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AN EMBEDDED 2-SPHERE IN IRREDUCIBLE 4-MANIFOLDS

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ABSTRACT. It has long been a question which homology class is represented by an embedded 2-sphere in a smooth 4-manifold. In this article we study the adjunction inequality, one of main results of Seiberg-Witten theory in smooth 4-manifolds, for an embedded 2-sphere. As a result, we give a criterion which homology class cannot be represented by an embedded 2-sphere in some cases.

1. Introduction

As gauge theory, in particular Seiberg-Witten theory, has revealed many remarkable facts in smooth 4-manifolds, it has also a powerful application in studying smoothly embedded surfaces in a smooth 4-manifold. For example, as a generalization of adjunction formula in complex geometry, one gets a similar formula in a smooth 4-manifold, called adjunction inequality. As we see in section 3, the adjunction inequality is a powerful tool to study the minimal genus of an embedded surface representing the same homology class in a smooth 4-manifold with non-trivial SW-basic classes and it also tells us an upper bound of intersection numbers of a given homology class with SW-basic classes. But the adjunction inequality is not known for a smoothly embedded 2-sphere. Hence "In which smooth 4-manifolds does the adjunction inequality hold for embedded 2-spheres?" is an interesting question. We are going to answer for this question in some cases. That is, if X is a minimal symplectic 4-manifold or a spin smooth 4-manifold having one

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SW-basic class with $b_2^+ > 1$, we prove that the adjunction inequality on X is still true for an embedded 2-sphere. Explicitly

THEOREM 1.1. *Suppose X is a minimal symplectic 4-manifold $b_2^+ > 1$, or a spin smooth 4-manifold with one SW-basic class and $b_2^+ > 1$. Then any homologically non-trivial, smoothly embedded 2-sphere S in X satisfies the adjunction inequality:*

$$-2 \geq [S] \cdot [S] + |K_X \cdot [S]|$$

where K_X is a canonical class or SW-basic class of X.

Furthermore, as a corollary of Theorem 1.1 above, we get a criterion that if X is a minimal symplectic 4-manifold $b_2^+ > 1$ or a spin smooth 4-manifold with one SW-basic class and $b_2^+ > 1$, then any non-trivial homology class $\alpha \in H_2(X : \mathbf{Z})$ satisfying $\alpha \cdot \alpha + |\alpha \cdot K_X| \geq 0$ cannot be represented by a smoothly embedded 2-sphere.

2. Seiberg-Witten equations

In this section we briefly review the basics of Seiberg-Witten equations introduced by N. Seiberg and E. Witten (cf. [11], [5]).

Let X be an oriented, closed Riemannian 4-manifold, and let L be a characteristic line bundle on X, i.e., $c_1(L)$ is an integral lift of $w_2(X)$. This determines a $Spin^c$-structure on X which induces a unique complex spinor bundle $W \cong W^+ \oplus W^-$, where $W^\pm$ is the associated $U(2)$-bundles on X. Then $W^\pm \cong S^\pm \otimes L^{1/2}$ and $\det(W^\pm) \cong L$, where $S^\pm$ is a (locally defined) spinor bundle on X. For simplicity we assume that $H^2(X; \mathbf{Z})$ has no 2-torsion so that the set $Spin^c(X)$ of $Spin^c$-structures on X is identified with the set of characteristic line bundles on X.

Note that the Levi-Civita connection on TX together with a unitary connection A on L induces a connection $\nabla_A : \Gamma(W^+) \to \Gamma(T^*X \otimes W^+)$. This connection, followed by Clifford multiplication, induces a $Spin^c$-Dirac operator $D_A : \Gamma(W^+) \to \Gamma(W^-)$. The Seiberg-Witten equations are the following pair of equations for a unitary connection A on L and a section Ψ of $\Gamma(W^+)$:

$$(1) \qquad \begin{cases} D_A\Psi & = & 0 \\ \rho(F_A^+) & = & i(\Psi \otimes \Psi^*)_0 \end{cases}$$

where F_A^+ is the self-dual part of the curvature of A and $(\Psi \otimes \Psi^*)_0$ is the trace-free part of $(\Psi \otimes \Psi^*)$ interpreted as an endomorphism of W^+.

The gauge group $\mathcal{G} := Aut(L) \cong Map(X, S^1)$ acts on the space $\mathcal{A}_X(L) \times \Gamma(W^+)$ by

$$g \cdot (A, \Psi) = (g \circ A \circ g^{-1}, g \cdot \Psi).$$

Since the set of solutions is invariant under the action, it induces an orbit space, called the *(Seiberg-Witten) moduli space*, denoted by $M_X(L)$, whose formal dimension is

$$\dim M_X(L) = \frac{1}{4}(c_1(L)^2 - 3\sigma(X) - 2e(X))$$

where $\sigma(X)$ is the signature of X and $e(X)$ is the Euler characteristic of X. Note that if $b^+(X) > 0$ and $M_X(L) \neq \phi$, then for a generic metric on X the moduli space $M_X(L)$ contains no reducible solutions, so that it is a compact, smooth manifold of the given dimension. Furthermore the moduli space $M_X(L)$ is orientable and its orientation is determined by a choice of orientation on $\det(H^0(X;\mathbf{R}) \oplus H^1(X;\mathbf{R}) \oplus H^2_+(X;\mathbf{R}))$.

DEFINITION. The *Seiberg-Witten invariant* for a smooth 4-manifold X is a function $SW_X : Spin^c(X) \to \mathbf{Z}$ defined by

$$SW_X(L) = \begin{cases} 0 & \text{if } \dim M_X(L) < 0 \text{ or odd} \\ \sum_{(A,\Psi) \in M_X(L)} sign(A, \Psi) & \text{if } \dim M_X(L) = 0 \\ \langle \beta^{d_L}, [M_X(L)] \rangle & \text{if } \dim M_X(L) := 2d_L > 0 \text{ and even.} \end{cases}$$

Here $sign(A, \Psi)$ is ± 1 determined by an orientation on $M_X(L)$, and $\beta \in H^2(M_X(L);\mathbf{Z})$ is the first chern class of the $U(1)$-bundle

$$\widetilde{M_X}(L) = \{\text{solutions}(A, \Psi)\}/Aut^0(L) \longrightarrow M_X(L)$$

where $Aut^0(L)$ consists of gauge transformations which are the identity on the fiber of L over a fixed basepoint in X. For convenience, we denote the Seiberg-Witten invariant for X by $SW_X = \sum_L SW_X(L) \cdot e^L$.

DEFINITION. Let X be an oriented, closed smooth 4-manifold with $b_2^+ > 1$. We say a cohomology class $c_1(L) \in H^2(X;\mathbf{Z})$ is a *Seiberg-Witten basic class* (for brevity, *SW-basic class*) for X if $SW_X(L) \neq 0$.

It turns out that the Seiberg-Witten theory has many powerful applications to smooth 4-manifolds. For example, if $b_2^+(X) > 1$, the Seiberg-Witten invariant $SW_X = \sum SW_X(L) \cdot e^L$ is a diffeomorphism invariant, i.e., SW_X does not depend on the choice of generic metric on X and generic perturbation of the Seiberg-Witten equation. Furthermore, only finitely many $Spin^c$-structures on X have a non-zero Seiberg-Witten invariant. It also measures to some extent whether a given smooth 4-manifold is irreducible or not. That is, since the Seiberg-Witten invariant for a connected sum manifold $X = X_1 \sharp X_2$ with $b_2^+(X_i) > 0$ $(i = 1, 2)$ is identically zero, $SW_X \neq 0$ implies that X is irreducible unless X is homeomorphic to a blow-up manifold. Note that a smooth 4-manifold X is called *irreducible* if X is not a connected sum of other manifolds except for a homotopy sphere.

3. Adjunction inequality for embedded 2-spheres

In this section we investigate the adjunction inequality for an embedded 2-sphere in some irreducible 4-manifolds. The adjunction inequality which is originated from Thom conjecture, the complex curves in $\mathbf{CP}^2$ minimize the genus in their homology class, was obtained initially by Kronheimer and Mrowka's hard work in Donaldson theory ([4]), and later it was proved by an easy argument using Seiberg-Witten theory. As we see below, the adjunction inequality is a powerful tool to study the minimal genus of an embedded surface representing the same homology class in a smooth 4-manifold with non-trivial SW-basic classes and it also tells us an upper bound of intersection numbers of a given homology class with SW-basic classes. But the adjunction inequality is not known for a smoothly embedded 2-sphere (See also below). In this section we prove that if X is a minimal symplectic 4-manifold or a spin smooth 4-manifold having one SW-basic class with $b_2^+ > 1$, then X satisfies the adjunction inequality for an embedded 2-sphere. Furthermore we give a criterion that some homology classes in such 4-manifolds can not be represented by a smoothly embedded 2-sphere. First we state the adjunction inequality:

THEOREM 3.1 (Adjunction inequality [1], [5]). *Suppose X is a smooth 4-manifold with $b_2^+ > 1$ and $SW_X \neq 0$. If Σ is a smoothly embedded,* *oriented surface in X representing non-trivial homology class $[\Sigma]$ with $[\Sigma]\cdot[\Sigma]\geq 0$, then for any SW-basic class K_X of X*

$$2\cdot \text{genus}(\Sigma)-2 \;\geq\; [\Sigma]\cdot[\Sigma]+|K_X\cdot[\Sigma]|$$

REMARKS. **1**. Fintushel and Stern obtained a similar adjunction inequality for an immersed 2-sphere ([2]), and recently Ozsváth and Szabó extended the same adjunction inequality to the embedded surfaces with genus >0 and a negative self-intersection number in case X is a SW-simple type ([6]).
2. This induces the genus minimizing problem of embedded surfaces representing the same homology class in a smooth 4-manifold, i.e.,

$$\text{genus}(\Sigma)\geq\; 1+\frac{[\Sigma]\cdot[\Sigma]+|K_X\cdot[\Sigma]|}{2}$$

Note that if X is a complex surface (a symplectic 4-manifold), then the minimal genus of an embedded surface is obtained by a complex curve (a symplectic curve).
3. An immediate corollary of the adjunction inequality above is that any smoothly embedded 2-sphere representing non-trivial homology class in a smooth 4-manifold with $b_2^+>1$ and $SW_X\neq 0$ should have a negative self-intersection number. But, as we noticed that the adjunction inequality above is true only for an embedded 2-sphere with a non-negative self-intersection number, Theorem 3.1 above and Ozsváth and Szabó's recent result do not say anything for an embedded 2-sphere.

Now let us try to prove our main result by using a fundamental proposition proved by Fintushel and Stern.

PROPOSITION 3.1 ([2]). *Suppose X is a smooth 4-manifold with an embedded 2-sphere S with self-intersection $-r<0$. Let L be a characteristic line bundle with $SW_X(L)\neq 0$ and write*

$$|S\cdot L|=kr+R\,, \qquad \text{with }\; 0\leq R\leq r-1\,.$$

If $k>0$, then

$$SW_X(L)=\begin{cases} SW_X(L+2S) & \text{if }\; L\cdot S>0\\ SW_X(L-2S) & \text{if }\; L\cdot S<0\,.\end{cases}$$

REMARK. Proposition 3.1 above can be also proved by computing dimensions of related moduli spaces of SW-equations, of which argument is appeared in the author's thesis ([7]).

THEOREM 3.2. *Suppose X is a spin, smooth 4-manifold with one SW-basic class K_X and $b_2^+ > 1$. Then any homologically non-trivial, smoothly embedded 2-sphere S in X satisfies the adjunction inequality:*

$$-2 \ \geq \ [S]\cdot[S] + |K_X \cdot [S]| \ .$$

Proof. Suppose that S is a homologically non-trivial, smoothly embedded 2-sphere in X such that

$$[S]\cdot[S] + |K_X \cdot [S]| \geq 0 \ .$$

Then by Proposition 3.1 above, we have

$$SW_X(K_X) = \begin{cases} SW_X(K_X + 2[S]) & \text{if } \ K_X\cdot[S] > 0 \\ SW_X(K_X - 2[S]) & \text{if } \ K_X\cdot[S] < 0 \ . \end{cases}$$

Since X has (up to sign) one SW-basic class and $[S] \neq 0$,

$$-K_X = K_X + 2[S] \ \ (\text{or } \ -K_X = K_X - 2[S])$$

holds, so that $K_X = \pm[S]$ and $[S]\cdot[S] + |K_X \cdot [S]| = 0$. Hence the Poincare dual $PD([S])$ of an embedded 2-sphere S is a SW-basic class. Now decompose and stretch the manifold X along a boundary of a tubular neighborhood N_S of S so that $X = X_0 \cup_L N_S$. Then since the boundary, which is a lens space L, and a tubular neighborhood of S admit a positive scalar curvature metric, only a reducible SW-solution exists on the neck $L \times \mathbf{R}$ and on the tubular neighborhood N_S. Hence the SW-invariants of the manifold depend only on the other side X_0. That is, $SW_X(K_X) = SW_{X_0}(K_X|_{X_0})$. But $K_X|_{X_0} = 0|_{X_0}$ and $0 \in H^2(X : \mathbf{Z})$ is also a characteristic class (X is spin). Hence we have

$$SW_X(K_X) = SW_{X_0}(K_X|_{X_0}) = SW_{X_0}(0|_{X_0}) = SW_X(0)$$

contradicting a hypothesis that X has one SW-basic class. Thus if S is a smoothly embedded 2-sphere with $[S] \neq 0$ in X, then it satisfies

$$0 > [S]\cdot[S] + |K_X \cdot [S]| \ .$$

Furthermore, since K_X is a characteristic class in $H_2(X : \mathbf{Z})$, $[S]\cdot[S] + |K_X \cdot [S]|$ is even, so that the adjunction inequality holds

$$-2 \ \geq \ [S]\cdot[S] + |K_X \cdot [S]| \ . \qquad \square$$

Note that the set of all irreducible 4-manifolds with one SW-basic class is quite a large class of smooth 4-manifolds. For example, every simply connected minimal complex surface of general type with $b_2^+ > 1$ is an irreducible 4-manifold with one SW-basic class ([11]), and there are also infinitely many irreducible 4-manifolds with one SW-basic class which cannot admit a complex structure in any orientation ([3], [7], [8]). In fact, by using the same technique as in the proof above, we can extend this result to minimal symplectic 4-manifolds. Explicitly,

THEOREM 3.3. *Suppose X is a minimal symplectic 4-manifold with a canonical class K_X and $b_2^+ > 1$. Then any homologically non-trivial, smoothly embedded 2-sphere S in X satisfies the adjunction inequality:*

$$-2 \geq [S] \cdot [S] + |K_X \cdot [S]| .$$

Proof. Since a minimal symplectic 4-manifold X with $b_2^+ > 1$ is SW-simple type and the class K_X^{-1} of X is also a SW-basic class ([9]), if $[S] \cdot [S] + |K_X^{-1} \cdot [S]| = [S] \cdot [S] + |K_X \cdot [S]| > 0$, Proposition 3.1 above implies that

$$SW_X(K_X^{-1}) = \begin{cases} SW_X(K_X^{-1} + 2[S]) & \text{if } K_X^{-1} \cdot [S] > 0 \\ SW_X(K_X^{-1} - 2[S]) & \text{if } K_X^{-1} \cdot [S] < 0 . \end{cases}$$

But

$$\begin{aligned} \dim M_X(K_X^{-1} \pm 2[S]) &= \frac{1}{4}[(K_X^{-1} \pm 2[S])^2 - (2e(X) + 3\sigma(X))] \\ &= \dim M_X(K_X^{-1}) + ([S] \cdot [S] \pm K_X \cdot [S]) \\ &> 0 \end{aligned}$$

which contradicts that X is SW-simple type. Hence $[S] \cdot [S] + |K_X \cdot [S]| \leq 0$. If $[S] \cdot [S] + |K_X \cdot [S]| = 0$, as in the proof of Theorem 3.2 above, there are two possibilities: either $K_X = \pm[S]$ or a new SW-basic class $K_X^{-1} \pm 2[S]$. But since the canonical class K_X of a minimal symplectic 4-manifold with $b_2^+ > 1$ has a non-negative square, we get a contradiction $0 \leq K_X \cdot K_X = [S] \cdot [S] < 0$. Furthermore, since $SW(K_X^{-1} + 2E) = Gr(E)$ and $E \cdot E \geq 0$ for a minimal symplectic manifold with $b_2^+ > 1$ ([10]), $K_X^{-1} \pm 2[S]$ cannot be a SW-basic class of X, either. Hence we have

$$0 > [S] \cdot [S] + |K_X \cdot [S]|$$

so that the adjunction inequality holds

$$-2 \geq [S] \cdot [S] + |K_X \cdot [S]| . \qquad \square$$

COROLLARY 3.1. *Suppose X is a closed, smooth 4-manifold with $b_2^+ > 1$ which satisfies $(K_X - K'_X)^2 > -4$, for all SW-basic classes K_X, K'_X of X. Then any homologically non-trivial, smoothly embedded 2-sphere S in X satisfies the adjunction inequality:*

$$-2 \geq [S] \cdot [S] + |K_X \cdot [S]| .$$

Proof. If $[S] \cdot [S] + |K_X \cdot [S]| \geq 0$, then by the same way as in the proof above both K_X and $K_X \pm 2[S]$ are SW-basic classes of X. But $(K_X - (K_X \pm 2[S]))^2 = 4[S] \cdot [S] \leq -4$ contradicts the hypothesis of X. $\square$

COROLLARY 3.2. *Suppose X is a minimal symplectic 4-manifold $b_2^+ > 1$, or a spin smooth 4-manifold with one SW-basic class and $b_2^+ > 1$. Then any non-trivial homology class $\alpha \in H_2(X : \mathbf{Z})$ satisfying $\alpha \cdot \alpha + |\alpha \cdot K_X| \geq 0$ cannot be represented by a smoothly embedded 2-sphere.*

EXAMPLE. Let $E(n)$ be a simply connected elliptic surface with holomorphic Euler characteristic n and no multiple fibers. The intersection form of $E(n)$ is

$$Q_{E(n)} = \begin{cases} (2n-1)(1) \oplus (10n-1)(-1), & n = \text{odd} \\ nE_8 \oplus (2n-1)H, & n = \text{even} \end{cases}$$

where E_8 is the rank 8 negative definite intersection form obtained by the Dynkin diagram of E_8 and H is the intersection form of $S^2 \times S^2$, and the canonical class of $E(n)$ is $K_{E(n)} = (n-2)f$, where f is a generic fiber which is represented by one of two generators in H. Then any element of the form $\alpha + \beta \in H_2(E(n) : \mathbf{Z})$, where α and β are homology classes lying in nE_8 and $(2n-1)H$ respectively such that $\alpha^2 + (n-2)|f \cdot \beta| \geq 0$ and $\beta^2 = 0$, cannot be represented by a smoothly embedded 2-sphere. Note that all such classes have negative self-intersection numbers.

We close this paper by suggesting the following question:

QUESTION. Is the adjunction inequality still true in general for an embedded 2-sphere in any irreducible 4-manifold?

References

[1] S. Donaldson, *The Seiberg-Witten equations and 4-manifold topology*, Bulletin of the Amer. Math. Soc. **33** (1996), no. 1, 45-70.

[2] R. Fintushel and R. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish Journal Math. **19** (1995), no. 2 , 145-157.

[3] ———, *Nonsymplectic 4-manifolds with one basic class*, preprint (1998).

[4] P. Kronheimer and T. Mrowka, *Gauge theory for embedded surfaces*, Topology **32** (1993), 773-826 .

[5] ———, *The genus of embedded surfaces in the projective plane*, Mathematical Research Letters **1** (1994), 797-808.

[6] P. Ozsváth and Z. Szabó, *The symplectic Thom conjecture*, preprint (1998).

[7] J. Park, *Seiberg-Witten invariants of generalised rational blow-downs*, Bulletin of the Austral. Math. Soc. **56** (1997), 363-384.

[8] ———, *Irreducible 4-manifolds with one SW-basic class*, in preparation (1999).

[9] C. Taubes, *From the Seiberg-Witten invariants and symplectic forms*, Mathematical Research Letters **1** (1994), 809-822.

[10] ———, *The Seiberg-Witten equations to pseudo-holomorphic curves*, Journal of the Amer. Math. Soc. **9** (1996), no. 3, 845-918.

[11] E. Witten, *Monopoles and four-manifolds*, Mathematical Research Letters **1** (1994), 769-796.

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