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ON THE PRINCIPAL IDEAL THEOREM

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Abstract. Let R be an integral domain with identity. In this paper we will show that if R is integrally closed or if t-dim$R \leq 1$, then $R[\{X_\alpha\}]$ satisfies the principal ideal theorem for each family $\{X_\alpha\}$ of algebraically independent indeterminates if and only if R is an S-domain and it satisfies the principal ideal theorem.

1. Introduction

Krull's principal ideal theorem [10, Theorem 142] states that for a nonunit element x of a Noetherian ring R, if P is a prime ideal of R which is minimal over xR, then the height of P is at most 1. Thus if R is a Noetherian domain, then each minimal prime ideal of a nonzero principal ideal has height one. Kaplansky [10, page 104] called Krull's principal ideal theorem the most important single theorem in the theory of Noetherian rings.

Let R be an integral domain. As [5], we say that R satisfies the *principal ideal theorem* (PIT) if each minimal prime ideal of a nonzero principal ideal of R has height one. Examples of integral domains satisfying PIT include Noetherian domains, Krull domains, one-dimensional domains.

An integral domain R is called an S-domain (the S stands for Seidenberg) if for each height one prime ideal P of R, the expansion $P[X]$ of P to $R[X]$ has again height one. For any integral domain R, $R[X]$ is an S-domain [1, Theorem 3.2]. Barucci, Anderson and Dobbs [5, Proposition 6.1] showed that if $R[X]$ satisfies PIT, then R satisfies PIT and R is an S-domain. They also showed the converse if R is a GCD-domain or if R has locally funneled spectrum [5, Theorem 6.5]. As the main

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result of this paper, in Theorem 4, we show that the converse holds for integrally closed domains. If the integral closure R' of R satisfies PIT, then R satisfies PIT [5, Corollary 5.2]. In Theorem 3, we generalize [5, Proposition 6.3] to an integral domain with t-dim$R \leq 1$ (note that if a GCD-domain R satisfies PIT, then t-dim$R \leq 1$). Example 9 shows that the converse of [5, Corollary 5.2] fails.

Recall that R is *atomic* if each nonzero nonunit of R is a product of irreducible elements. Anderson, Chapman and Smith [3, Theorem 2.6] showed that if R is an atomic integral domain, then each nonzero prime ideal of R is a union of height one prime ideals of R if and only if R satisfies PIT, and that if one (hence two) of these two conditions is satisfied, then each nonzero nonunit of R is contained in a height one prime ideal of R. They [3, Example 2.8] also showed that Int($\mathbb{Z}$)=$\{f(x) \in \mathbb{Q}[X] | f(r) \in \mathbb{Z}$ for all $r \in \mathbb{Z}\}$ is atomic and each nonzero nonunit of Int($\mathbb{Z}$) is contained in a height one prime ideal of R but Int($\mathbb{Z}$) does not satisfy PIT.

Throughout this paper, R denotes an integral domain with quotient field K and R' the integral closure. For an ideal I of R, $I_v = (I^{-1})^{-1}$, $I_t = \cup\{J_v | J$ is a finitely generated subideal of $I\}$. If $I = I_t$ (resp. $I = I_v$), I is said to be a t-ideal (resp. divisorial). By the Zorn's lemma argument, one can easily show that each t-ideal is contained in a maximal t-ideal which is an ideal maximal among proper t-ideals and each maximal t-ideal is a prime ideal. Our notation will be essentially that of [7, 10].

2. Main Results

It is well-known that a minimal prime ideal of a t-ideal is also a t-ideal. Using this fact, we have the following result (cf. [5, Proposition 3.1], [5, Proposition 4.1]).

PROPOSITION 1. *An integral domain R satisfies PIT if and only if R_P satisfies PIT for each maximal t-ideal P of R.*

Proof. Let Q be a prime ideal of R which is minimal over a principal ideal rR. Since rR is a t-ideal, Q is a t-ideal. Thus there is a maximal t-ideal P of R containing Q. Since R_P satisfies PIT and QR_P is minimal over rR_P, ht$Q$=ht(QR_P) = 1. Hence R satisfies PIT. The converse is obvious. □

LEMMA 2. (cf. [8, Proposition 1.1]) *Suppose that Q is a proper prime ideal of $R[X]$ which is minimal over a nonzero principal ideal $fR[X]$ of $R[X]$. If $P := Q \cap R \neq 0$, then $Q = P[X]$.*

Proof. Let R' be the integral closure of R. Then $R'[X]$ is the integral closure of $R[X]$. Choose a prime ideal Q' of $R'[X]$ such that $Q' \cap R[X] = Q$ [10, Theorem 44]. Since Q is minimal over $fR[X]$, Q' is minimal over $fR'[X]$ and so Q' is a t-ideal. Since Q' is a t-ideal and $Q' \cap R' \supseteq P$, $Q' = (Q' \cap R')[X]$ [9, Lemma 4.1]. Thus $Q = (Q' \cap R')[X] \cap R[X] = (Q' \cap R)[X] = P[X]$. □

THEOREM 3. (cf. [2, Lemma 4.10]) *Let R be an integral domain with t-dim$R \leq 1$. Then the following statements are equivalent.*

1. *R is an S-domain.*
2. *$R[X]$ satisfies PIT.*
3. *$R[X_1, \cdots, X_n]$ satisfies PIT for each positive integer n.*
4. *$R[\{X_\alpha\}]$ satisfies PIT for each family $\{X_\alpha\}$ of algebraically independent indeterminates.*

Proof. If t-dim$R = 0$, then R is a field. So the result is trivial. Thus we assume that t-dim$R = 1$.

(1) $\Rightarrow$ (2) Let f be a nonzero element of $R[X]$, and let Q be a prime ideal of $R[X]$ which is minimal over $fR[X]$. If $Q \cap R = 0$, ht$Q = 1$ [10, Theorem 36]. If $P := Q \cap R \neq 0$, $Q = P[X]$ by Lemma 2. Since $P[X]$ is a t-ideal of $R[X]$, P is a t-ideal of R [9, Corollary 2.3]. Thus ht$Q$=ht$(P[X])$ = ht$P = 1$.

(2) $\Rightarrow$ (1) [5, Proposition 6.1].

(2) $\Rightarrow$ (3) $R[X_1]$ is an S-domain [1, Theorem 3.2]. If Q is a maximal t-ideal of $R[X_1]$, then $Q \cap R = 0$ or $Q = P[X_1]$ where $P := Q \cap R \neq 0$ [8, Proposition 1.1]. Thus ht$Q = 1$ and hence t-dim$R[X_1] = 1$. By induction on n and the proof (1) $\Rightarrow$ (2), we get that $R[X_1, \ldots, X_n]$ satisfies PIT.

(3) $\Rightarrow$ (2) Clear.

(3) $\Leftrightarrow$ (4) [5, Proposition 6.4]. □

REMARK. If a Prüfer v-multiplication domain R satisfies PIT, then R_P satisfies PIT for each maximal t-ideal P of R by Proposition 1. Since R_P is a valuation domain [9, Theorem 3.2], dim$R_P = 1$ and hence t-dim$R = 1$. Since a GCD-domain is a Prüfer v-multiplication domain, [5, Proposition 6.3] is a special case of Theorem 3.

THEOREM 4. *Let R be an integrally closed domain. Then $R[X]$ satisfies PIT if (and only if) R satisfies PIT and R is an S-domain.*

Proof. Suppose that Q is a proper prime ideal of $R[X]$ which is minimal over a nonzero principal ideal $fR[X]$ of $R[X]$. If $Q \cap R = 0$, then $\mathrm{ht}Q = 1$ [10, Theorem 36]. Thus we assume that $P := Q \cap R \neq 0$. By Lemma 2, $Q = P[X]$. Since R_P is an integrally closed S-domain, it satisfies PIT and $\mathrm{ht}Q = \mathrm{ht}(QR[X]_{(R-P)})$, we may assume that R is a quasi-local domain with maximal ideal P.

Since $K[X]$ is a UFD, $f = f_1^{e_1} \dots f_n^{e_n}$, where each f_i is a prime element of $K[X]$. Choose nonzero elements $b_i \in R$ such that $g_i := b_i f_i \in R[X]$. Let $b = b_1^{e_1} \dots b_n^{e_n}$, then $bf = g_1^{e_1} \dots g_n^{e_n}$. So $bf \in g_i R[X] \subseteq g_i A_{g_i}^{-1}[X] = g_i K[X] \cap R[X]$ by the fact that R is integrally closed [7, Corollary 34.9]. Since $g_i A_{g_i}^{-1}[X]$ is a prime ideal of $R[X]$ and $b \notin g_i A_{g_i}^{-1}[X]$, $f \in g_i A_{g_i}^{-1}[X]$.

Since $P[X]$ is minimal over $fR[X]$, $g_i A_{g_i}^{-1}[X] \nsubseteq P[X]$. So there is $h_i \in A_{g_i}^{-1}[X]$ with $g_i h_i \notin P[X]$. Since $R = A_{g_i h_i} \subseteq A_{g_i} A_{h_i} \subseteq R$, A_{g_i} is invertible. By [7, Theorem 28.1], A_f is invertible. Since R is quasi-local, A_f is principal. Since $fR[X] \subseteq A_f R[X] \subseteq P[X]$, P is minimal over A_f and hence $\mathrm{ht}Q = \mathrm{ht}(P[X]) = \mathrm{ht}P = 1$. Hence $R[X]$ satisfies PIT. □

The following result is an immediate consequence of Theorem 4, [1, Theorem 3.2] and [5, Proposition 6.4].

COROLLARY 5. *Let R be an integrally closed domain. Then $R[\{X_\alpha\}]$ satisfies PIT for each family $\{X_\alpha\}$ of algebraically independent indeterminates if and only if R satisfies PIT and R is an S-domain.*

An integral domain R is called a *half-factorial domain* (HFD) if R is atomic, and given any two irreducible factorizations $a = a_1 a_2 \dots a_n = b_1 b_2 \dots b_m$ of an element $a \in R$, then $n = m$.

COROLLARY 6. *Let $R[X]$ be an HFD. Then $R[X]$ satisfies PIT if and only if R satisfies PIT and R is an S-domain.*

Proof. Suppose that $R[X]$ is an HFD. Then R is integrally closed [6, Theorem 2.2]. By Theorem 4, the proof is completed. □

Let $S \subseteq T$ be integral domains. Then the pair S, T satisfies the *going-down theorem* (GD) if $P \supsetneq P_0$ is a chain of prime ideals of S and if Q is a prime ideal of T such that $Q \cap S = P$, then there exists a prime ideal Q_0 of T such that $Q_0 \subsetneq Q$ and $Q_0 \cap S = P_0$ [10, p. 28].

COROLLARY 7. *Suppose that $R \subseteq R'$ satisfies GD. Then $R[X]$ satisfies PIT if (and only if) R satisfies PIT and R is an S-domain.*

Proof. Let the notation be as Theorem 4 and let P' be a prime ideal of R' lying over P. Since $f \in P'[X]$ and $R'[X]$ is integral over $R[X]$, $P'[X]$ is minimal over $fR'[X]$. By the same way as the proof of Theorem 4, we can show that $A_f R'_{P'}$ is principal, i.e., $A_f R'_{P'} = aR'_{P'}$ for some coefficient a of f [7, Proposition 7.4]. So P' is minimal over aR'. By GD, P is minimal over aR. Hence ht$(P[X])$=ht$P = 1$. □

PROPOSITION 8. *If R' is an S-domain, then R is an S-domain.*

Proof. For a height one prime ideal P of R, R'_S is the integral closure of R_P where $S = R - P$ [7, Proposition 10.2]. Since R'_S is an S-domain and ht$(P[X])$ = ht$(PR_P[X])$, we may assume that R is a quasi-local domain with maximal ideal P and dim$R = 1$. Since R' is an S-domain of dimension 1, dim$R'[X] = 2$ and hence dim$R[X] = 2$. Thus ht$(P[X])$ = 1. □

In the following example we show that the converse of Proposition 8 is not true. Barucci, Anderson and Dobbs [5, Corollary 5.2] proved that if R' satisfies PIT, then R satisfies PIT. The following example also shows that the converse fails or serves as a counterexample to the converse.

EXAMPLE 9. Let K be a field, $D_1 = K[X, Y]$ the polynomial ring in two indeterminates over K, $M_1 = (X)$ and $N_1 = (X - 1, Y)$, then M_1 and N_1 are prime ideals of D_1, respectively, of height 1 and 2, and if $S = D_1 - (M_1 \cup N_1)$, then $D = (D_1)_S$ is a two-dimensional semi-local domain with two maximal ideals $M = (M_1)_S$ and $N = (N_1)_S$ such that ht$M = 1$, ht$N = 2$.

Let $R' = K + M$, then R' is a two-dimensional semi-quasi-local domain with two maximal ideals M and $N' = N \cap R'$ such that $R'_{N'} = D_N = K[X, Y]_{(X-1,Y)}$ is a Krull domain (See [4, page 3]). Moreover, we claim that $R := K + (M \cap N')$ is an integral domain such that:

1. R is a quasi-local, two-dimensional S-domain with quotient field $K(X, Y)$ and $M \cap N'$ is the maximal ideal of R. (See [4]).
2. R' is the integral closure of R and R' is not an S-domain. (See [4]).
3. $R[Z]$ satisfies PIT but $R'[Z]$ does not satisfy PIT, where Z is an indeterminate over $K(X, Y)$.
4. R and R' satisfy PIT.

Proof. (3) Let f be a nonzero element of $R[Z]$, and let Q be a prime ideal of $R[Z]$ which is minimal over $fR[Z]$. If $Q \cap R = 0$, ht$Q = 1$ [10, Theorem 36]. If $P := Q \cap R \neq 0$, $Q = P[Z] \subseteq (M \cap N')[Z]$ by Lemma 2. Let P' be a prime ideal of R' lying over P with $P' \subseteq N'$. Then $P'[Z]$ is minimal over $fR'[Z]$ since $R'[Z]$ is integral over $R[Z]$. Thus $P'R'_{N'}[Z]$ is minimal over $fR'_{N'}[Z]$. Since $R'_{N'}[Z]$ is a Krull domain [7, Theorem 43.11 (3)], htP' =ht$(P'R'_{N'})$=ht$(P'R'_{N'}[Z]) = 1$. Since ht$N' = 2$, $P' \subsetneq N'$. So ht$P = 1$ and ht$Q$= ht$(P[Z]) = 1$.

Since R' is not an S-domain, $R'[Z]$ does not satisfy PIT [5, Proposition 6.1].

(4) Since $R[Z]$ satisfies PIT, R satisfies PIT. Also $R'_{N'}$ satisfies PIT since $R'_{N'}$ is a Krull domain, and R'_M satisfies PIT since dimR'_M=ht$M$ = 1. By [5, Proposition 3.1], R' satisfies PIT. □

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