

DISK-HOMOGENEOUS RIEMANNIAN MANIFOLDS

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ABSTRACT. We introduce the notion of strongly k -disk homogeneous space and establish a characterization theorem. More specifically, we prove that any analytic Riemannian manifold (M, g) of dimension n which is strongly k -disk homogeneous with $2 \leq k \leq n - 1$ is a space of constant curvature. Its Kähler analog is obtained. The total mean curvature homogeneity of geodesic sphere in k -disk is also considered.

1. introduction

Let (M, g) be an analytic Riemannian manifold of dimension n , and let $i(m)$ denote the injectivity radius at a point $m \in M$. We shall denote by $B_m(r)$ the geodesic ball with center m and radius $r < i(m)$ in (M, g) . For any unit vector $x \in T_m M$, we shall denote by $D_m^x(r)$ the geodesic disk with center m and radius $r < i(m)$ which is perpendicular to x . Further, we shall denote by $V_m(r)$ the n -dimensional volume of $B_m(r)$ and by $V_m^x(r)$ the $(n - 1)$ -dimensional volume of $D_m^x(r)$. The Riemannian manifold (M, g) will be said to be *disk-homogeneous* if, at each point $m \in M$, $V_m^x(r)$ does not depend on the unit vector $x \in T_m M$, and (M, g) will be said to be *strongly disk-homogeneous* if it is disk-homogeneous at each point $m \in M$ and if, in addition, $V_m^x(r)$ does not depend on the point of $m \in M$. The Euclidean space and any rank one symmetric space are strongly disk-homogeneous. The converse for low dimensional cases was considered in [5]. In fact Kowalski and Vanhecke proved the following theorem.

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THEOREM 1.1. *Any Riemannian manifold of dimension ≤ 4 which is strongly disk-homogeneous is either locally Euclidean, or it is locally isometric to a rank one symmetric space.*

It is the purpose of this paper to introduce the notion of strongly k -disk homogeneous space and to establish the similar characterization as Theorem 1.1. For a k -dimensional subspace of $T_m M$ at a point $m \in M$ generated by a set $\{e_1, \dots, e_k\}$ of orthonormal vectors, we shall denote by $D_m^k(r; e_1, \dots, e_k)$ the geodesic k -disk with center m which is given by

$$D_m^k(r; e_1, \dots, e_k) = \{p \in M : d(p, m) \leq r\} \cap \exp_m(\text{Span}\{e_1, \dots, e_k\}),$$

where $\exp_m : T_m M \rightarrow M$ is the exponential map at m . Further, we shall denote by $V_m^k(r; e_1, \dots, e_k)$ the k -dimensional volume of $D_m^k(r; e_1, \dots, e_k)$. (M, g) will be said to be *strongly k -disk homogeneous* if $V_m^k(r; e_1, \dots, e_k)$ does not depend on the choice of $\{e_1, \dots, e_k\}$ or on the point of $m \in M$. Note that a strongly $(n - 1)$ -disk homogeneous space is a strongly disk-homogeneous space in the usual sense. Then the main result of this paper is stated as follows.

THEOREM 1.2. *Any analytic Riemannian manifold (M, g) of dimension n which is strongly k -disk homogeneous for $2 \leq k \leq n - 2$ is a space of constant curvature.*

Theorem 1.2 is proved in §3 after reviewing some preliminaries in §2. The Kähler analog of Theorem 1.2 is proved in §4, and the total mean curvature homogeneity of geodesic sphere in k -disk is considered in §5.

2. Preliminaries

Let (M, g) be an n -dimensional Riemannian manifold of class C^ω and let $m \in M$. Let $r > 0$ be so small that the exponential map \exp_m is defined on a ball of radius r in the tangent space $T_m M$. Further, let (x_1, \dots, x_n) be a system of normal coordinates of M at m such that $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ at m forms an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_m M$, and ω the volume form of M , defined locally up to a sign. We write

$$g_{pq} = g\left(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q}\right), \quad \omega_{1\dots n} = \omega\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right).$$

Then $g_{pq}(m) = \delta_{pq}$. In [3] the following power series expansion is derived for g_{pq} :

$$(2.1) \quad g_{pq} = \delta_{pq} - \frac{1}{3} \sum_{i,j=1}^n R_{ipjq}(m)x_i x_j - \frac{1}{6} \sum_{i,j,k=1}^n \nabla_i R_{jpkq}(m)x_i x_j x_k + \dots,$$

where ∇ and R are the Riemannian connection and the Riemannian curvature tensor of M respectively. We also denote the Ricci tensor and the scalar curvature tensor by ρ and τ , respectively. The power series expansions for $\omega_{1\dots n}$ and for the volume $S_m(r)$ of the geodesic sphere $G_m(r) = \{p \in M : d(m,p) = r\}$ are derived in [3]. For our purpose we need the power series expansion of the geodesic k -disk $D_m^k(r; e_1, \dots, e_k)$. To this end we will use the results and the methods of derivation in [3]. Let $\tilde{\omega}$ be the induced volume form on $D_m^k(r; e_1, \dots, e_k)$ and write

$$\tilde{\omega}_{1\dots k} = \tilde{\omega} \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right).$$

Since

$$(2.2) \quad \tilde{\omega}_{1\dots k}^2 = \det \begin{bmatrix} g_{11} & \dots & g_{1k} \\ g_{21} & \dots & g_{2k} \\ \dots & \dots & \dots \\ g_{k1} & \dots & g_{kk} \end{bmatrix},$$

we get the power series expansion for $\tilde{\omega}_{1\dots k}$ by combining (2.1) and (2.2)

$$(2.3) \quad \tilde{\omega}_{1\dots k} = 1 - \frac{1}{6} \sum_{i,j=1}^k (R_{1i1j} + R_{2i2j} + \dots + R_{kikj})(m)x_i x_j + \dots$$

Note that

$$(2.4) \quad V_m^k(r; e_1, \dots, e_k) = \int_0^r \int_{S^{k-1}(1)} r^{k-1} \tilde{\omega}_{1\dots k}(\exp_m(tu)) du dt.$$

We can expand $\tilde{\omega}_{1\dots k}(\exp_m(ru))$ in a power series in r , where the coefficients are homogeneous polynomials in the $a_i = x_i/r$. Thus

$$\begin{aligned} & \tilde{\omega}_{1\dots k}(\exp_m(ru)) \\ &= 1 - \frac{1}{6} \left(\sum_{i,j=1}^k (R_{1i1j} + R_{2i2j} + \dots + R_{kikj})(m)a_i a_j \right) r^2 + \dots \end{aligned}$$

Furthermore,

$$(2.5) \quad \int_{S^{k-1}(1)} 1 \, du = \text{volume}(S^{k-1}(1)) = 2\Gamma\left(\frac{1}{2}\right)^k \Gamma\left(\frac{k}{2}\right)^{-1}.$$

Next

$$\begin{aligned} & \sum_{i,j=1}^k \int_{S^{k-1}(1)} (R_{1i1j} + R_{2i2j} + \cdots + R_{kiki})(m) a_i a_j \, du \\ &= \sum_{i=1}^k (R_{1i1i} + R_{2i2i} + \cdots + R_{kiki})(m) \int_{S^{k-1}(1)} a_i^2 \, du \\ &= \frac{2}{k} \Gamma\left(\frac{1}{2}\right)^k \Gamma\left(\frac{k}{2}\right)^{-1} \sum_{i=1}^k (R_{1i1i} + R_{2i2i} + \cdots + R_{kiki})(m), \end{aligned}$$

because $a_1^2 + \cdots + a_k^2 = 1$. Thus we have the power series expansion

$$(2.6) \quad V_m^k(r; e_1, \dots, e_k) = \frac{\alpha_k r^k}{k} \left\{ 1 - \frac{1}{6(k+2)} \left(\sum_{i=1}^k (R_{1i1i} + R_{2i2i} + \cdots + R_{kiki})(m) \right) r^2 + \cdots \right\},$$

where α_k is given by (2.5).

3. Characterization of disk homogeneous Riemannian manifold

In this section we prove Theorem 1.2 which characterizes a disk-homogeneous Riemannian manifold.

Proof. We only do the case $k = 4$ since the other cases are similar. From the assumption that M is strongly k -disk homogeneous we see that the coefficient of r^{k+2} in (2.6) is independent of m and $\{e_1, \dots, e_k\}$. Thus by replacing the index $k = 4$ to j and summing up over $4 \leq j \leq n$ we get

$$(3.1) \quad \begin{aligned} & (n-5)(R_{1212} + R_{1313} + R_{2323})(m) + 2(\rho_{11} + \rho_{22} + \rho_{33})(m) \\ &= \text{constant}. \end{aligned}$$

Similarly, replacing the index 3 to j in (3.1) and summing up over $3 \leq j \leq n$, we get

$$(3.2) \quad (n-5)(n-4)R_{1212}(m) + 2(n-4)(\rho_{11} + \rho_{22})(m) + \tau(m) = \text{constant}.$$

Once again we get from (3.2)

$$(3.3) \quad (n-4)(n-3)\rho_{11}(m) + \tau(m) = \text{constant}.$$

Finally we obtain from (3.3)

$$(3.4) \quad \tau(m) = \text{constant}.$$

Then the relations (3.1)–(3.4) and the strongly k -disk homogeneous assumption lead us to conclude that

$$(3.5) \quad \rho_{ii}(m) = \text{constant}, \quad R_{ijij}(m) = \text{constant}, \quad 1 \leq i, j \leq n.$$

This completes the proof of Theorem 1.2. □

4. Real and complex disk homogeneous Kähler manifold

In this section we consider the Kähler analog of Theorem 1.2. Let M be a Kähler manifold of complex dimension n with the almost complex structure J . We shall say that M is *strongly real k -disk homogeneous* if the volume of geodesic k -disk of radius r does not depend on the center m or the real k -frame generating the geodesic k -disk. Similarly M is called *strongly complex k -disk homogeneous* if the volume of geodesic $2k$ -disk of radius r does not depend on the center m or the complex k -frame generating the geodesic $2k$ -disk. In terms of an orthonormal basis $\{e_1, Je_1, \dots, e_n, Je_n\}$ (or briefly $\{e_1, e_{1*}, \dots, e_n, e_{n*}\}$) of $T_m M$, this implies that if M is strongly real k -disk homogeneous then $V_m^k(r; e_{i_1}, \dots, e_{i_k})$ does not depend on the choice of k -frame $\{e_{i_1}, \dots, e_{i_k}\}$ or the point of $m \in M$. Similarly, if M is strongly complex k -disk homogeneous, then $V_m^{2k}(r; e_{i_1}, e_{i_1*}, \dots, e_{i_k}, e_{i_k*})$ does not depend on the choice of $2k$ -frame $\{e_{i_1}, e_{i_1*}, \dots, e_{i_k}, e_{i_k*}\}$ or the point of $m \in M$.

THEOREM 4.1. *Let M be a Kähler manifold of complex dimension n and suppose that M is strongly real and complex k -disk homogeneous for $2 \leq k \leq n-1$. Then M has constant holomorphic sectional curvature.*

Proof. We only prove the case $k = 3$ since the other cases are similar. From the assumption that M is strongly real 3-disk homogeneous we see that the coefficient

$$\sum_{i=1}^3 (R_{1i1i} + R_{2i2i} + R_{3i3i})$$

of r^5 in (2.6) with $k = 3$ is constant. Thus, by replacing the index 3 to j and summing up over $3 \leq j \leq n$, we get

$$(4.1) \quad (n - 3)R_{1212} + (R_{1212} + R_{1313} + \cdots + R_{1n1n}) = \text{constant}.$$

It follows that

$$(4.2) \quad R_{1212} + R_{1313} + \cdots + R_{1n1n} = \text{constant}, \quad R_{1212} = \text{constant}.$$

This implies that any anti-holomorphic sectional curvature

$$(4.3) \quad R_{ijij} = \text{constant}, \quad 1 \leq i \neq j \leq n.$$

Next from the assumption that M is strongly complex 3-disk homogeneous we see that

$$(4.4) \quad \sum_{i,j=1}^3 (R_{ijij} + R_{i^*j^*i^*j^*} + R_{i^*j^*i^*j^*} + R_{i^*j^*i^*j^*}) = \text{constant}.$$

Combining (4.3) and (4.4) we obtain

$$(4.5) \quad R_{11^*11^*} + R_{22^*22^*} + R_{33^*33^*} = \text{constant}$$

which gives subsequently

$$(4.6) \quad (n - 3)(R_{11^*11^*} + R_{22^*22^*}) + (R_{11^*11^*} + \cdots + R_{nn^*nn^*}) = \text{constant},$$

$$(4.7) \quad (n - 3)R_{11^*11^*} + 2(R_{11^*11^*} + \cdots + R_{nn^*nn^*}) = \text{constant},$$

$$(4.8) \quad R_{11^*11^*} + \cdots + R_{nn^*nn^*} = \text{constant}.$$

From (4.7) and (4.8), we get $R_{11^*11^*} = \text{constant}$ which implies

$$(4.9) \quad R_{ii^*ii^*} = \text{constant}, \quad 1 \leq i \leq n.$$

Thus M has constant holomorphic sectional curvature. □

5. Total mean curvature homogeneity of geodesic $(k - 1)$ -spheres

In this section we consider the total mean curvature homogeneity of geodesic sphere in k -disk. The power series expansion for the mean curvature at the point $\exp_m(ru)$ of the geodesic $(k-1)$ -sphere $S_m^{k-1}(r; e_1, \dots, e_k)$ in $D_m^k(\bar{r}; e_1, \dots, e_k)$ is given by

$$(5.1) \quad h(\exp_m(ru)) = \frac{1}{k-1} \left\{ \frac{k-1}{r} - \frac{r}{3} \sum_{i,j=1}^k \rho_{ij} a_i a_j + \dots \right\} (m),$$

where

$$(5.2) \quad u = \sum_{i=1}^k a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \sum_{i=1}^k a_i^2 = 1.$$

The total mean curvature $H_m^k(r; e_1, \dots, e_k)$ of the geodesic $(k-1)$ -sphere $S_m^{k-1}(r; e_1, \dots, e_k)$ in $D_m^k(\bar{r}; e_1, \dots, e_k)$ is given by

$$(5.3) \quad H_m^k(r; e_1, \dots, e_k) = r^{k-1} \int_{S^{k-1}(1)} (h^{k-1} \tilde{\omega}_{1\dots k})(\exp_m(ru)) \, du.$$

Thus the power series expansion for the total mean curvature $H_m^k(r; e_1, \dots, e_k)$ is given by

$$(5.4) \quad H_m^k(r; e_1, \dots, e_k) = c_{k-1} \left\{ 1 - \frac{1}{2k} \left(\sum_{i=1}^k (R_{1i1i} + R_{2i2i} + \dots + R_{ki ki})(m) \right) r^2 + \dots \right\}.$$

Then we have the following theorem from the proof of Theorem 1.2 in §3.

THEOREM 5.1. *Let M be a Riemannian manifold of dimension n and suppose that for all $m \in M$ and all sufficiently small r the geodesic $(k-1)$ -sphere $S_m^{k-1}(r; e_1, \dots, e_k)$ in $D_m^k(\bar{r}; e_1, \dots, e_k)$ has the total mean curvature $H_m^k(r; e_1, \dots, e_k)$ independent of m and $\{e_1, \dots, e_k\}$. Then M has constant sectional curvature.*

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