Bull. Korean Math. Soc. **36** (1999), No. 4, pp. 661–669

A STUDY ON REIDEMEISTER OPERATION

Seoung Ho Lee

Abstract. L. Degui introduced an upper bound of Reidemeister number. In this paper we give a simple proof of Degui's Theorem.

1. Introduction

All spaces considered here will be connected compact polyhedra and thus admit their universal covering spaces.

Let $f : X \to X$ be a given self-map and $p : \widetilde{X} \to X$ the universal covering projection of X. A lifting of f is a map $\tilde{f} : \widetilde{X} \to \widetilde{X}$ such that the diagram commutes

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & X \end{array}$$

Two liftings $\tilde{f}$ and $\tilde{f}'$ of f are said to be conjugate if there exists $\gamma \in \pi_X$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$ where π_X is the group of covering translations of $\widetilde{X}$. The equivalence classes by conjugacy are called *lifting classes of* f, denoted by $[\tilde{f}] = \{\gamma \circ \tilde{f} \circ \gamma^{-1} | \gamma \in \pi_X\}$. The number of lifting classes of f is called the *Reidemeister number of* f, denoted by $R(f)$. A lower bound of $R(f)$ has been obtained in [9,10] as follows:

$$|\mathrm{Coker}(1 - f_{1*})| \leq R(f)$$

where $f_{1*} : H_1(X) \to H_1(X)$ is the homomorphism induced by f and $H_1(X)$ is the 1-dimensional homology group of X.

Received August 25, 1998.
1991 Mathematics Subject Classification: 55M20.
Key words and phrases: Reidemeister operation; exact sequence.

In [2, Theorem 3], L. Degui obtained an upper bound of $R(f)$ as follows:

$$R(f) \leq |\mathrm{Coker}(1 - f_{1*})||H|$$

where H is the commutator subgroup of the fundamental group $\pi_1(X)$.

The purpose of this paper is to present a simple proof of Theorem 3 in [2] and Theorem 1 in [12] using the theory of P. R. Heath [3]. We work only with the fundamental group and the universal covering space rather than with the fundamental group, the universal covering space and the regular covering spaces.

In this first section we define the Reidemeister operation of a pair of homomorphisms and prove some of its algebraic properties. In the second section we give a simple proof of Degui's Theorem.

2. The Reidemeister Operation of a Pair of Homomorphisms

Let G, G' be groups and $f, g : G \to G'$ a pair of homomorphisms. We write composition in groups additively.

DEFINITION 2.1. The *Reidemeister operation of* (f, g) is the left action of G on G' given by

$$(\alpha, \beta') \mapsto f(\alpha) + \beta' - g(\alpha).$$

Let $f - g : G \to G'$ denote the *function* defined by $(f - g)(\alpha) = f(\alpha) - g(\alpha)$; then by a slight abuse we write the set of orbits of the operation as $\mathrm{Coker}(f - g)$ with elements $[\alpha']$ for $\alpha' \in G'$ (cf. [3,12]). We observe that if $j : G' \to \mathrm{Coker}(f - g)$ denote the quotient function defined by $j(\alpha') = [\alpha']$, then $j(\alpha') = j(\beta')$ if and only if there is a $\gamma \in G$ with $\alpha' = f(\gamma) + \beta' - g(\gamma)$, and there is then an exact sequence (with the obvious base points)

$$(2.2) \qquad 0 \to \Phi(f, g) \to G \xrightarrow{f-g} G' \xrightarrow{j} \mathrm{Coker}(f - g) \to 0,$$

of groups and based sets, where $\Phi(f, g)$ is the subgroup of G consisting of those α for which $f(\alpha) = g(\alpha)$. Note that since $f - g$ need not be a homomorphism, Coker$(f - g)$ need not be the quotient of G' by a subgroup.

The order $\sharp$Coker$(f - g)$ of the orbit set is called the *Reidemeister number of* (f, g) on G' and is written $R(f, g)$.

PROPOSITION 2.3. *If G' is abelian, $f - g$ is a homomorphism and Coker$(f - g)$ has a canonical group structure in which j is a homomorphism.*

Proof. Define an operation on Coker$(f - g)$ by

$$[\alpha'] * [\beta'] = [\alpha' + \beta'].$$

Let $[\alpha'] = [\alpha'_1]$ and $[\beta'] = [\beta'_1]$. Then there exist γ, γ_1 such that $\alpha'_1 = f(\gamma) + \alpha' - g(\gamma), \beta'_1 = f(\gamma_1) + \beta' - g(\gamma_1)$. Since G' is abelian,

$$\begin{aligned}[\alpha'_1 + \beta'_1] &= [f(\gamma) + \alpha'_1 - g(\gamma) + f(\gamma_1) + \beta' - g(\gamma_1)] \\ &= [f(\gamma + \gamma_1) + (\alpha' + \beta') - g(\gamma + \gamma_1)] \\ &= [\alpha' + \beta'].\end{aligned}$$

Thus the operation $*$ is well-defined. It is easy to show that Coker$(f-g)$ has a canonical group structure and j is a homomorphism. □

LEMMA 2.4. *For all $\alpha \in G, \beta' \in G'$, $[f(\alpha) + \beta'] = [\beta' + g(\alpha)]$. In particular $[f(\alpha)] = [g(\alpha)]$ for all $\alpha \in G$.*

Proof. $[f(\alpha) + \beta'] = [f(-\alpha) + (f(\alpha) + \beta') - g(-\alpha)] = [\beta' + g(\alpha)]$.□

We consider next the naturality of Reidemeister operations of pairs. Suppose we are given a commutative diagram of groups and homomorphisms

$$(2.5)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{j} & G & \xrightarrow{q} & \bar{G} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f|_H, g|_H} & & \downarrow{\scriptstyle f,g} & & \downarrow{\scriptstyle \bar{f},\bar{g}} & & \\ 0 & \longrightarrow & H' & \xrightarrow{j'} & G' & \xrightarrow{q'} & \bar{G}' & \longrightarrow & 0 \end{array}$$

in which the rows are exact and $f|_H, g|_H$ are the restrictions of f, g to H. Then q restricts to a homomorphism $\dot{q} : \Phi(f, g) \to \Phi(\bar{f}, \bar{g})$ also denoted by q: further q' induces a function q'_* :Coker$(f - g)$ $\to$Coker$(\bar{f} - \bar{g})$ in the obvious way. Thus we have

THEOREM 2.6. *In the above situation there is an exact sequence*

$$0 \to \Phi(f|_H, g|_H) \to \Phi(f,g) \xrightarrow{q} \Phi(\bar{f},\bar{g}) \xrightarrow{\delta} Coker(f|_H - g|_H)$$
$$\xrightarrow{j'_*} Coker(f-g) \xrightarrow{q'_*} Coker(\bar{f}-\bar{g}) \to 0$$

of groups and based sets in which δ *is given by*

$$\delta(\bar{\alpha}) = [f(\alpha) - g(\alpha)] \text{ where } q(\alpha) = \bar{\alpha}.$$

Furthermore, if G' *is abelian, then the sequence can be regarded as an exact sequence of groups.*

Proof. The result can be proved by arranging sequences of the form of 2.2 for $f|_H - g|_H, f-g, \bar{f}-\bar{g}$ on a grid and using those of type 2.5 to connect them. First we show that Imj'_* =Kerq'_*. Since $q'_*j'_*([h]) = [\bar{0}]$, we have Im$j'_*$ ⊂Kerq'_*. Conversely, if $q'_*([\alpha']) = [\bar{0}]$, then there exists $\bar{\alpha}_1 \in \bar{G}$ such that $\bar{0} = \bar{f}(\bar{\alpha}_1) + q'(\alpha') - \bar{g}(\bar{\alpha}_1)$, and $q'(f(\alpha_1) + \alpha' - g(\alpha_1)) = \bar{0}$, there exists $h' \in H'$ such that $j'(h') = f(\alpha_1) + \alpha' - g(\alpha_1)$, ; that is, $j'_*([h']) = [f(\alpha_1) + \alpha' - g(\alpha_1)] = [\alpha']$. Thus we have Ker$q'_*$ ⊂ Imj'_*. The other terms fall out easily. Furthermore, if G' is abelian, then H' and $\bar{G}'$ are abelian. Using Proposition 2.3, the sequence can be regarded as an exact sequence of groups (see Ker-Coker Sequence in [13]). □

The function q'_* :Coker$(f-g)$ →Coker$(\bar{f}-\bar{g})$ in Theorem 2.6 is surjective so

$$|\text{Coker}(\bar{f}-\bar{g})| \le R(f,g). \tag{2.7}$$

If f is the identity homomorphism, we have the same result of [3].

COROLLARY 2.8 [3]. *There exists an exact sequence*

$$0 \to \Phi(g|_H) \to \Phi(g) \xrightarrow{q} \Phi(\bar{g}) \xrightarrow{\delta} Coker(1 - g|_H)$$
$$\xrightarrow{j_*} Coker(1-g) \xrightarrow{q_*} Coker(1-\bar{g}) \to 0$$

of groups and based sets in which δ *is given by*

$$\delta(\bar{\alpha}) = [\alpha - g(\alpha)] \text{ where } q(\alpha) = \bar{\alpha}.$$

Furthermore, if G' is abelian, then the sequence can be regarded as an exact sequence of groups.

Let $\alpha' \in G'$, then the Reidemeister operation of (f, g) on H' induces an operation(on the left) of H on $H' + \alpha'$ by restriction, i.e., $H \times (H' + \alpha') \longrightarrow H' + \alpha'$ defined by $(h, k' + \alpha') \mapsto f|_H(h) + (k' + \alpha') - g|_H(h)$. Since $q'(f(h) + k' + \alpha' - g(h) - \alpha') = \bar{0}$, this action is well-defined. Let $\mathrm{Orb}(H' + \alpha')$ denote the orbits of this operation with elements $[k' + \alpha']$ for $k' \in H'$, then the inclusion of H' into G' induces a function

$$j'_* : \mathrm{Orb}(H' + \alpha') \to \mathrm{Coker}(f - g)$$

defined by $[k' + \alpha'] \mapsto [j'(k') + \alpha']$.

PROPOSITION 2.9. *In the above situation we have*

$$j'_*(Orb(H' + \alpha')) = {q'}_*^{-1}(q'_*([\alpha'])).$$

Proof. First, since $q'_* j'_*([k' + \alpha']) = [q'j'(k') + q'(\alpha')] = q'_*([\alpha'])$, then we have $j'_*(\mathrm{Orb}(H' + \alpha')) \subset {q'}_*^{-1}(q'_*([\alpha']))$. Conversely, if $q'_*([\beta']) = q'_*([\alpha'])$, then we show that $[\beta'] = [j'(h') + \alpha']$ for some $h' \in H'$. Since $[\bar{\beta}'] = [\bar{\alpha}']$, there exists $\bar{\gamma} \in \bar{G}$ such that $\bar{\beta}' = \bar{f}(\bar{\gamma}) + \bar{\alpha}' - \bar{g}(\bar{\gamma})$. But $q'(\beta' - (f(\gamma) + \alpha' - g(\gamma)) = \bar{0}$, and by the exactness, there exists $k' \in H'$ such that $j'(k') = \beta' - (f(\gamma) + \alpha' - g(\gamma))$. Thus by Lemma 2.4, we get

$$\begin{aligned} [\beta'] &= [j'(k') + f(\gamma) + \alpha' - g(\gamma)] \\ &= [f(-\gamma) + j'(k') + f(\gamma) + \alpha'] \\ &= [j'(h') + \alpha'] \qquad \text{for some } h' \in H'. \end{aligned}$$

□

COROLLARY 2.10. *$R(f, g) \leq |H'||Coker(\bar{f} - \bar{g})|$.*

Proof. Since ${q'}_*^{-1}(q'_*([\alpha'])) \leq |\mathrm{Orb}(H' + \alpha')| \leq |H'|$, we get the conclusion. □

We are now ready to compare the Reidemeister number of homomorphisms $f, g : G \to G'$ with the Reidemeister number of $H(f), H(g)$:

$H(G) \to H(G')$ where H is the abelianization functor from groups to abelian groups. For any group G the sequence

$$(2.11) \qquad 0 \to H \to G \xrightarrow{\natural} H(G) \to 0$$

is exact (see for example [4; p. 55, 9.2(8)]) where H is the commutator subgroup of G and $\natural$ is a natural projection. Moreover, there is a commutative diagram of groups and homomorphisms

$$(2.12) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{j} & G & \xrightarrow{\natural} & H(G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f|_H, g|_H} & & \downarrow{\scriptstyle f,g} & & \downarrow{\scriptstyle H(f),H(g)} & & \\ 0 & \longrightarrow & H' & \xrightarrow{j'} & G' & \xrightarrow{\natural'} & H(G') & \longrightarrow & 0 \end{array}$$

where H' is the commutator subgroup of G'. Thus by Theorem 2.6

PROPOSITION 2.13. *We get an exact sequence as follows:*

$$0 \to \Phi(f|_H, g|_H) \to \Phi(f,g) \xrightarrow{\natural} \Phi(H(f), H(g)) \xrightarrow{\delta} Coker(f|_H - g|_H)$$
$$\xrightarrow{j'_*} Coker(f-g) \xrightarrow{\natural'_*} Coker(H(f) - H(g)) \to 0$$

By (2.7) and Corollary 2.10, we have

COROLLARY 2.14. $|Coker(H(f) - H(g))| \leq R(f,g) \leq |H'||Coker(H(f) - H(g))|$.

If f is the identity homomorphism, we have

COROLLARY 2.15. $|Coker(1-H(g)) \leq R(g) \leq |H||Coker(1-H(g))|$.

3. Reidemeister Number of a Pair of Continuous Maps

Let $f, g : X \to Y$ be a pair of maps. Fix universal coverings $P : \widetilde{X} \to X, Q : \widetilde{Y} \to Y$. Denoted by $\pi_X := \pi_1(X), \pi_Y := \pi_1(Y)$ the groups of natural transformations of $\widetilde{X}$ and $\widetilde{Y}$ respectively. Let lift(f,g) be the set of all pairs of liftings $(\tilde{f},\tilde{g})$ for which the following diagram commutes .

$$(3.1)\qquad \begin{array}{ccc} \widetilde{X} & \xrightarrow{(\tilde{f},\tilde{g})} & \widetilde{Y} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ X & \xrightarrow{(f,g)} & Y \end{array}$$

Two lifting pairs $(\tilde{f},\tilde{g})$ and $(\tilde{f}',\tilde{g}')$ are said to be conjugate if there exists $\gamma\in\pi_X$ and $\gamma'\in\pi_Y$ such that $(\tilde{f}',\tilde{g}')=\gamma'\circ(\tilde{f},\tilde{g})\circ\gamma^{-1}$. The equivalence classes by conjugacy are called *lifting classes of* (f,g) *on* $\widetilde{Y}$ and the lifting class of $(\tilde{f},\tilde{g})$ is denoted by $[\tilde{f},\tilde{g}]=\{\gamma'\circ(\tilde{f},\tilde{g})\circ\gamma^{-1}|\gamma\in\pi_X,\gamma'\in\pi_Y\}$. The number of lifting classes of (f,g) is called the *Reidemeister number of* (f,g), denoted $R(f,g)$ (see [6,8,11,12]).

Let $(\tilde{f},\tilde{g})$ be a reference lifting of (f,g) corresponding $\langle w\rangle$ where w is a path from $f(x_0)$ to $g(x_0)$. The Reidemeister operation of (f,g) is the left action of π_X on π_Y given by

$$(\gamma,\alpha')\mapsto f_\pi(\gamma)+\alpha'-\tilde{g_\pi}(\gamma)$$

where f_π is the induced homomorphism of f, w_* is the isomorphism induced by the path w and $\tilde{g_\pi}=w_*\circ g_\pi$ (see [7,12]).

THEOREM 3.2 [12]. *The lifting classes of* (f,g) *are in 1-1 correspondence with orbits in* π_Y, *the lifting class* $[\tilde{f},\alpha'\circ\tilde{g}]$ *corresponding to the* $[\alpha']\in$ *Coker*$(f_\pi-\tilde{g_\pi})$.

Then we have

$$(3.3)\qquad R(f,g)=R(f_\pi,\tilde{g}_\pi).$$

By [5], we have an exact sequence

$$0\to H\xrightarrow{j}\pi_1(X,x_0)\xrightarrow{\theta}H_1(X)\to 0$$

where H is the commutator subgroup of $\pi_1(X, x_0)$. Furthermore, by [9; p. 45 Lemma 1.13] we have a commutative diagram as follows:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(f_\pi, \tilde{g}_\pi)} & \pi_1(Y, f(x_0)) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ H_1(X) & \xrightarrow{(f_{1*}, g_{1*})} & H_1(Y) \end{array} \tag{3.4}$$

Thus θ is an abelianization functor. By Proposition 2.13, we have

PROPOSITION 3.5. *There is an exact sequence as follows:*

$$0 \to \Phi(f_\pi|_H, \tilde{g}_\pi|_H) \to \Phi(f_\pi, \tilde{g}_\pi) \xrightarrow{\theta} \Phi(f_{1*}, g_{1*}) \xrightarrow{\delta} Coker(f_\pi|_H - \tilde{g}_\pi|_H)$$
$$\xrightarrow{j'_*} Coker(f_\pi - \tilde{g}_\pi) \xrightarrow{\theta'_*} Coker(f_{1*} - g_{1*}) \to 0$$

By Corollary 2.14 and (3.3), we have

COROLLARY 3.6 [12]. $|Coker(f_{1*} - g_{1*})| \leq R(f, g) \leq |H'||Coker(f_{1*} - g_{1*})|$ *where* H' *is the commutator subgroup of* $\pi_1(Y, f(x_0))$.

If f is the identity map, then

COROLLARY 3.7 [2]. $|Coker(1 - g_{1*})| \leq R(g) \leq |H||Coker(1 - g_{1*})|$ *where* H *is the commutator subgroup of* $\pi_1(X, g(x_0))$.

References

[1] R. F. Brown, *The Lefchetz Fixed Point Theorem*, Scott, Foresman and Glenview, Ill, 1971.

[2] L. Degui, *Remarks on the Reidemeister numbers*, Bull. Korean Math. Soc. **33** (1996), 397-409.

[3] P. R. Heath, *Product formula for Nielsen numbers of fibre maps*, Pacific J. Math. **117** (1985), no. 2, 267-289.

[4] H. Herrlich and G. E. Strecker, *Category Theory*, Allyn and Bacon Inc.,. 1973.

[5] S. T. Hu, *Homotopy Theory*, Academic Press Inc., 1959.

[6] C. G. Jang and S. Lee, *A relative Nielsen number in the coincidence theory*, J. Korean Math. Soc. **32** (1995), 171-181.

[7] J. Jezierski, *The Nielsen number product formula for coincidences*, Fund. Math. **134** (1989), 183-212.

[8] ______, *A relative coincidence Nielsen number*, To appear.
[9] B. J. Jiang, *Lectures on Nielsen Fixed Point Theory*, Contemp. Math. 14, Amer. Math. Soc., Providence, 1959.
[10] T. H. Kiang, *The Theory of Fixed Point Classes*, Scientific Press. Peking, 1979.
[11] S. H. Lee, *A relative Nielsen coincidence number for the complement, I*, J. Korean Math. Soc. **33** (1996), no. 4, 709-716.
[12] ______ and S. D. Baek, *Remarks on the Reidemeister numbers for coincidences*, Comm. Korean Math. **13** (1998), no. 1, 109-121.
[13] D. G. Northcott, *A First Course of Homological Algebra*, Cambridge University Press, 1973.

Department of Mathematics, Mokwon University, Taejon 301-729, Korea
E-mail: lsh271@mokwon.ac.kr