \circ T^{\frac{t}{2}}) = deg(T^{\frac{t}{2}}) \cdot deg(T^{\frac{t}{2}}) = 1$. Hence the case $deg(T^t) = -1$ does not occurs.

LEMMA 6. (1) *Let $T : S^1 \to S^1$ be a continuous map. For a continuous map $S : S^1 \to S^1$, if $d(S,T) < \frac{1}{2}$, then there exist liftings S_F and T_F of S and T, respectively, such that $d(S_F, T_F) = d(S,T)$.*

(2) *For any two continuous maps $S, T : S^1 \to S^1$, if $d(S,T) < \frac{1}{2}$, then $deg(S) = deg(T)$.*

(3) *For a continuous map $S : S^1 \to S^1$, there exists $\epsilon > 0$ such that for a continuous map $T : S^1 \to S^1$ with $S \circ T = T \circ S$ and $d(S,T) < \epsilon$, there exist liftings S_F and T_F of S and T respectively, such that $d(S_F, T_F) = d(S,T)$ and $S_F \circ T_F = T_F \circ S_F$.*

Proof. (1) : Take $x_0 \in R$ such that $d(S(p(x_0)), T(p(x_0))) = d(S,T)$. Then we can take liftings S_F and T_F of S and T, respectively such that

$$|S_F(x_0) - T_F(x_0)| = d(S(p(x_0)), T(p(x_0))) = d(S,T) < \frac{1}{2}.$$

Then we can prove $d(S_F, T_F) = d(S,T)$. Indeed, suppose that there exists $y \in R$ such that $|S_F(y) - T_F(y)| > d(S,T)$. Since a map $x \mapsto |S_F(x) - T_F(x)|$ is continuous, by the intermediate value theorem, there exists x_1 between x_0 and y such that $d(S,T) < |S_F(x_1) - T_F(x_1)| < \frac{1}{2}$. Then

$$\begin{aligned} d(S(p(x_1)), T(p(x_1))) &= d(p(S_F(x_1)), p(T_F(x_1))) \\ &= |S_F(x_1) - T_F(x_1)| > d(S,T). \end{aligned}$$

This is a contradiction.

(2) : By (1), there exist liftings S_F and T_F of S and T, respectively, with $d(S_F, T_F) = d(S,T)$. Let $deg(S) = m$ and $deg(T) = n$. Then we have for $x \in R$, $S_F(x+1) = S_F(x) + m$ and $T_F(x+1) = T_F(x) + n$. Thus

$$\begin{aligned} \frac{1}{2} &> |S_F(x+1) - T_F(x+1)| \\ &\geq -|S_F(x) - T_F(x)| + |m-n| \\ &\geq |m-n| - \frac{1}{2}. \end{aligned}$$

Since $|m-n|$ is constant and less than 1, we can obtain $m = n$. Thus we have $deg(S) = deg(T)$.

(3) : Since S is uniformly continuous, there exists $0 < \epsilon < \frac{1}{2}$ such that for $z, w \in S^1$

$$d(z,w) < \epsilon \text{ implies } \ d(S(z), S(w)) < \frac{1}{2}.$$

Let $T : S^1 \to S^1$ be a continuous map with $S \circ T = T \circ S$ and $d(S,T) < \epsilon$. By (1), there exist liftings S_F and T_F of S and T, respectively, such that $d(S_F, T_F) = d(S,T) < \epsilon$. Since S_F is a lifting of S, we know that for all $x, y \in R$

$$|x-y| < \epsilon \text{ implies } \ |S_F(x) - S_F(y)| < \frac{1}{2}.$$

We know that $S_F \circ T_F$ and $T_F \circ S_F$ are liftings of $S \circ T$ and $T \circ S$, respectively, by observing the following diagram.

$$\begin{array}{ccccccc} R & \xrightarrow{S_F} & R & \xrightarrow{T_F} & R & \xrightarrow{S_F} & R \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow[S]{} & S^1 & \xrightarrow[T]{} & S^1 & \xrightarrow[S]{} & S^1 \end{array}$$

Then for all $x \in R$,

$$|T_F \circ S_F(x) - S_F \circ T_F(x)|$$

$$\leq |T_F(S_F(x)) - S_F(S_F(x))| + |S_F(S_F(x)) - S_F(T_F(x))| < \epsilon + \frac{1}{2} < 1.$$

Since $S_F \circ T_F$ and $T_F \circ S_F$ are liftings of the same map $S \circ T = T \circ S$, we have $S_F \circ T_F(x) = T_F \circ S_F(x)$. □

LEMMA 7. *Let $T \in C^0(S^1, S^1)$ with $deg(T) = 1$. Then there exists $\epsilon > 0$ such that for a continuous map $S : S^1 \to S^1$ with $S \circ T = T \circ S$ and $d(S,T) < \epsilon$, and for all $x \in R$*

$$|\rho_x(S_F) - \rho_x(T_F)| \leq d(S,T).$$

Proof. By Lemma 6 (3), there exist $\epsilon > 0$ such that there are liftings S_F and T_F of S and T, respectively with $S_F \circ T_F = T_F \circ S_F$ and $d(S_F, T_F) = d(S,T) < \epsilon$. Hence for $x \in R$,

$$\begin{aligned} &S_F^n(x) - x \\ &= S_F^n(x) - T_F(S_F^{n-1}(x)) + S_F^{n-1}(T_F(x)) - T_F^2(S_F^{n-2}(x)) \\ &\quad + S_F^{n-2}(T_F^2(x)) - \cdots - T_F^n(x) + T_F^n(x) - x. \end{aligned}$$

Therefore we have

$$|S_F^n(x) - x| \leq nd(S,T) + |T_F^n(x) - x|. \tag{1}$$

Now dividing inequality (1) by n and taking limit supremum as $n \to \infty$, we have $\rho_x(S_F) \leq d(S,T) + \rho_x(T_F)$. And by interchanging the roles of S_F and T_F, we have $\rho_x(T_F) \leq d(S,T) + \rho_x(S_F)$. Consequently, we have

$$|\rho_x(S_F) - \rho_x(T_F)| \leq d(S,T).$$

□

LEMMA 8. *Let $T, S : S^1 \to S^1$ be continuous maps with degree 1. Suppose that T is a rotation map. Then*

$$|\rho_x(S_F) - \rho(T_F)| \leq d(S,T).$$

Proof. Let $T_F : R \to R$ be a lifting of T with $T_F(x) = x + \rho(T_F)$ for all $x \in R$. Then we can take a lifting $S_F : R \to R$ of $S : S^1 \to S^1$ with $d(S_F, T_F) = d(S,T)$. Then we have $T_F^n(x) = x + n\rho(T_F)$ and hence

$$\begin{aligned} & |T_F^k(S_F^{n-k}(x)) - T_F^{k+1}(S_F^{n-k-1}(x))| \\ = \; & |S_F^{n-k}(x) + k\rho(T_F) - (T_F(S_F^{n-k-1}(x)) + k\rho(T_F))| \\ = \; & |S_F(S_F^{n-k-1}(x)) - T_F(S_F^{n-k-1}(x))| \\ \leq \; & d(S,T) \end{aligned}$$

for $0 \leq k < n$. Therefore we have

$$\begin{aligned} (2) \quad & |S_F^n(x) - (x + n\rho(T_F))| \\ \leq \; & |S_F^n(x) - T_F(S_F^{n-1}(x))| + |T_F(S_F^{n-1}(x)) - T_F^2(S_F^{n-2}(x))| + \cdots \\ & + |T_F^n(x) - (x + n\rho(T_F))| \\ \leq \; & nd(S,T). \end{aligned}$$

Now dividing the inequality (2) by n and taking limit supremum as $n \to \infty$, we have $|\rho_x(S_F) - \rho(T_F)| \leq d(S,T)$. □

Remark that if $\{T^t\}_{t \in R}$ is a flow of homeomorphisms, then $\rho(T^t)$ is an one-point.

Let $\{T^t\}_{t\in R}, \{S^t\}_{t \in R}$ be two flows on the circle S^1. We say that two flows $\{T^t\}_{t\in R}$ and $\{S^t\}_{t\in R}$ are *topologically conjugate* if there exists a homeomorphism $\varphi : S^1 \to S^1$ such that $S^t = \varphi^{-1} \circ T^t \circ \varphi$.

A continuous flow $\{T^t\}_{t\in R}$ of homeomorphisms is said to be *positively equicontinuous* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for $z, w \in S^1$, $d(z,w) < \delta$ implies $d(T^t(z), T^t(w)) < \epsilon$ for all $t \geq 0$.

THEOREM 9. *Let $\{T^t\}_{t\in R}$ be a positively equicontinuous flow of homeomorphisms on S^1. Then either $\{T^t\}_{t\in R}$ is the trivial flow, or $\{T^t\}_{t\in R}$ is topologically conjugate to a continuous flow of rotation maps of S^1. Indeed,*

(1) *for all $t \in R$, $T^t = id_{S^1}$; or*

(2) *there exist a homeomorphism $\varphi : S^1 \to S^1$ and a continuous map $c : R \to S^1$ such that*

$$(*) \qquad \varphi^{-1} \circ T^t \circ \varphi(z) = c(t)z, \quad t \in R, \ z \in S^1$$

$$(**) \qquad c(s+t) = c(s)c(t), \quad s, t \in R.$$

Proof. Suppose that $T^{t_0} \neq id_{S^1}$ for some $t_0 > 0$. Then we have $\rho(T^{t_0})$ is not 1. Since the flow $\{T^t\}_{t\in R}$ is continuous and $\rho(T^0) = 1$, there exists $s \in (0, t_0)$ such that $\rho(T^s) = e^{2\pi i\rho}$ where ρ is irrational. Now by Lemma 3, there exists a homeomorphism $\varphi : S^1 \to S^1$ such that $\varphi^{-1} \circ T^s \circ \varphi$ is an irrational rotation map. Then by Theorem 1, $\varphi^{-1} \circ T^t \circ \varphi$ is also a rotation map for all $t \in R$. Hence there is a map $c : R \to S^1$ such that $\varphi^{-1} \circ T^t \circ \varphi(z) = c(t)z$ for all $t \in R, z \in S^1$. The continuity of c follows from the continuity of $(t, z) \mapsto T^t(z)$ and $c(t) = \frac{\varphi^{-1} \circ T^t \circ \varphi(z)}{z}$. Also we have

$$\begin{aligned} c(t+s) &= \frac{\varphi^{-1} \circ T^{t+s} \circ \varphi(z)}{z} \\ &= \frac{\varphi^{-1} \circ T^t \circ \varphi \circ \varphi^{-1} \circ T^s \circ \varphi(z)}{z} \\ &= \frac{\varphi^{-1} \circ T^t \circ \varphi(c(s)z)}{z} \\ &= \frac{c(t)c(s)z}{z} = c(t)c(s) \end{aligned}$$

which proves $(**)$. □

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