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EQUIVARIANT HOMOTOPY EQUIVALENCES AND A FORGETFUL MAP

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ABSTRACT. We consider the forgetful map from the group of equivariant self equivalences to the group of non-equivariant self equivalences. A sufficient condition for this forgetful map being a monomorphism is obtained. Several examples are given.

1. Introduction

Let (P, q, B, G) be a principal G-bundle over B. Then G acts on P freely and $P/G = B$. Let $aut_G(P)$ be the space of unbased G-equivariant homotopy equivalences from P to P of the total space P of the given principal G-bundle. We work in the category of connected CW-complexes.

The path component of $aut_G(P)$ forms a group where the multiplication is given by the composition of maps. That is, we define

$$\mathcal{F}_G(P) = \pi_0(aut_G(P)).$$

Two maps f, g in $aut_G(P)$ are in the same path component if there is a G-equivariant homotopy between f and g. We call this group the group of unbased G-equivariant self equivalences.

On the other side, we consider the space of unbased self homotopy equivalences from P to P, which is denoted by $aut(P)$. Two maps are in the same path component if there is a homotopy between them. By the same consideration as above, we define

$$\mathcal{F}(P) = \pi_0(aut(P)).$$

We call this group the group of unbased self homotopy equivalences.

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The author has been interested in the natural map which forgets a G-action on P, that is, a forgetful homomorphism

$$F : \mathcal{F}_G(P) \to \mathcal{F}(P) \tag{1.1}$$

and raised the following problem in 1988 (see [5, p. 206, Problem 13]):

When is the homomorphism F of (1.1) a monomorphism?

This problem seems to be difficult, because the generators of $\mathcal{F}_G(P)$ are not known in general, even if we have the group structure of $\mathcal{F}_G(P)$ (see [3], [7]).

An example where F is not a monomorphism, is given by [8] in 1996 as in the following.

Let $(G/T, q, BT, G)$ be a principal G-bundle, where G is a compact connected Lie group, which is not a torus and T is a maximal torus of G. It is shown that $\mathcal{F}_G(G/T)$ is an infinite group. But $\mathcal{F}(G/T)$ is a finite group. So

$$F : \mathcal{F}_G(G/T) \to \mathcal{F}(G/T)$$

cannot be a monomorphism.

On the contrary, many examples where F might be a monomorphism, could be found in [3], [7] by the calculations of the group of G-equivariant self equivalences.

For example, let (E_G, q, B_G, G) be a universal principal G-bundle, then $\mathcal{F}_G(E_G) = 1$ (see [7, Example 3.1]). So

$$\mathcal{F}_G(E_G) \to \mathcal{F}(E_G)$$

is a monomorphism.

In this note we consider the sufficient condition that the homomorphism F of (1.1) is a monomorphism and several examples are given.

2. Bundle Map Theory

Let f be a G-equivariant map from P to P. Then one has the induced map on $\bar{f}$ on B such that $qf = \bar{f}q$.

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{\bar{f}} & B \end{array}$$

One has naturally the following map

$$\Phi : aut_G(P) \to aut(B), \Phi(f) = \bar{f}.$$

This construction determines a Serre fibration with fibre the space $I_G(P)$ of unbased bundle equivalences over B (cf. [3], [4], [7]):

$$(2.1) \qquad I_G(P) \to aut_G(P) \to aut(B)$$

By the Gottlieb's theorem [1],

$$(2.2) \qquad \pi_0(I_G(P)) = \pi_1(map(B, B_G), k),$$

where $k : B \to B_G$ is a classifying map and B_G is a classifying space.

So we have the following exact sequence of groups and homomorphisms by (2.1) and (2.2) (see [3], [4], [7])

$$(2.3) \qquad \pi_1(map(B, B_G), k) \to \mathcal{F}_G(P) \to \mathcal{F}_k(B) \to 0,$$

where $\mathcal{F}_k(B) = \{\bar{f} \in \mathcal{F}(B); k\bar{f} \simeq k\}$.

3. $K(\pi, n)$-action

Let $\mathcal{R}_q(P)$ be the group of homotopy classes of q-retracting equivalences, that is, an element f of $\mathcal{F}(P)$ for which there is an element $\bar{f}$ of $\mathcal{F}(B)$ satisfying $qf \simeq \bar{f}q$ (see [6, p. 645]). $\mathcal{R}_q(P)$ is a subgroup of $\mathcal{F}(P)$ and $F(\mathcal{F}_G(P))$ is a subgroup of $\mathcal{R}_q(P)$.

If the structure group of the principal G-bundle (P, q, B, G) is an Eilenberg-MacLane space $K(\pi, n) = G (n \geq 1)$

$$(3.1) \qquad \begin{array}{ccc} P & \longrightarrow & E_{K(\pi,n)} \\ {\scriptstyle q}\downarrow & & \downarrow \\ B & \xrightarrow{k} & B_{K(\pi,n)}, \end{array}$$

then $B_{K(\pi,n)} = K(\pi, n+1)$ and

$$\pi_1(map(B, B_{K(\pi,n)}), k) = \pi_1(map(B, K(\pi, n+1)), *) = H^n(B, \pi).$$

Therefore if $H^n(B;\pi) = 0$, we have the following diagram by (3.1) and (2.3)

(3.2)
$$
\begin{array}{ccc}
\mathcal{F}_{K(\pi,n)}(P) & \xrightarrow{F} & \mathcal{R}_q(P) \\
\cong \downarrow & \nearrow^{F'} & \\
\mathcal{F}_k(B) & \xrightarrow{i} & \mathcal{F}(B),
\end{array}
$$

where i is an inclusion and $F' : \mathcal{F}_k(B) \to \mathcal{R}_q(P)$.

THEOREM 3.3. *Let $(P, q, B, K(\pi, n))$ be a principal $K(\pi, n)$-bundle, we assume $H^n(B, \pi) = 0$. The forgetful map $F : \mathcal{F}_{K(\pi,n)}(P) \to \mathcal{R}_q(P) \subset \mathcal{F}(P)$ is a monomorphism, if a map $G : \mathcal{R}_q(P) \to \mathcal{F}(B)$ which commutes the following diagram (3.4) exists.*

(3.4)
$$
\begin{array}{ccc}
\mathcal{F}_{K(\pi,n)}(P) & \xrightarrow{F} & \mathcal{R}_q(P) \\
\cong \downarrow & \nearrow^{F'} & \downarrow G \\
\mathcal{F}_k(B) & \xrightarrow{i} & \mathcal{F}(B)
\end{array}
$$

Proof. Since $GF' = i$ and i is a monomorphism(inclusion), F' is a monomorphism. $\mathcal{F}_{K(\pi,n)}(P)$ is isomorphic to $\mathcal{F}_k(B)$. So F is a monomorphism. □

Let $\mathcal{E}(B)$ be the group of based self homotopy equivalences of B, and we denote by $\mathcal{E}_\#(B)$ the subgroup of $\mathcal{E}(B)$ consisting of classes that induce the identity automorphism of all homotopy groups.

COROLLARY 3.5. *Let $(P_k, q, B, K(\pi, n))$ be a principal $K(\pi, n)$-bundle, where $\pi_i(B) = 0 (i \leq n)$, π is a free abelian group and $\mathcal{E}_\#(B) = 1$. Then*

$$F : \mathcal{F}_{K(\pi,n)}(P_k) \to \mathcal{F}(P_k)$$

is a monomorphism for any k.

Proof. By the homotopy exact sequence of the given principal bundle, we have $\pi_i(B) = \pi_i(P_k) (i \geq n+2)$, $\pi_{n+1}(B) = \pi_{n+1}(P) \oplus \pi'$ (π' is a free subgroup of π) and $\pi_i(B) = 0 (i \leq n)$. $H^n(B, \pi) = 0$, since $\pi_i(B) = 0 (i \leq n)$. Since $\mathcal{E}_\#(B) = 1$, the induced map on the base space B is determined uniquely for the given self homotopy equivalence in $\mathcal{R}_q(P)$. So a map $G : \mathcal{R}_q(P) \to \mathcal{F}(B)$ which commutes the diagram (3.4) exists. □

4. Examples

EXAMPLE 4.1. For the trivial bundle $(T^n \times B, q, B, T^n)$, where T^n is an n-dimensional torus $(n \geq 1)$ and $\pi_1(B) = 0$,

$$F : \mathcal{F}_{T^n}(T^n \times B) \to \mathcal{F}(T^n \times B)$$

is a monomorphism.

Proof. For a given homotopy equivalence $f : T^n \times B \to T^n \times B$, we define $\bar{f} : B \to B$ by $qfi(i : B \to S^1 \times B, i(b) = (*, b))$. Since $\pi_1(B) = 0$, $\bar{f} : B \to B$ induces an isomorphism on homotopy groups. So $\bar{f}$ is a homotopy equivalence. Therefore $G : \mathcal{R}_q(T^n \times B) = \mathcal{F}(T^n \times B) \to \mathcal{F}(B)$ exists. Since $K(Z^n, 1) = T^n$ and $H^1(B; Z^n) = 0$ by assumption. The result follows by Theorem 3.3. □

EXAMPLE 4.2. For the trivial bundle $(K(\pi, n) \times B, q, B, K(\pi, n))$ with $\pi_i(B) = 0$ $(i \leq n)$,

$$F : \mathcal{F}_{K(\pi,n)}(K(\pi, n) \times B) \to \mathcal{F}(K(\pi, n) \times B)$$

is a monomorphism.

Proof. Since $\pi_i(B) = 0$ $(i \leq n)$, $H^n(B, \pi) = 0$. The proof is similar to example 4.1. □

EXAMPLE 4.3. Let (P_k, q, B, T^n) be a principal T^n-bundle $(n \geq 1)$ with $\pi_1(B) = 0$ and $\mathcal{E}_\#(B) = 1$. For example, take $B = P^m(C)(m \geq 1)$ (complex projective space) (see [2, p.32]). Then

$$F : \mathcal{F}_{T^n}(P_k) \to \mathcal{F}(P_k)$$

is a monomorphism for any k.

Proof. This is obtained from Corollary 3.5 for n=1. □

EXAMPLE 4.4. Let $(P, q, B, K(\pi, n))$ be a principal $K(\pi, n)$-bundle with $\pi_i(B) = 0$ $(i \geq n)$ and $H^n(B; \pi) = 0$. Then

$$F : \mathcal{F}_{K(\pi,n)}(P) \to \mathcal{F}(P)$$

is a monomorphism.

Proof. Since $\pi_i(B) = 0 (i \geq n)$, it is easy to see that a map $G : \mathcal{F}(P) \to \mathcal{F}(B)$ exists by the elementary homotopy theory (e.g. Postnikov tower) and the diagram (3.4) is commutative. So the result is obtained from Theorem 3.3. □

PROBLEM 1. For any S^1-bundle (P_k, q, B, S^1) with $\pi_1(B) = 0$, is the forgetful map

$$F : \mathcal{F}_{S^1}(P_k) \to \mathcal{F}(P_k)$$

a monomorphism?

References

[1] D. H. Gottlieb, *Applications of bundle map theory*, Trans. Amer. Math. Soc. **171** (1972), 23–50.

[2] P. J. Kahn, *Self-equivalences of $(n-1)$-connected $2n$-manifolds*, Math. Ann. **180** (1969), 26–47.

[3] H. Ōshima and K. Tsukiyama, *On the group of equivariant self equivalences of free actions*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **22** (1986), 905–923.

[4] ———, *Bundle map theory in the category of weak Hausdorff k-spaces*, Mem. Fac. Sci. Eng. Shimane Univ. **31** (1998), 27–55.

[5] R. A. Piccinini, *Groups of self-equivalences and related topics*, Springer Lecture Notes in Math. **1425** (1988).

[6] J. W. Rutter, *Self-equivalences and principal morphisms*, Proc. London Math. Soc. **20** (1970), 644-658.

[7] K. Tsukiyama, *Equivariant self equivalences of principal fibre bundles*, Math. Proc. Camb. Phil. Soc. **98** (1985), 87–92.

[8] ———, *An example of self homotopy equivalences*, J. Math. Soc. Japan **48** (1996), 317–319.

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