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SPECTRALLY BOUNDED JORDAN DERIVATIONS ON BANACH ALGEBRAS

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ABSTRACT. Every spectrally bounded Jordan derivation on a unital Banach algebra maps into its Jacobson radical.

1. Introduction

Throughout this paper A will represent an associative algebra over a complex field $\mathbb{C}$, and $rad(A)$ the Jacobson radical of A. We also write $[x, y]$ for the commutator $xy - xy$. A linear mapping $d : A \to A$ is called a derivation if $d(xy) = d(x)y + xd(y) \quad (x,\ y \in A)$. A linear mapping $d : A \to A$ is called a Jordan derivation if $d(x \cdot y) = x \cdot d(y) + d(x) \cdot y$ $(x,\ y \in A)$, where $a \cdot b$ denotes the Jordan product $ab + ba$. Obviously, every derivation is a Jordan derivation. The converse in general is not true. Brešar proved that if R is a 2-torsion free semiprime ring, then every additive Jordan derivation $d : R \to R$ is an additive derivation [2].

Let us introduce the background of our investigation. In 1955 Singer and Wermer obtained a fundamental result which started investigation into the ranges of derivations on Banach algebras [6]. The result states that every bounded derivation on a commutative Banach algebra maps into the Jacobson radical. In the same paper they conjectured that the assumption of boundedness is not necessary. This is called the Singer-Wermer conjecture. In 1988 Thomas [7] proved the conjecture. The so-called non-commutative Singer-Wermer conjecture states that every derivation on a Banach algebra A such that $[d(x), x] \in rad(A)$ for all

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$x \in A$ maps A into its Jacobson radical $rad(A)$. In [4, p. 246] Mathieu showed that this conjecture is equivalent to the conjecture that every derivation d on a Banach algebra A such that $[d(x), x] \in rad(A)$ for all $x \in A$ is spectrally bounded (i.e., $r(d(x)) \leq Mr(x)$ for all $x \in A$ and for some $M \geq 0$). Recently, Brešar and Mathieu proved that every spectrally bounded derivation on a unital Banach algebra maps into its Jacobson radical [3, Theorem 2.5]. It is the main purpose of the present note to show that the above Brešar and Mathieu's result is true for a Jordan derivation as well.

2. The Results

DEFINITION 2.1. The *unitization* of a normed algebra A over $\mathbb{C}$, denoted by $A + \mathbb{C}$, is the normed algebra consisting of the set $A \times \mathbb{C}$ with addition, scalar multiplication and product defined(for all $x, y \in A, \alpha, \beta \in \mathbb{C}$)by

$$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta),$$
$$\beta(x, \alpha) = (\beta x, \beta\alpha),$$
$$(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta),$$

and with the norm defined by $||(x, \alpha)|| = ||x|| + |\alpha|$.

REMARKS. (i) It is a routine matter to verify that $A + \mathbb{C}$ is a normed algebra with unit element $(0, 1)$, that $||(0, 1)|| = 1$, and that the mapping $x \mapsto (x, 0)$ is an isometric isomorphism of A onto a subalgebra of $A + \mathbb{C}$.
(ii) Let d be a Jordan derivation on A. Then we can define a Jordan derivation d_1 on the unitization $A + \mathbb{C}$ of A by $d_1(x, \lambda) = (d(x), 0)$, $(x \in A, \lambda \in \mathbb{C})$.

DEFINITION 2.2. Let A and B be complex Banach algebras. A linear mapping $T : A \to B$ is called *spectrally bounded* (*spectrally infinitesimal*) if there is a constant $M \geq 0$ such that $r(T(x)) \leq Mr(x)$ $(r(T(x)) = 0)$ for all $x \in A$. If $r(T(x)) = r(x)$ for all $x \in A$, we say that T is a *spectral isometry*. If $r(x) = 0$, then x is called *quasinilpotent*. (Herein, $r(x) = \lim_{n\to\infty} ||x^n||^{\frac{1}{n}}$ denotes the spectral radius of the element

x and $Q(A)$ the set of all quasinilpotent elements of A).
Observe that the canonical epimorphism $\phi : A \to A/rad(A)$ is a spectral isometry.

The following lemma is a crucial ingredient in our proof of the main theorem.

LEMMA 2.3 *Every spectrally bounded Jordan derivation d on a Banach algebra A leaves each primitive ideal of A invariant.*

Proof. Suppose that $r(d(x)) \leq Mr(x)$ for all $x \in A$ and some $M \geq 0$. Let P be a primitive ideal of A. Then A has an irreducible representation $\pi : A \to L(X)$ on a Banach space X with kernel P, where $L(X)$ is the algebra of all linear mappings on X. An application of [5, Lemma 3.1] yields that $d^n(x^n) - n!(d(x))^n \in P$ for all $x \in P$ and $n \in \mathbb{N}$. Therefore we obtain that

$$\begin{aligned} r(d(x)+P) &= r(d(x)^n + P)^{1/n} \\ &= (n!)^{-1/n}\, r(d^n(x^n)+P)^{1/n} \\ &\leq (n!)^{-1/n}\, r(d^n(x^n))^{1/n} \\ &\leq (n!)^{-1/n} M r(x) \qquad\qquad (\, x \in P,\ n \in \mathbb{N}) \end{aligned}$$

whence $d(x)+P \in Q(A/P)$ for each $x \in P$. Let $x \in P$ and suppose that $d(x) \notin P$. We first note that the normed division algebra $\mathcal{D} = \{T \in L(X) : aT\xi = T(a\xi),\ a \in A,\ \xi \in X\}$ is $\mathbb{C}1$ [1, p. 128]. If $\pi(d(x)) \in \mathbb{C}1$, then $a\pi(d(x))\xi = \pi(d(x))(a\xi)$ for all $a \in A$ and $\xi \in X$, and hence $ad(x)\xi = d(x)a\xi$ for all $a \in A$ and $\xi \in X$. This means that $ad(x) - d(x)a \in P$ for all $a \in A$. So $r((a+P)(d(x)+P)) \leq r(a+P)r(d(x)+P) = 0$ for all $a \in A$ by the quasinilpotency of $d(x)+P$ wherefore $d(x)+P \in rad(A/P) = \{0\}$. Thus we obtain that $d(x) \in P$. This contradiction implies that $\pi(d(x)) \notin \mathbb{C}1$. Then $\{1, \pi(d(x))\}$ is linearly independent whence there exists $\xi_0 \in X$ such that $\{\xi_0, \pi(d(x))\xi_0\}$, that is, $\{\xi_0, d(x)\xi_0\}$ is linearly independent. Now, by the Jacobson density theorem, we choose a y in A such that $y\xi_0 = \xi_0$ and $yd(x)\xi_0 = \xi_0 - d(x)\xi_0$. Then $(d(x)\cdot y)\xi_0 = d(x)y\xi_0 + yd(x)\xi_0 = \xi_0$, so $d(x)\cdot y + P$ is not quasinilpotent. However $d(x)\cdot y+P$ is quasinilpotent since $d(x)\cdot y + P = (d(x\cdot y) - x\cdot d(y)) + P \in d(P) + P$ $(x \in P,\ y \in A)$. This contradiction shows that $d(x) \in P$ and completes the proof. □

The next lemma is an immediate consequence of [3, Theorem 2.5].

LEMMA 2.4. *Let d be a spectrally bounded derivation on a semisimple unital Banach algebra A. Then $d = 0$ on A.*

Now we prove our main theorem.

THEOREM 2.5. *Let d be a spectrally bounded Jordan derivation on a unital Banach algebra A. Then $d(A) \subseteq rad(A)$.*

Proof. Suppose that $r(d(x)) \leq Mr(x)$ for all $x \in A$ and some $M \geq 0$. By Lemma 2.3, $d(rad(A)) \subseteq rad(A)$, so such a Jordan derivation d induces a Jordan derivation $\bar{d}$ on a semisimple Banach algebra $A/rad(A)$, and $r(\bar{d}(\phi(x))) = r(\phi(d(x))) = r(d(x)) \leq Mr(x) = Mr(\phi(x))$ $(x \in A)$ shows that $\bar{d}$ is spectrally bounded with the same constant as d. Also, since $\bar{d}$ is a Jordan derivation on a semisimple Banach algebra, $\bar{d}$ is a derivation [2]. Hence we see that $\bar{d}$ is a spectrally bounded derivation on a semisimple unital Banach algebra $A/rad(A)$, and Lemma 2.4 implies that $\bar{d} = 0$ on $A/rad(A)$. We complete the proof. □

COROLLARY 2.6. *Let d be a spectrally infinitesimal Jordan derivation on a Banach algebra A. Then $d(A) \subseteq rad(A)$.*

Proof. By Remarks, we can suppose that A is unital, and so Theorem 2.5 guarantees the conclusion. □

If U, V are subsets of an algebra A, then $[U, V]$ denotes the linear span of the commutators $[x, y]$ $(x \in U,\ y \in V)$.

COROLLARY 2.7. *Let d be a Jordan derivation on a Banach algebra A with the property $[[A, A], A] = \{0\}$. Then $d(A) \subseteq rad(A)$.*

Proof. By the assumption, $(A,\ \cdot)$ is an associative commutative Banach algebra with norm $|||x||| = 2||x||$ $(x \in A)$ under the Jordan product $\cdot$. Hence d is a derivation on a commutative Banach algebra $(A,\ \cdot)$. This implies that $d(A,\ \cdot) \subseteq rad(A,\ \cdot)$ by [7]. Since $(A,\ \cdot)$ is commutative, we see that $rad(A,\ \cdot) = Q(A,\ \cdot)$. A simple calculation shows that $Q(A,\ \cdot) = Q(A)$. Then we obtain that $d(A) \subseteq Q(A)$ (i.e., d is spectrally infinitesimal) since $d(A, \cdot) = d(A)$. From Corollary 2.6 it is immediate that $d(A) \subseteq rad(A)$. This completes the proof of the corollary. □

EXAMPLE. Let
$$A = \left\{ \begin{pmatrix} w & 0 & 0 \\ x & w & 0 \\ y & z & w \end{pmatrix} : w,\ x,\ y,\ z \in \mathbb{C} \right\}.$$
Then A is a non-commutative associative algebra under the usual matrix addition, product and scalar multiplication. Define an algebra norm $||\cdot||$ on A by $||X|| = 3\max\{|w|,\ |x|,\ |y|,\ |z|\}$ $(X \in A)$. We now see that A is a Banach algebra satisfying the property $[[A, A], A] = \{O\}$ under this norm, where O is the zero matrix in A.

COROLLARY 2.8. *Let d be a derivation on a Banach algebra A. If $d^2(x) = 0$ for all $x \in A$, then $d(A) \subseteq rad(A)$.*

Proof. Assume that $d^2(x) = 0$ for all $x \in A$. Note that $d(A) \subseteq Q(A)$, i.e., d is spectrally infinitesimal by [8, Theorem 2.9]. Hence we obtain that $d(A) \subseteq rad(A)$ by Corollary 2.6. We complete the proof. □

COROLLARY 2.9. *Let d be a derivation on a semisimple Banach algebra A. Then $d^2 = 0$ on A implies $d = 0$ on A.*

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