Bull. Korean Math. Soc. **36** (1999), No. 4, pp. 693–699

NOTES ON THE REGULAR MODULES

KEIVAN MOHAJER AND SIAMAK YASSEMI

ABSTRACT. It is a well-known result that a commutative ring R is von Neumann regular if and only if for any maximal ideal $\mathfrak{m}$ of R the R-module $R/\mathfrak{m}$ is flat. In this note we bring a generalization of this result for modules.

0. Introduction

Let R be a commutative ring with non-zero identity. Recall that an element $a \in R$ is said to be regular if there exists $x \in R$ such that $a^2x = a$, and R is said to be von Neumann regular if each of its elements is regular.

The familiar notion of a von Neumann regular ring has a generalization for modules. In [4], Fieldhouse defines a regular module as one whose submodules are all pure (the submodule N of M is said pure submodule if the inclusion $0 \to N \to M$ remains exact upon tensoring by any R-module).

In [10], Xu showed that R is a von Neumann regular ring if and only if for any maximal ideal $\mathfrak{m}$ of R the R-module $R/\mathfrak{m}$ is flat. We show the same result (in some sense) for modules; The R-module M is flat and regular if and only if for any maximal element $\mathfrak{m}$ of support of M the R-module $R/\mathfrak{m}$ is flat.

Received October 8, 1998.
1991 Mathematics Subject Classification: 16E50; 16D60.
Key words and phrases: von Neumann regular; flat module.

1. Regular Modules

Let M be an R–module. The support of M is denoted by Supp(M) and it is defined by

$$\begin{aligned}&\mathrm{Supp}(M)\\&=\{\mathfrak{p}\in\mathrm{Spec}(R)|\mathfrak{p}\supseteq\mathrm{Ann}(N)\text{ for some cyclic submodule }N\text{ of }M\}.\end{aligned}$$

Note that this definition is equivalent with the classical definition of support (cf. [7, page 26]) that is

$$\mathrm{Supp}(M)=\{\mathfrak{p}\in\mathrm{Spec}(R)|M_{\mathfrak{p}}\neq 0\}.$$

The Jacobson radical of M is denoted by J(M) and it is the intersection of all elements in MaxSupp(M).

PROPOSITION 1.1. *Let M be an R–module. Then the following are equivalent:*

(i) *M is regular*
(ii) *For any submodule N of M, MaxSupp(N) = Supp(N) and Ann(N) = J(N).*
(iii) *For any $x\in M$ MaxSupp(Rx) = Supp(Rx) and Ann(Rx) = J(Rx).*

Proof. (i)⇒(ii). Let $\mathfrak{p}\in$ Supp(N). Then there exists a non–zero $x\in N$ such that Ann(x) $\subseteq\mathfrak{p}$. By [2; Page 315] we know that R/Ann(x) is a von Neumann regular ring and hence $\mathfrak{p}\in$ Max(R). Therefore MaxSupp(N) = Supp(N). Since R/Ann(x) is a von Neumann regular ring we have that

$$\mathrm{Ann}(N)=\cap_{x\in N}\mathrm{Ann}(x)=\cap_{\mathfrak{m}\in\mathrm{Supp}(N)}\mathfrak{m}=\mathrm{J}(N).$$

(ii)⇒(iii). It is obvious.

(iii)⇒(i). Let $x\in M$ is a non–zero element. Since MaxSupp(Rx) = Supp(Rx) and Ann(Rx) = J(Rx) we have that R/Ann(x) is a von Neumann regular ring, cf. [6; Theorem 1.16]. Now the assertion follows from [2; Page 315]. □

THEOREM 1.2. *Let M be an R–module. Then the following are equivalent:*

(i) *M is regular*
(ii) *$M_{\mathfrak{p}}$ is a semisimple $R_{\mathfrak{p}}$–module for any $\mathfrak{p}\in$ Supp(M).*

(iii) $M_{\mathfrak{m}}$ *is a semisimple* $R_{\mathfrak{m}}$*–module for any* $\mathfrak{m} \in$ *MaxSupp*(M).

Proof. (i)$\Rightarrow$(ii). Let $\mathfrak{p} \in \mathrm{Supp}(M)$. Let $N_{\mathfrak{p}}$ be an arbitrary $R_{\mathfrak{p}}$–submodule of $M_{\mathfrak{p}}$. For any $R_{\mathfrak{p}}$–module L we have the exact sequence $0 \to L \otimes_R N \to L \otimes_R M$. Therefore $0 \to (L \otimes_R N)_{\mathfrak{p}} \to (L \otimes_R M)_{\mathfrak{p}}$ is exact and hence $0 \to L \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \to L \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ is exact. Thus $M_{\mathfrak{p}}$ is a regular $R_{\mathfrak{p}}$– module. By proposition (1.1) we know that $\mathrm{Supp}(M_{\mathfrak{p}}) = \{\mathfrak{p}R_{\mathfrak{p}}\}$ and $\mathrm{Ann}(M_{\mathfrak{p}}) = \mathfrak{p}R_{\mathfrak{p}}$ and hence $M_{\mathfrak{p}}$ is a semisimple $R_{\mathfrak{p}}$–module.

(ii)$\Rightarrow$(iii). It is obvious.

(iii)$\Rightarrow$(i). We know that any semisimple module is regular. Therefore $M_{\mathfrak{m}}$ is regular for any $\mathfrak{m} \in \mathrm{MaxSupp}(M)$. Now it is easy to see that M is regular. □

2. F–regular Modules

The R–module M is called F–regular if M is a flat and regular module. Note that any von Neumann regular ring is an F–regular as an R–module.

LEMMA 2.1. *The following are equivalent:*

(i) M *is F–regular.*

(ii) *For any* $\mathfrak{m} \in$ *MaxSupp*(M), $R/\mathfrak{m}$ *is a flat* R*–module.*

(iii) *For any* $\mathfrak{m} \in$ *MaxSupp*(M), $R/\mathfrak{m}$ *is an injective* R*–module.*

(iv) *For any* $\mathfrak{m} \in$ *MaxSupp*(M), $R_{\mathfrak{m}}$ *is a field.*

(v) *For any* R*–module* N *with Supp*$(N) \subseteq$ *Supp*(M), N *is a flat* R*–module.*

(vi) *For any cyclic* R*–module* C *with Supp*$(C) \subseteq$ *Supp*(M), C *is a flat* R*–module.*

Proof. (i)$\Rightarrow$(ii). Let M be an F–regular module. Set $\mathfrak{m} \in \mathrm{MaxSupp}(M)$. There exists a non–zero element $x \in M$ such that $\mathrm{Ann}(x) \subseteq \mathfrak{m}$. Consider the surjective homomorphism $\varphi : Rx \to R/\mathfrak{m}$. Since Rx is a pure submodule of the flat module M, we have that M/Rx is flat (cf. [8; 3.55]) and hence Rx is flat. Since Rx is regular we have that $\mathrm{Ker}(\varphi)$ is a pure submodule of Rx. Thus $R/\mathfrak{m}$ is flat, cf. [8; 3.55].

(ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(ii) are well–known, cf. [10; 1.1].

(ii)$\Rightarrow$(iv). Since $R/\mathfrak{m}$ is a flat R–module we have that $(R/\mathfrak{m})_{\mathfrak{m}} \cong R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$–module and hence $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$–module, cf. [1; Chap. II, § 3 Exercise 3(e)]. Therefore $\mathfrak{m}R_{\mathfrak{m}} = 0$ and so the assertion holds.

(iv)$\Rightarrow$(v). Let N be an R–module with $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$. Then for any $\mathfrak{m} \in \operatorname{MaxSupp}(N)$, $R_{\mathfrak{m}}$ is a field and hence $N_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$–module. Therefore N is a flat R–module.

(v)$\Rightarrow$(vi). It is clear.

(vi)$\Rightarrow$(ii). Set $\mathfrak{m} \in \operatorname{MaxSupp}(M)$. Now the assertion follows from the fact that $\operatorname{Supp}(R/\mathfrak{m}) = \{\mathfrak{m}\}$.

(v)$\Rightarrow$(i). We have M is flat. Let N be an arbitrary submodule of M. Since $\operatorname{Supp}(M/N) \subseteq \operatorname{Supp}(M)$ we have that M/N is flat and hence N is a pure submodule of M. Thus M is regular. □

COROLLARY 2.2. *The following are equivalent:*

(i) *R is a von Neumann regular ring.*
(ii) *Every R–module is F–regular.*
(iii) *Every cyclic R–module is F–regular.*

Proof. (i)$\Rightarrow$(ii). Since $\operatorname{MaxSupp}(M) \subseteq \operatorname{MaxSpec}(R)$ we have that $R/\mathfrak{m}$ is a flat R–module for any $\mathfrak{m} \in \operatorname{MaxSupp}(M)$.

(ii)$\Rightarrow$(iii). This is clear.

(iii)$\Rightarrow$(i). Since R is a cyclic R–module we have that R is a regular R–module and hence R is a von Neumann regular ring. □

COROLLARY 2.3. *The following are hold:*

(a) *Every submodule and homomorphic image of an F–regular module is F–regular.*
(b) *If $R/\operatorname{Ann}(M)$ is an F–regular R–module then M is F–regular.*
(c) *If M is a finitely generated F–regular module then $R/\operatorname{Ann}(M)$ is a (von Neumann) regular ring.*

Proof. (a) For any submodule N of M and any homomorphic image L of M we have $\operatorname{MaxSupp}(N) \subseteq \operatorname{MaxSupp}(M)$ and $\operatorname{MaxSupp}(L) \subseteq \operatorname{MaxSupp}(M)$.

(b) The assertion holds from the fact that $\operatorname{MaxSupp}(M) \subseteq \operatorname{MaxSupp}(R/\operatorname{Ann}(M))$.

(c) We have that $\operatorname{MaxSupp}(M) = \operatorname{MaxSpec}(R/\operatorname{Ann}(M))$. □

REMARK 2.4. In part (b) of corollary 2.3 we can not change the condition F–regular module for $R/\mathrm{Ann}(M)$ to a von Neumann regular ring. For example let $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ such that $M = R/\mathfrak{m}$ is not a flat R–module. Then $R/\mathrm{Ann}(M)$ is a von Neumann regular ring but M is not an F–regular module.

THEOREM 2.5. *Let M and N be R–modules and M be a finitely presented module. If M or N are F–regular then $Hom(M, N)$ is F–regular.*

Proof. For any $\mathfrak{p} \in \mathrm{Spec}(R)$, $(\mathrm{Hom}(M, N))_{\mathfrak{p}} \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$, cf. [8; 3.84]. Therefore $\mathrm{Supp}(\mathrm{Hom}(M, N)) \subseteq \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$ and hence $\mathrm{MaxSupp}(\mathrm{Hom}(M, N)) \subseteq \mathrm{MaxSupp}(M) \cap \mathrm{MaxSupp}(N)$. Now the assertion follows from lemma 2.1. □

THEOREM 2.6. *Let M and N are R–modules. If M or N is F–regular then $M \otimes N$ is F–regular.*

Proof. We know that $\mathrm{Supp}(M \otimes N) \subseteq \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. Therefore

$$\mathrm{MaxSupp}(M \otimes N) \subseteq \mathrm{MaxSupp}(M) \cap \mathrm{MaxSupp}(N).$$

Now the assertion follows from lemma 2.1. □

THEOREM 2.7. *Let M be F–regular and let S be a multiplicative closed subset of R. Then $S^{-1}M$ is F–regular as an R–module and as an $S^{-1}R$–module.*

Proof. By theorem 2.6 we have $\mathrm{S}^{-1}M$ is F–regular as an R–module. Now set $\mathfrak{q} \in \mathrm{MaxSupp}_{\mathrm{S}^{-1}R}\mathrm{S}^{-1}M$. then there exists $\mathfrak{m} \in \mathrm{MaxSupp}(M)$ such that $\mathfrak{m} \cap \mathrm{S}$ is empty and $\mathrm{S}^{-1}\mathfrak{m} = \mathfrak{q}$. Since M is F–regular we have that $R/\mathfrak{m}$ is a flat R–module and hence $\mathrm{S}^{-1}R/\mathfrak{q} = \mathrm{S}^{-1}R/\mathrm{S}^{-1}\mathfrak{m} \cong \mathrm{S}^{-1}(R/\mathfrak{m})$ is a flat $\mathrm{S}^{-1}R$–module. □

The pure dimension of the R–module M is denoted by pure $\dim_R(M)$ and it is the least integer n such that for any finitely presented R–module P, the R–module $\mathrm{Ext}_R^{n+1}(P, M)$ is zero, cf. [5]. The R–module M is called absolutely pure if the pure dimension of M is zero. If R is a coherent ring and M is a regular module then we will show that the flat and the pure dimension of M are equal. First we bring a lemma.

LEMMA 2.8. *Let R be a coherent ring and let M be an R–module. Then the following conditions are equivalent;*

(i) *M is absolutely pure.*
(ii) *$M_\mathfrak{p}$ is an absolutely pure $R_\mathfrak{p}$–module for each $\mathfrak{p} \in Spec(R)$.*
(iii) *$M_\mathfrak{m}$ is an absolutely pure $R_\mathfrak{m}$–module for each $\mathfrak{m} \in Max(R)$.*

Proof. (i)⇒(ii). Let K' be an arbitrary finitely generated $R_\mathfrak{p}$ submodule of a finitely generated free $R_\mathfrak{p}$–module L'. There exist a finitely generated R–module K and a finitely generated free R–module L such that $K' = K_\mathfrak{p}$ and $L' = L_\mathfrak{p}$. Since M is an absolutely pure then $\mathrm{Hom}(L, M) \to \mathrm{Hom}(K, M) \to 0$ is an exact sequence and hence $(\mathrm{Hom}(L, M))_\mathfrak{p} \to (\mathrm{Hom}(K, M))_\mathfrak{p} \to 0$ is exact. We know that K is finitely presented, cf. [5, 2.3.2]. Therefore $\mathrm{Hom}_{R_\mathfrak{p}}(L', M_\mathfrak{p}) \to \mathrm{Hom}_{R_\mathfrak{p}}(K', M_\mathfrak{p}) \to 0$ is an exact sequence. Thus $M_\mathfrak{p}$ is absolutely pure.

(ii)⇒(iii). It is clear.

(iii)⇒(i). Let K be a finitely generated submodule of a finitely generated free module L. Then for each $\mathfrak{m} \in \mathrm{Max}(R)$, the sequence $\mathrm{Hom}_{R_\mathfrak{m}}(L_\mathfrak{m}, M_\mathfrak{m}) \to \mathrm{Hom}_{R_\mathfrak{m}}(K_\mathfrak{m}, M_\mathfrak{m}) \to 0$ is exact. By the same reason as above, for any $\mathfrak{m} \in \mathrm{Max}(R)$ we have the exact sequence $(\mathrm{Hom}(L, M))_\mathfrak{m} \to (\mathrm{Hom}(K, M))_\mathfrak{m} \to 0$ and hence the sequence $\mathrm{Hom}(L, M) \to \mathrm{Hom}(K, M) \to 0$ is exact. Thus M is absolutely pure. □

COROLLARY 2.9. *Let R be a coherent ring. Then for any R–module M*

$$\textit{pure dim}_R(M) = \sup\{\textit{pure dim}_{R_\mathfrak{m}}(M_\mathfrak{m}) | \mathfrak{m} \in \textit{MaxSupp}(M)\}.$$

THEOREM 2.10. *Let R be a coherent ring and let M be a regular R–module. Then flat $\dim_R(M)$ = pure $\dim_R(M)$.*

Proof. "≥" Let M be a regular module with flat $\dim_R(M) = n$. By theorem 1.2 we know that $M_\mathfrak{m}$ is a semisimple $R_\mathfrak{m}$–module, for any $\mathfrak{m} \in \mathrm{Max}(R)$. Since flat $\dim_{R_\mathfrak{m}} M_\mathfrak{m} \leq n$, we have that each simple direct summand of $M_\mathfrak{m}$ has flat dimension not greater than n and hence has injective dimension not greater than n. Since every injective module is absolutely pure, we have that pure dimension is not greater than injective dimension. Thus the pure dimension of any simple direct summand of

$M_{\mathfrak{m}}$ is not greater than n. Therefore pure $\dim_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \leq n$, cf. [9; corollary 2.4]. Now the assertion follows from corollary 2.9.

"$\leq$" Suppose that pure $\dim_R(M) = n$. By corollary 2.9 for any $\mathfrak{m} \in \mathrm{Max}(R)$, pure $\dim_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \leq n$. therefore for any simple direct summand S of $M_{\mathfrak{m}}$ we have pure $\dim_{R_{\mathfrak{m}}} S \leq n$. By [2; Corollary 2] the character module $S^+ = \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})$ is semisimple and consists of copies of S. By using [3; Theorem 1] we have flat $\dim_{R_{\mathfrak{m}}} S^+ \leq n$ and hence flat $\dim_{R_{\mathfrak{m}}} S \leq n$. Thus flat $\dim_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \leq n$. Now the assertion holds. □

ACKNOWLEDGMENTS. The authors would like to thank E. Enochs, University of Kentucky, for his invaluable help. The second author was supported in part by the University of Tehran. The authors would also like to thank the University of Tehran for the facilities offered during the preparation of this paper.

References

[1] N. Bourbaki, *Commutative algebra*, Hermann, Paris, 1972.
[2] T. J. Cheatham and J. R. Smith, *Regular and semisimple modules*, Pacific J. Math. **65** (2) (1976), 315–323.
[3] T. J. Cheatham and D. R. Stone, *Flat and projective character modules*, Proc. Amer. Math. Soc. **81** (2) (1981), 175–177.
[4] D. J. Fieldhouse, *Purity and flatness*, Ph. D. Thesis, McGill University, 1967.
[5] S. Glaz, *Commutative coherent rings*, Lecture notes in Math., vol. 1371, Springer Verlag, 1989.
[6] K. R. Goodearl, *Von Neumann regular rings*, Pitman, 1979.
[7] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.
[8] J. Rotman, *An introduction to homological algebra*, Academic Press, New York, 1979.
[9] B. Stenstrom, *Coherent rings and FP–injective modules*, J. London Math. Soc. **2** (2) (1970), 323–329.
[10] J. Xu, *Flatness and injectivity of simple modules over a commutative ring*, Commun. in Algebra **19** (2) (1991), 535–537.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEHRAN, P.O. BOX 13145–448, TEHRAN, IRAN
E-mail: yassemi@khayam.ut.ac.ir