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FIBREWISE INFINITE SYMMETRIC PRODUCTS AND M-CATEGORY

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ABSTRACT. Using a base–point free version of the infinite symmetric product we define a fibrewise infinite symmetric product for any fibration $E \to B$. The construction works for any commutative ring R with unit and is denoted by $R_f(E) \to B$. For any pointed space B let $G_i(B) \to B$ be the i–th Ganea fibration. Defining M_R–cat $(B) := \inf\{i \mid R_f(G_i(B)) \to B$ admits a section$\}$ we obtain an approximation to the Lusternik–Schnirelmann category of B which satisfies e.g. a product formula. In particular, if B is a 1–connected rational space of finite rational type, then $M_{\mathbb{Q}}$–cat (B) coincides with the well–known (purely algebraically defined) M–category of B which in fact is equal to cat (B) by a result of K. Hess. All the constructions more generally apply to the Ganea category of maps.

0. Introduction

Let $\mathcal{S}$ be the category of simplicial sets. It carries the structure of a closed model category where the weak equivalences are the maps which turn into homotopy equivalences by realization and where the cofibrations are the injective maps [16]. Any map $Y \to X$ factors as a weak equivalence followed by a fibration which we call 'the' associated fibration.

We recall the Ganea construction [10]. Given a map $p: Y \to X$ with X pointed by $*$ we let $G_0(p): G_0(p,Y) \to X$ be the associated fibration; suppose the fibration $G_i(p): G_i(p,Y) \to X$ with fibre F_i is defined, $i \geq 0$, then let $G'_{i+1}(p,Y) \to X$ be the map $\pi: G_i(p,Y) \cup_{F_i} C(F_i) \to X$ where $C(F_i)$ is the cone on F_i and $\pi \mid G_i(p,Y) = G_i(p), \pi \mid C(F_i) = *$ and

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define $G_{i+1}(p): G_{i+1}(p, Y) \to X$ as the associated fibration. Recall that $\mathrm{cat}(X) := \inf\{i \mid G_i(* \to X)$ admits a section $\}$. In this absolute case we write $G_i(X) \to X$ for the fibration $G_i(* \to X)$.

Let now $p: E \to B$ be a map of 1-connected rational spaces of finite type over $\mathbb{Q}$. Let M_B be a Sullivan model of B and $M_B \xrightarrow{p^*} M_B \otimes_\tau M'$ a KS-extension modeling p (the index τ should remind that the differential on the tensor product is twisted).

We say that p has an M-section, if there is an M_B-module map $r: M_B \otimes_\tau M' \to M_B$ with $r \circ p^* = \mathrm{id}$. Define $M\text{-cat}(p) := \inf\{i \mid G_i(p)$ admits an M-section $\}$. In particular, $M\text{-cat}(B) := M\text{-cat}(* \to B)$ has been introduced in [12] as an algebraic approximation to $\mathrm{cat}(B)$. According to [13] one has $M\text{-cat}(B) = \mathrm{cat}(B)$.

Let R be a non-trivial commutative ring with unit. For $X \in \mathcal{S}$ let $R \otimes X$ be the free module generated by X and let $R(X) \subset R \otimes X$ be the affine subspace consisting of linear combinations of simplices with coefficient sum 1. If $R = \mathbb{Z}$, then $R(X)$ is a basepoint free version of the infinite symmetric product on X (see [2]). Given a fibration $Y \to X$ in $\mathcal{S}$ there is a fibrewise version of $R(-)$ denoted by $R_f(Y) \to X$ according to [7].

Our main result says that $M\text{-cat}(p) \leq k$, if and only if $\mathbb{Q}_f(G_k(p, E)) \to B$ admits a section.

By [13] this implies that for B a simply connected rational space of finite rational type $G_k(B) \to B$ admits a section if and only if $\mathbb{Q}_f(G_k(B)) \to B$ admits one.

Thus for any R as above, any map $g: Y \to X$ we may define an invariant $M_R\text{-cat}(g) := \inf\{i \mid R_f(G_i(g, Y)) \to X$ admits a section $\}$. In section 4 we will comment this notion and prove e.g. a product formula.

In section 1 we recall the construction of $R_f(Y) \to X$. In section 2 we study an algebraic construction serving as a bridge to the spatial construction (over $\mathbb{Q}$). In section 3 we prove the main result.

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1. Construction of $R_f(Y) \to X$

Let R be a non–trivial commutative ring with unit 1. For $X \in \mathcal{S}$ let $R \otimes X$ be the free module generated by X; in particular, $R \otimes X$ is a simplicial abelian group. According to [2] we consider $R(X) \subsetneq R \otimes X$, $(R(X))_n := \{\sum \lambda_i x_i \mid x_i \in X_n, \sum \lambda_i = 1, \lambda_i \neq 0$ for only finitely many indices $i\}$. Given a base point $* \in X$ the composition $R(X) \to R \otimes X \to R \otimes X / R \otimes *$ is a weak equivalence. (Moreover $R(X)$ can be identified with kernel $(R \otimes X \to R \otimes *)$).

There is a natural map $X \to R(X)$, $x \mapsto 1x$, hence $R(\_)$ is a coaugmented functor $\mathcal{S} \to \mathcal{S}$ in the sense of [7]. It takes weak equivalences to weak equivalences and its value on a point space is the point space. Therefore there is a fibrewise version of the functor by [7]. We recall its definition:

Let $\pi : Y \to X$ be a fibration. Denote by I the simplex category of X [4]; its objects are the simplices $\sigma : \Delta[n] \to X$ ($\Delta[n]$ the standard n–simplex), the morphisms $(\Delta[n], \sigma) \to (\Delta[m], \tau)$ are the order preserving maps $\alpha : \Delta[n] \to \Delta[m]$ with $\alpha^*(\tau) = \sigma$. Let $\widetilde{X}$ denote the functor $I \to \mathcal{S}$ given on objects by $\widetilde{X}((\Delta[n], \sigma)) = \Delta[n]$; let $\widetilde{Y} : I \to \mathcal{S}$ be the functor with $\widetilde{Y}((\Delta[n], \sigma)) := \sigma^*(Y)$, the pullback of $Y \to X$ by σ. We then have canonical weak equivalences hocolim $\widetilde{X} \xrightarrow{\simeq} \mathrm{colim}\widetilde{X} \xrightarrow{\simeq} X$ and hocolim $\widetilde{Y} \xrightarrow{\simeq} \mathrm{colim}\widetilde{Y} \xrightarrow{\simeq} Y$.

We now form the following diagram:

$$\begin{tikzcd}
 & & \mathrm{hocolim}\, \widetilde{Y} \arrow[dl] \arrow[dr, "\simeq"] & \\
\mathrm{hocolim}\, R \circ \widetilde{Y} \arrow[d] & P \arrow[l] \arrow[r] \arrow[d] & R_f(Y) \arrow[dr] & Y \arrow[l] \arrow[d, "\pi"] \\
\mathrm{hocolim}\, R \circ \widetilde{X} & \mathrm{hocolim}\, \widetilde{X} \arrow[l] \arrow[rr, "\simeq"] & & X .
\end{tikzcd}$$

Here $R \circ \widetilde{Y} : I \to \mathcal{S}$ is the composition of the functors $\widetilde{Y}$ and $R(\_)$ (similarly for $R \circ \widetilde{X}$). The lower left square is a homotopy pullback diagram. The upper triangle is a homotopy pushout defining $R_f(Y) \to$

X. The homotopy fibre of $R_f(Y) \to X$ over a component of X is weakly equivalent to any $R(\sigma^*(Y))$, σ a simplex of the component.

2. The Universal A–algebra of an A–module

Let R be a commutative field throughout this section.

Denote by Mod the category of graded (in degrees ≥ 0) R–modules. Let $dMod$ be the category of graded differential modules over R, the differential having degree 1.

Let Alg be the category of differential commutative associative graded algebras over R with unit.

For any R–module $M \in Mod$ we denote by $\Lambda(M)$ the free R–algebra on M. If $M \in dMod$, then $\Lambda(M)$ is given the unique algebra differential extending the differential on M.

Given $A \in Alg$. An A–module is an object $M \in dMod$ together with a structure map $A \otimes_R M \to M$ satisfying the usual properties (comp. [11]).

DEFINITION 1. Let $A \in$ Alg. An A–algebra is an algebra $B \in$ Alg together with a morphism (in Alg) $j: A \to B$.

Note that an A–algebra is an A–module in the obvious way.

PROPOSITION 1. *Given an A–module M, there exists an A–algebra U_M and an A–module map $\iota: M \to U_M$ such that for any A–algebra B and A–module map $\alpha: M \to B$ there exists a unique A–algebra morphism $a: U_M \to B$ with $a \circ \iota = \alpha$.*

Proof. Denote by am the image of $a \otimes m \in A \otimes M$ under the structure map $A \otimes M \to M$.

Set $U_M := (A \otimes \Lambda(M))/W$ where W is the differential ideal generated by $\{a \otimes m - 1 \otimes am \mid a \in A, m \in M\}$. Define $\iota: M \to U_M$ by $\iota(m) := [1 \otimes m]$ (where $[\_]$ means coset with respect to W).

We note that U_M is an A–algebra with $A \to U_M$ given by $A \to A \otimes \Lambda(X) \to A \otimes \Lambda(X)/W$. The map ι is an A–module map; for given $a \in A$, $m \in M$, $\iota(am) = [1 \otimes am] = [a \otimes m] = a[1 \otimes m] = a\iota(m)$.

To check the universal property is straightforward. $\square$

We shall need a variant of this construction.

DEFINITION 2. An R–module $M \in \mathrm{Mod}$ is pointed, if there is a distinguished element – which we call 1_M – in M^0. In case $M \in d\mathrm{Mod}$ we require $d(1_M) = 0$.

An algebra A is always pointed by its unit element 1_A.

DEFINITION 3. Given a pointed R–module M, let $\hat{\Lambda}(M)$ be the algebra $\Lambda(M)/J$ where J is the differential ideal generated by $1 - 1_M$ (1 the unit element of $\Lambda(M)$).

REMARK 1. The functor $\hat{\Lambda}(\_)$ is left adjoint to the forgetful from Alg to the category of pointed modules.

The universal construction of Proposition 1 has an obvious analogue in the pointed category. We give the construction as a definition.

DEFINITION 4. Let M be a pointed A–module. Then we set $\hat{U}_M := (A \otimes \hat{\Lambda}(M))/W$ where W is the differential ideal generated by $\{a \otimes [m] - 1 \otimes [am] \mid a \in A, [m] \text{ coset of } m \in M \text{ in } \hat{\Lambda}(M)\}$.

DEFINITION 5. An A–module M is called semi–free [9], if there exists a free R–module $X = \bigoplus_{i=0}^{\infty} X_i$ (each X_i a graded module) such that $M \cong A \otimes X$ and $d_M(1 \otimes X_i) \subset A \otimes (\bigoplus_{j<i} X_j)$, all i.

The semi–free module M is pointed, if it is pointed by an element $1_A \otimes 1_X$ with $1_X \in X_0^0$; the module $X \in Mod$ is pointed by 1_X.

PROPOSITION 2. *Let $M = A \otimes X$ be a semi–free A–module (res. a pointed semi–free A–module). Then $U_M \cong A \otimes_\tau \Lambda(X)$ (resp. $\hat{U}_M \cong A \otimes_\tau \hat{\Lambda}(X)$). (Note: The symbol $(\_)_\tau$ indicates that the differential is twisted; it is the unique algebra differential extending the differential on M).*

Proof. We consider only the first case. The A–module map $A \otimes X \to A \otimes_\tau \Lambda(X)$ induces an A–algebra morphism $U_M \cong (A \otimes \Lambda(A \otimes X))/W \to A \otimes_\tau \Lambda(X)$ which is surjective. Since any element in $A \otimes \Lambda(A \otimes X)$ is equivalent modulo W to one in the subset $A \otimes_\tau \Lambda(X)$, this map is also injective. □

REMARK 2. Let $M \in d\mathrm{Mod}$ be pointed and augmented, i.e., there is given a module map $c\colon M \to R$ with $c(1_M) = 1_R$. Let $\overline{M} := \mathrm{kernel}(c)$. Then $\hat{\Lambda}(M) \cong \Lambda(\overline{M})$.

COROLLARY. *Let $M_B \xrightarrow{p^*} M_B \otimes_\tau M'$ be a KS–extension modeling the fibration $p\colon E \to B$ (as in the Introduction). Then an M_B–module map $r\colon M_B \otimes_\tau M' \to M_B$ with $r \circ p^* = \mathrm{id}$ exists, if and only if an algebra map $\rho\colon M_B \otimes_\tau \hat{\Lambda}(M') \to M_B$ with $\rho \circ (\iota \circ p^*) = \mathrm{id}$ exists where $\iota\colon M_B \otimes_\tau M' \to M_B \otimes_\tau \hat{\Lambda}(M')$ is the canonical inclusion.*

3. The Main Result

Let $p\colon E \to B$ be a fibration of 1–connected rational spaces of finite type over $\mathbb{Q}$. Let M_B be a Sullivan model of B and let the KS–extension $M_B \to M_B \otimes_\tau M'$ model p.

THEOREM. *The KS–extension $M_B \to M_B \otimes_\tau \hat{\Lambda}(M')$ is a model of the fibration $\mathbb{Q}_f(E) \to B$.*

To give the proof we have to recall a few facts from rational homotopy theory.

Denote by Δ the usual category of standard simplices $\Delta[n]$. For each n let A^*_n be the algebra of polynomial differential forms on $\Delta[n]$ with rational coefficients. The collection A^*_n, $n = 0, 1, 2, \ldots$ constitutes a simplicial object in Alg to be denoted by $A^*_\bullet$. We recall that $A^*_\bullet$ is a cohomology theory in the sense of [3] and that $Z^n(A^*_\bullet)$ is in a canonical way an Eilenberg–Maclane space $K(\mathbb{Q}, n)$. (Here $Z^n(A^*_k)$ is the module of cycles of degree n in the differential algebra A^*_k).

Given $X \in \mathcal{S}$ one defines $A^*(X) := \{f\colon X \to A^*_\bullet \mid f \text{ simplicial}\}$; for $M \in Alg$ we set $\|M\| := \{g\colon M \to A^*_\bullet \mid g \text{ differential algebra map}\}$, so that $\|M\|_n := \{g\colon M \to A^*_n \mid g \text{ map of differential algebras}\}$.

If $M_X \xrightarrow{\kappa} A^*(X)$ is a cofibrant model, then we obtain a simplicial map $X \xrightarrow{\bar{\kappa}} \|M_X\|$ by $\sigma \mapsto \sigma^* \circ \kappa$, where $\sigma \in X_n$ is viewed as a simplicial map $\sigma\colon \Delta[n] \to X$. If X is 1–connected rational of finite type over $\mathbb{Q}$, this map $\bar{\kappa}$ is a weak equivalence (see [1]).

PROPOSITION 3. *Let $X \in \mathcal{S}$ be 1–connected rational of finite type over $\mathbb{Q}$. Let $M_X \xrightarrow{\kappa} A^*(X)$ be a cofibrant model. Then there is a canonical weak equivalence $\mathbb{Q}(X) \to \|\hat{\Lambda}(M_X)\|$.*

Proof. (a) We first find a weak equivalence

$$\mathbb{Q} \otimes X \to \|\Lambda(M_X)\|.$$

We look at the following diagram:

$$\begin{tikzcd} \|\Lambda(M_X)\| & \mathbb{Q} \otimes X \arrow[l, dashed, "\overline{\overline{\kappa}}"'] \\ \|M_X\| \arrow[u] & X . \arrow[l, "\overline{\kappa}"'] \arrow[u] \end{tikzcd}$$

Observe that $\|\Lambda(M_X)\| = Alg(\Lambda(M_X), A^*_\bullet) = Cochain(M_X, A^*_\bullet)$ is a simplicial abelian group. (Here $Cochain(-,-)$ denotes the set of cochain morphisms). Hence there is a unique morphism $\overline{\overline{\kappa}}$ of simplicial abelian groups making the diagram commute. We want to show that $\overline{\overline{\kappa}}$ induces isomorphisms of homotopy groups. Recall that $\pi_*(\mathbb{Q} \otimes X)$ is canonically identified with $H_*(X, \mathbb{Q})$. On the other hand $Cochain(M_X, A^*_\bullet)$ is weakly equivalent to $Cochain(\bigoplus_{i \geq 0} H^i(X, \mathbb{Q}), A^*_\bullet) = \prod_{i \geq 0} Cochain(H^i(X, \mathbb{Q}), Z^i(A^*_\bullet))$. Recall that $Z^i(A^*_\bullet)$ is a $K(\mathbb{Q}, i)$; therefore $\pi_n(Cochain(H^i(X, \mathbb{Q}), Z^i(A^*_\bullet)))$ is canonically isomorphic to $H_i(X, \mathbb{Q})$ for $n = i$ and is zero for $n \neq i$. Moreover, the composition $X \to \|M_X\| \to \|\Lambda(M_X)\|$ identifies $H_*(X, \mathbb{Q})$ with $\pi_*(\|\Lambda(M_X)\|)$. Hence $\overline{\overline{\kappa}}$ is a weak equivalence.

(b) Choose a base point $* \in X$. Then M_X inherits an augmentation $M_X \to A^*(X) \to A^*(*)$; let $\overline{M}_X$ be the augmentation ideal. We now consider the following diagram:

$$\begin{tikzcd}
Alg(\Lambda(\overline{M}_X), A^*_\bullet) \arrow[r, phantom, "\simeq"] \arrow[ddr, "\simeq"'] & Alg(\hat{\Lambda}(M_X), A^*_\bullet) \arrow[d] & R(X) \arrow[l, dashed] \arrow[d] \arrow[dd, bend left, "\simeq"] \\
 & Cochain(M_X, A^*_\bullet) \arrow[d] & R \otimes X \arrow[l, "\overline{\overline{\kappa}}"'] \arrow[d] \\
 & Cochain(\overline{M}_X, A^*_\bullet) & R \otimes X / R \otimes * \arrow[l, "\simeq"]
\end{tikzcd}$$

We only need to check that $\overline{\overline{\kappa}}$ induces a map $R(X) \to Alg(\hat{\Lambda}(M_X), A^*_\bullet)$.

A linear combination $\Sigma\lambda_i\sigma_i$ of simplices in $(R \otimes X)_n$ with $\Sigma\lambda_i = 1$ is mapped to the element given by $\Sigma\lambda_i(\sigma_i^* \circ \kappa) \mid M_X$ in *Cochain* $(M_X, A^*_\bullet)$. Obviously the corresponding algebra map $\Lambda(M_X) \to A^*_\bullet$ vanishes on the element $(1 - 1_{M_X})$. □

Proof of the Theorem. Let $\mathbb{Q} - \mathcal{S}_1$ be the category of 1–connected rational spaces of finite type over $\mathbb{Q}$. We first choose a functorial cofibrant model construction $\mathbb{Q} - \mathcal{S}_1 \to Alg, X \mapsto M_X$. For the arrows $M_X \to M_Y$ we then choose a functorial KS–model to be denoted by $M_X \to M_X \otimes M'_Y$. (We drop the index τ reminding that the differential on the tensor product is twisted). These choices are possible according to [1].

Let $E \to B$ be a fibration, E, $B \in \mathbb{Q} - \mathcal{S}_1$. As in section 1 let I be the simplex category of B and $\tilde{B}, \tilde{E}$ the corresponding functors.

For an n–simplex $\sigma \in B_n$ we denote the KS–model of $\sigma^*(E) \to \Delta[n]$ by $M_B(\sigma) \to M_B(\sigma) \otimes M'_E(\sigma)$. To perform the fibrewise construction of R on $E \to B$ we may as well perform it on $\|M_B \otimes M'_E\| \to \|M_B\|$ using the maps $\|M_B(\sigma) \otimes M'_E(\sigma)\| \to \|M_B(\sigma)\|$ as building blocks. So the essential part of the construction is given by the diagram

$$\begin{tikzcd}
E \arrow[r, "\simeq"] \arrow[d] & \|M_B \otimes M'_E\| \arrow[d] & \text{hocolim } \|M_B(\sigma) \otimes M'_E(\sigma)\| \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & \text{hocolim } R(\|M_B(\sigma) \otimes M'_E(\sigma)\|) \arrow[d] \\
B \arrow[r, "\simeq"] & \|M_B\| & \text{hocolim } \|M_B(\sigma)\| \arrow[l, "\simeq"'] \arrow[r] & \text{hocolim } R(\|M_B(\sigma)\|)
\end{tikzcd},$$

and it suffices to show that the upper row of the following diagram is constituted by weak equivalences:

$$\begin{tikzcd}
\|M_B \otimes \hat{\Lambda}(M'_E)\| \arrow[d] & \operatorname{hocolim} \|M_B(\sigma) \otimes \hat{\Lambda}(M'_E(\sigma))\| \arrow[l, "(1)"'] \arrow[r, "(2)"] \arrow[d] & \operatorname{hocolim} \|\hat{\Lambda}(M_B(\sigma) \otimes M'_E(\sigma))\| \arrow[d] & \operatorname{hocolim} R(\|M_B(\sigma) \otimes M'_E(\sigma)\|) \arrow[l, "\simeq"'] \arrow[d] \\
\|M_B\| & \operatorname{hocolim} \|(M_B(\sigma)\| \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \operatorname{hocolim} \|\hat{\Lambda}(M_B(\sigma))\| & \operatorname{hocolim} R(\|M_B(\sigma)\|) \arrow[l, "\simeq"']
\end{tikzcd}$$

The weak equivalences on the right are provided by Proposition 3. The arrows (1),(2) map the homotopy fibres of the three vertical morphism to the left, $\|\hat{\Lambda}(M'_E)\|$, $\|M_B(\sigma) \otimes \hat{\Lambda}(M'_E(\sigma))\|$ and $\|\hat{\Lambda}(M_B(\sigma) \otimes M'_E(\sigma))\|$ resp. by weak equivalences. Hence (1),(2) are weak equivalences as required. $\square$

4. Comments

First we give the definition of the invariant M_R–cat mentioned in the introduction in more detail.

DEFINITION 6. Let R be a commutative ring with unit. Let X, Y be pointed spaces and $\pi : Y \to X$ a map.

(a) We say that π has an M_R–section, if $R_f(\pi', Y') \to X$ has a section (where $\pi' : Y' \to X$ is the associated fibration).

(b) $M_R\text{–cat}\,(\pi) := \inf\{k \mid G_k(\pi) : G_k(\pi, Y) \to X$ has an M_R–section $\}$.

(c) $M_R\text{–cat}\,(B) := M_R\text{–cat}\,(* \to B)$.

Several facts have to be mentioned:

WARNING 1. For fields $\mathcal{K}$ there exists already a notion $M_{\mathcal{K}}\text{–cat}\,(B)$ ([12,15]). We do not know whether it coincides with the one given here for $\mathcal{K} \neq \mathbb{Q}$.

WARNING 2. Let $g: Y \to X$ be a map in $\mathbb{Q} - \mathcal{S}_1$ with model $M_X \to M_Y$. There is then a notion of category of the M_X–module M_Y [15]. We do not think that it coincides with $M_{\mathbb{Q}}$–cat(g) in general. But of course this is true for $g: * \to X$.

WARNING 3. There is another notion of category of a map which we can transfer into our setting: Let B be pointed and $g: E \to B$ a map. Then define $\overline{M}_R$–cat$(g) := \inf\{k \mid g$ factors over $R_f(G_k(B)) \to B\}$. For $R = \mathbb{Q}$ and g in $\mathbb{Q} - \mathcal{S}_1$ this invariant coincides with $M_{\mathbb{Q}}$–cat(g) as studied in [15]. But this may not be true for the corresponding invariant $M_{\mathcal{K}}$–cat(g) of [15] for $\mathcal{K} \neq \mathbb{Q}$.

PROPOSITION 4. *Let B_1, B_2 be pointed spaces. Then* M_R–cat$(B_1 \times B_2) \leq M_R$–cat$(B_1) + M_R$–cat$(B_2)$.

Proof. Let M_R–cat$(B_1) \leq k$, M_R–cat$(B_2) \leq \ell$. Then $G_k(B_1) \to B_1$ and $G_\ell(B_2) \to B_2$ have M_R–sections. The next Proposition implies that $G_k(B_1) \times G_\ell(B_2) \to B_1 \times B_2$ has an M_R–section. Hence the statement follows from the existence of a map $G_k(B_1) \times G_\ell(B_2) \to G_{k+\ell}(B_1 \times B_2)$ over $B_1 \times B_2$ (by [14] or [8]). □

PROPOSITION 5. *Let $E_1 \to B_1$, $E_2 \to B_2$ be fibrations with M_R–sections. Then $E_1 \times E_2 \to B_1 \times B_2$ has an M_R–section.*

Proof. Let I_1, I_2 and I be the simplex categories of B_1, B_2 resp. $B_1 \times B_2$. Let the functors $\widetilde{B}_1, \widetilde{E}_1, \widetilde{B}_2, \widetilde{E}_2$, $\widetilde{B_1 \times B_2}$ and $\widetilde{E_1 \times E_2}$ be defined as in Section 1. Note that we have canonical natural transformations $I \to I_1$, $I \to I_2$. Thus we obtain morphisms

$$\operatorname*{hocolim}_{I} \widetilde{E_1 \times E_2} \longrightarrow \operatorname*{hocolim}_{I_1 \times I_2} \widetilde{E}_1 \times \widetilde{E}_2 \longrightarrow (\operatorname*{hocolim}_{I_1} \widetilde{E}_1) \times (\operatorname*{hocolim}_{I_2} \widetilde{E}_2)$$

and

$$\operatorname*{hocolim}_{I} \widetilde{B_1 \times B_2} \longrightarrow \operatorname*{hocolim}_{I_1 \times I_2} \widetilde{B}_1 \times \widetilde{B}_2 \longrightarrow (\operatorname*{hocolim}_{I_1} \widetilde{B}_1) \times (\operatorname*{hocolim}_{I_2} \widetilde{B}_2) .$$

The two arrows to the right are weak equivalences because of the Fubini Theorem [4], hence the arrows to the left are weak equivalences, because there are canonical weak equivalences (hocolim $\widetilde{E_1 \times E_2}) \simeq E_1 \times E_2 \simeq$ (hocolim $\widetilde{E}_1)\times$(hocolim $\widetilde{E}_2$) and similarly (hocolim $\widetilde{B_1 \times B_2}) \simeq B_1 \times B_2 \simeq$ (hocolim $\widetilde{B}_1$) $\times$ (hocolim $\widetilde{B}_2$).

The essential step in the construction of $R_f(E_1 \times E_2) \to B_1 \times B_2$ consists in applying the functor $R(-)$ to the 'building blocks' in a suitable hocolim representation. For these building blocks we now take the collection of arrows given by the natural transformation $\widetilde{E}_1 \times \widetilde{E}_2 \to \widetilde{B}_1 \times \widetilde{B}_2$. We are thus lead to the diagram:

$$\begin{array}{ccccc}
\operatorname*{hocolim}_{I_1 \times I_2} R \circ (\widetilde{E}_1 \times \widetilde{E}_2) & \xleftarrow{\ \overline{\alpha}\ } & \operatorname*{hocolim}_{I_1 \times I_2} (R \circ \widetilde{E}_1) \times (R \circ \widetilde{E}_2) & \xrightarrow{\ \overline{\beta}\ } & \\
\downarrow & & \downarrow & & \\
\operatorname*{hocolim}_{I_1 \times I_2} R \circ (\widetilde{B}_1 \times \widetilde{B}_2) & \xleftarrow[\ \overline{\alpha}\ ]{} & \operatorname*{hocolim}_{I_1 \times I_2} (R \circ \widetilde{B}_1) \times (R \circ \widetilde{B}_2) & \xrightarrow{\ \overline{\beta}\ } &
\end{array}$$

$$\begin{array}{cc}
\xrightarrow[\simeq]{\ \overline{\beta}\ } & \operatorname*{hocolim}_{I_1} R \circ \widetilde{E}_1 \times \operatorname*{hocolim}_{I_2} R \circ \widetilde{E}_2 \\
 & \downarrow \\
\xrightarrow[\simeq]{\ \overline{\beta}\ } & \operatorname*{hocolim}_{I_1} R \circ \widetilde{B}_1 \times \operatorname*{hocolim}_{I_2} R \circ \widetilde{B}_2
\end{array}$$

REMARK 3. Why do we not have the corresponding result of Proposition 4 for maps? In fact, there is a result, but it is more complicated.

Suppose that $p_1 : E_1 \to B_1$, $p_2 : E_2 \to B_2$ have M_R–cat $(p_1) \leq k$ resp. M_R–cat $(p_2) \leq \ell$. Suppose that cat (E_1), cat (E_2) are finite. Then there is a map [8] from $G_k(p_1) \times G_\ell(p_2)$ towards $G_{n(k,\ell)}(p_1 \times p_2)$ over $B_1 \times B_2$, where $n(k,\ell) = k + \ell + \max\{\operatorname{cat}(E_1), \operatorname{cat}(E_2)\}$. Therefore we only obtain M_R–cat $(p_1 \times p_2) \leq M_R$–cat $(p_1) + M_R$–cat $(p_2) + \max\{\operatorname{cat}(E_1), \operatorname{cat}(E_2)\}$.

REMARK 4. The constructions of this section can similarly be done for any coaugmented functor $T : \mathcal{S} \to \mathcal{S}$ that admits a fibrewise version and a natural transformation $T(X) \times T(Y) \to T(X \times Y)$, $X, Y \in \mathcal{S}$, compatible with the coaugmentations.

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