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# ANOTHER CHARACTERIZATION OF ROUND SPHERES

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ABSTRACT. A characterization of geodesic spheres in the simply connected space forms in terms of the ratio of the Gauss-Kronecker curvature and the (usual) mean curvature is given: An immersion of $n$ dimensional compact oriented manifold without boundary into the $n+1$ dimensional Euclidean space, hyperbolic space or open half sphere is a totally umbilic immersion if the mean curvature $H_1$ does not vanish and the ratio $H_n/H_1$ of the Gauss-Kronecker curvature $H_n$ and $H_1$ is constant.

## 1. Introduction

Let $M^n$ be an immersed submanifold of $N^{n+1}$ and let $H_k$ denote the $k$-th mean curvature function of $M^n$, that is, $H_k$ is the $k$-th elementary symmetric polynomial of principal curvatures of $M^n$ divided by $\binom{n}{k}$. For instance, $H_1$ is the usual mean curvature and $H_n$ is the Gauss-Kronecker curvature.

In [5], we obtained the following characterization of round spheres in the simply connected space forms in terms of the mean curvature functions $H_k$:

THEOREM A. *Let $N^{n+1}$ be one of the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$ or the open half sphere $\mathbb{S}^{n+1}_+$ and $\phi : M^n \to N^{n+1}$ be an isometric immersion of a compact oriented n-dimensional manifold without boundary $M^n$. If $H_{k-1}$ does not vanish and the ratio*

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*$H_k/H_{k-1}$ of two* consecutive *mean curvatures is a* constant *for some $k = 2, \ldots, n$, then $\phi(M^n)$ is a geodesic hypersphere.*

While in the known characterizations of round spheres we need to assume that a mean curvature function is constant joint with some extra global conditions, for example, convexity [10], star-shapedness [4], or embeddedness [1], [6], [7], [8], [9], the above theorem shows that the mean curvature function itself is enough to characterize round spheres (cf. [3]).

In this note, we consider the other extreme case and prove the following theorem:

THEOREM B. *Let $N^{n+1}$ be one of the Euclidean space $\mathbb{R}^{n+1}$, the hyperbolic space $\mathbb{H}^{n+1}$ or the open half sphere $\mathbb{S}^{n+1}_+$ and $\phi : M^n \to N^{n+1}$ be an isometric immersion of a compact oriented n-dimensional manifold without boundary $M^n$. If $H_1$ does not vanish and the ratio $H_n/H_1$ of the Gauss-Kronecker curvature and the usual mean curvature is a constant, then $\phi(M^n)$ is a geodesic hypersphere.*

We cannot expect the same result for the whole sphere $\mathbb{S}^{n+1}$. For example, $H_1$ and $H_2$ of the torus

$$\mathbb{S}^1(a) \times \mathbb{S}^1(b) \subset \mathbb{S}^3, \quad a^2 + b^2 = 1, \quad a \neq b$$

are nonzero constants.

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## 2. Proof

We use the hyperboloid model for $\mathbb{H}^{n+1}$ and the usual embedding of $\mathbb{S}^{n+1}$ into $\mathbb{R}^{n+2}$. Let $\eta$ denote a unit normal field on $M^n$. We use the following Minkowski formula (for proof, see [6]) where $\langle\ ,\ \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^{n+1}$ (on $\mathbb{R}^{n+2}$) when $N^{n+1}$ is $\mathbb{R}^{n+1}$ (when $N^{n+1}$ is $\mathbb{S}^{n+1}_+$) and the Lorentzian inner product on $\mathbb{R}^{n+2}$ when $N^{n+1}$ is $\mathbb{H}^{n+1}$.

LEMMA A. *Set $H_0 = 1$. Then the following identities hold for every $k = 1, \dots, n$.*

(1) *When $N^{n+1}$ is $\mathbb{R}^{n+1}$,*

$$\int_M (H_{k-1} + H_k\langle \phi, \eta \rangle)\, dM = 0.$$

(2) *When $N^{n+1}$ is $\mathbb{H}^{n+1}$,*

$$\int_M (H_{k-1}\langle \phi, p \rangle + H_k\langle \eta, p \rangle)\, dM = 0 \quad \text{for any } p \in \mathbb{R}^{n+2}.$$

(3) *When $N^{n+1}$ is $\mathbb{S}^{n+1}_{+}$,*

$$\int_M (H_{k-1}\langle \phi, p \rangle - H_k\langle \eta, p \rangle)\, dM = 0 \quad \text{for any } p \in \mathbb{R}^{n+2}.$$

We also use the following inequalities for higher order mean curvatures:

LEMMA B. *Suppose that all the principal curvatures are positive. Then, for every $k = 1, 2, \dots, n$, the followings hold:*

(1) *Every $k-$th mean curvature function $H_k$ is positive.*
(2) *The equality $H_1 H_{n-1} = H_n$ holds only at umbilical points.*
(3) $H_k/H_{k-1} \leq H_{k-1}/H_{k-2}$.
(4) *For every $l < k$,* $H_k/H_l \leq H_{k-1}/H_{l-1}$.

*Proof.* (1) is clear.
(2) is the equality case for the arithmetic$-$geometric mean inequality.
For (3), see, for example, Section 12 of [2].
From (3), we have

$$H_k/H_{k-1} \leq H_{k-1}/H_{k-2} \leq \cdots \leq H_{l+1}/H_l \leq H_l/H_{l-1}.$$

Hence (4) holds. □

Now, assume $H_n/H_1 = \alpha$ for a constant number $\alpha$.

**(2.1) Proof when $N^{n+1} = \mathbb{R}^{n+1}$:** Since $M^n$ is compact, one can find a point in $M^n$ where all the principal curvatures are positive. Then $H_n, H_1$ are positive at that point. Since $H_n/H_1$ is constant on $M^n$ and since $H_1$ does not vanish on $M^n$ by assumption, $H_1$ and $H_n$ are positive on $M^n$. Then $\alpha > 0$ and from the inequality (4) of Lemma B, we have

$$(*) \qquad 0 < \alpha = H_n/H_1 \leq H_{n-1} \ (= H_{n-1}/H_0).$$

Since $H_n = \alpha H_1$, we have by Lemma A,

$$\begin{aligned} 0 &= \int_M (H_{n-1} + H_n \langle \phi, \eta \rangle)\, dM \\ &= \int_M (H_{n-1} + \alpha H_1 \langle \phi, \eta \rangle)\, dM, \end{aligned}$$

that is,

$$(1) \qquad \int_M H_{n-1}\, dM = \int_M (-\alpha H_1 \langle \phi, \eta \rangle)\, dM.$$

On the other hand, since $\alpha$ is constant, we also have by Lemma A,

$$\int_M \alpha (1 + H_1 \langle \phi, \eta \rangle)\, dM = 0,$$

that is,

$$(2) \qquad \int_M \alpha\, dM = \int_M (-\alpha H_1 \langle \phi, \eta \rangle)\, dM.$$

From (1) and (2), we have

$$\int_M (H_{n-1} - \alpha)\, dM = 0.$$

Since we have from $(*)$,

$$H_{n-1} - \alpha \geq 0,$$

it follows that

$$H_{n-1} = \alpha = H_n / H_1$$

everywhere on $M^n$. Now, by (2) of lemma B, every point is an umbilical point, that is, $\phi(M^n)$ is a geodesic hypersphere.

(2.2) **Proof when** $N^{n+1} = \mathbb{H}^{n+1}$: At a point of $M^n$ where the distance function of $\mathbb{H}^{n+1}$ attains its maximum, all the principal curvatures are positive. Then $(*)$ also holds in this case and $H_1, H_n$ are positive on $M^n$. Since $H_n = \alpha H_1$, we have

$$\begin{aligned} 0 &= \int_M (H_{n-1}\langle \phi, p\rangle + H_n \langle \eta, p\rangle)\, dM \\ &= \int_M (H_{n-1}\langle \phi, p\rangle + \alpha H_1 \langle \eta, p\rangle)\, dM, \end{aligned}$$

that is,

$$\int_M H_{n-1}\langle \phi, p\rangle\, dM = \int_M (-\alpha H_1 \langle \eta, p\rangle)\, dM.$$

Since $\alpha$ is constant, it also holds that

$$\int_M \alpha(\langle \phi, p\rangle + H_1\langle \eta, p\rangle)\, dM = 0,$$

then, it follows that

$$\int_M (H_{n-1} - \alpha)\langle \phi, p\rangle\, dM = 0.$$

Now, if we take $p = (1, 0, \ldots, 0) \in \mathbb{R}^{n+2}$, then the sign of $\langle \phi, p\rangle$ does not change on $M^n$. Since $H_{n-1} - \alpha \geq 0$ from $(*)$, we have $H_{n-1} - \alpha = 0$ everywhere on $M^n$. Then every point is an umbilical point as in (2.1). Hence $\phi(M^n)$ is a geodesic hypersphere.

(2.3) **Proof when** $N^{n+1} = \mathbb{S}^{n+1}_+$: Let $c \in \mathbb{S}^{n+1}$ be the centre of $\mathbb{S}^{n+1}_+$. Then at a point of $M^n$ where the height function $\langle \phi, c\rangle$ attains its minimum, all the principal curvatures are positive because $M^n$ lies in the open half sphere with the centre $c$. Then $(*)$ holds and the equality in $(*)$ holds only at umbilical points. Proceeding as in (2.2), we have

$$\int_M (H_{n-1} - \alpha)\langle \phi, p\rangle \, dM = 0.$$

Since $M^n$ lies in the open half sphere, for $p = c$, $\langle \phi, c\rangle$ is positive on $M^n$. Then, since $H_{n-1} - \alpha \geq 0$ by $(*)$, it follows that $H_{n-1} - \alpha = 0$ everywhere on $M^n$. Now arguing in the same way as above we can see that $\phi(M^n)$ is a geodesic hypersphere.

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