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ν-PATHS OF ARCS IN REGULAR MULTIPARTITE TOURNAMENTS

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ABSTRACT. A ν-path of an arc xy in a multipartite tournament T is an oriented path in $T - y$ which starts at x such that y does not dominate the end vertex of the path. We show that if T is a regular n-partite ($n \geq 7$) tournament, then every arc of T has a ν-path of length m for all m satisfying $2 \leq m \leq n-2$. Our result extends the corresponding result for regular tournaments, due to Alspach, Reid and Roselle [2] in 1974, to regular multipartite tournaments.

1. Introduction

The vertex set of a digraph D is denoted by $V(D)$. If xy is an arc of a digraph D, then we say that x *dominates* y. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A dominates B, denoted by $A \to B$. The *outset* $N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *inset* $N^-(x)$ is the set of vertices dominating x. A digraph D is said to be *regular* if there is an integer r such that $|N^+(x)| = |N^-(x)| = r$ holds for every $x \in V(D)$.

A digraph obtained by replacing each edge of a complete n-partite graph with an arc or a pair of mutually opposite arcs is called a *semicomplete n-partite digraph* or a *semicomplete multipartite digraph*. A *multipartite tournament* is a semicomplete multipartite digraph without

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a cycle of length 2, and a tournament is an n-partite tournament having exactly n vertices.

Paths and cycles in a digraph are always assumed to be directed. A *bypath* of an arc xy is a path from x to y. Alspach, Reid and Roselle [2] proved that every arc of a regular tournament with $n \geq 7$ vertices has bypaths of all lengths ℓ, $3 \leq \ell \leq n-1$. Further results about bypaths in tournaments (respectively, in local tournaments) can be found in [5] and [6] (respectively, in [4]).

It is not difficult to construct a regular n-partite ($n \geq 3$) tournament T such that T contains an arc having no bypath of length ℓ for some ℓ with $3 \leq \ell \leq n-1$. So, the result in [2] cannot be extended to multipartite tournaments in this way.

Note that the concept of bypaths defined as above has another representation, i.e., an arc xy of a tournament T has a bypath of length ℓ if and only if $T - y$ contains a path of length $\ell - 1$ which starts at x, and y does not dominate the end vertex of the path.

In general, we define a *ν-path* of an arc xy in a digraph D as a path in $D - y$ which starts at x such that y dominates the end vertex of the path only if the end vertex also dominates y. Thus, the concept of ν-paths in digraphs is a generalization of that of bypaths in tournaments.

In this paper, we prove the following theorem, and it is clear that our result generalizes the result of [2] for regular tournaments.

THEOREM. *Let T be a regular n-partite tournament with $n \geq 7$. Then every arc of T has a ν-path of length m for all m satisfying $2 \leq m \leq n-2$.*

2. Proof of the theorem

Let $V_0, V_1, \cdots, V_{n-1}$ be the partite sets of T. From the regularity of T, it is not difficult to check that all partite sets of T have the same cardinality, say k. So, it is clear that

$$|N^+(x)| = |N^-(x)| = \frac{(n-1)k}{2} \quad \text{for each } x \in V(T).$$

Let a_1a_0 be an arbitrary arc of T and assume without loss of generality that $a_i \in V_i$ for $i = 0, 1$. We first show that a_1a_0 has a ν-path of length 2. Since $n \geq 7$, there are at least two vertices b, c in $N^+(a_1) - V_0$ such that $T[\{a_0, b, c\}]$ is a tournament. Without loss of generality, we assume $b \rightarrow c$. If $c \rightarrow x_0$ for some $x_0 \in V_0$, then a_1bc (when $x_0 = a_0$) or a_1cx_0 (when $x_0 \neq a_0$) is a desired ν-path of a_1a_0. So, we may assume that $V_0 \rightarrow c$. Now, we see from the regularity of T that there exists a vertex x with $c \rightarrow x \rightarrow a_0$, and hence, a_1cx is a ν-path of a_1a_0.

Suppose that a_1a_0 has a ν-path P of length $m-1$ (say $P = a_1a_2\cdots a_m$), but it has no ν-path of length m for some m satisfying $3 \leq m < n-1$. Let

$$\begin{aligned} A &= \{\, x \mid x \in V_i,\ V_i \cap V(P) = \emptyset,\ x \rightarrow a_0,\ 2 \leq i \leq n-1\}, \\ B &= \{\, y \mid y \in V_j,\ V_j \cap V(P) = \emptyset,\ a_0 \rightarrow y,\ 2 \leq j \leq n-1\}. \end{aligned}$$

Let $P_0 = \{a_0, a_1, a_2, \cdots, a_m\}$. It is clear that $A \cup B \neq \emptyset$ and every vertex in $A \cup B$ is adjacent with all vertices in P_0. If T is a tournament, then the theorem holds by [2]. So, we prove the theorem for $k \geq 2$ and consider the following two cases:

Case 1. $A \neq \emptyset$.

From the initial hypothesis that a_1a_0 has no ν-path of length m, we see that $A \rightarrow a_m$, and consequently, $A \rightarrow P$ holds.

Let a be an arbitrary vertex of A. Since T is regular, it is easy to check that there is a vertex a' such that $a_{m-2} \rightarrow a'$, but $a \not\rightarrow a'$. Clearly, $a' \notin P_0$. If a and a' are adjacent, then we have $a' \rightarrow a$, and hence, a_1a_0 has a ν-path $a_1 \cdots a_{m-2}a'aa_m$ of length m, a contradiction. Assume now that a and a' belong to the same partite set of T. Since $|N^+(a')| = |N^+(a)|$ and $a \rightarrow a_{m-2} \rightarrow a'$, there is a vertex u with $a' \rightarrow u \rightarrow a$. Obviously, $u \notin P_0$. Thus, a_1a_0 has a ν-path $a_1 \cdots a_{m-2}a'ua$ of length m, a contradiction.

Case 2. $A = \emptyset$.

In this case we have $B \neq \emptyset$. Assume without loss of generality that $V_{n-1} \subsetneq B$. From the regularity of T and the definition of B, it is not difficult to check that every arc from a_0 to B is in a cycle of length 3.

(a): $B \rightarrow a_1$.

If there is a vertex $b \in B$ with $a_1 \rightarrow b$, then $a_i \rightarrow b$ for all $i \geq 2$. Since a_0b is in a cycle of length 3, there is a vertex x such that $b \rightarrow x \rightarrow a_0$. It is easy to see that $x \notin V(P)$. But now, $a_1a_2 \cdots a_{m-1}bx$ is a ν-path of a_1a_0 which is of length m, a contradiction.

(b): $B \rightarrow a_2$.

Assume, on the contrary, that there is a vertex $b \in B$ such that $a_2 \to b$, then $a_i \to b$ for all $i \geq 3$. Thus, we have $V_0 \to b$. It follows that a_0 is adjacent with each vertex of $N^+(b)$, and furthermore, $a_0 \to (N^+(b) - a_1) \cup V_j$, where V_j is the partite set of T which contains b. From the assumption that $|V_j| = k \geq 2$, we have $|N^+(a_0)| > |N^+(b)|$, a contradiction to the regularity of T.

(c): $m \geq 4$.

If $m = 3$, then, by (b) and the assumption that $n \geq 7$, we have $|N^-(a_2)| > |N^+(a_2)|$, a contradiction.

(d): $a_m \to B$.

If there is a vertex $b \in B$ (assume without loss of generality that $b \in V_{n-1}$) such that $b \to a_m$, then we have $b \to \{a_3, \cdots, a_{m-1}\}$. Since $V_{n-1} \to a_2$, b is adjacent to each vertex of $N^+(a_2)$. Since $|N^+(b) \cap P_0| = m$ and $|N^+(a_2) \cap P_0| \leq m - 1$, there is a vertex $x \notin V(P)$ such that $a_2 \to x \to b$. Hence, $a_1 a_2 x b a_4 \cdots a_m$ is of length m, a contradiction.

(e): $a_{m-1} \to B$.

Suppose, on the contrary, that $b \to a_{m-1}$ for some $b \in B$ (assume without loss of generality that $b \in V_{n-1}$). If there is a vertex $u \in N^+(a_1) \backslash P_0$ with $u \to b$, then $a_1 u b a_3 \cdots a_m$ is of length m, a contradiction. Hence, we have that $b \to N^+(a_1) \backslash P_0$. From $V_{n-1} \to a_1$ and the regularity of T we conclude that

$$(1) \qquad |N^+(a_1) \cap P_0| \geq |N^+(b) \cap (P_0 \cup B)| \geq m - 1.$$

Suppose that $m \geq 5$. Then it is easy to see that $b \to N^+(a_2) \backslash P_0$. Since $|N^+(a_2)| = |N^+(b)|$ and $B \to a_2$, $|N^+(a_2) \cap P_0| \geq |N^+(b) \cap (P_0 \cup B)| \geq m - 1$ holds. This implies that

$$(2) \qquad a_2 \to \{a_0, a_3, a_4, \cdots, a_m\} \text{ and } N^+(b) \cap B = \emptyset.$$

It is a simple matter to verify by (2) that $a_0 \to \{a_3, a_4, \cdots, a_{m-1}\}$, and furthermore, $a_1 \not\to a_3$ (otherwise, $a_1 a_3 \cdots a_m b a_2$ yields a contradiction). So, by (1), the following holds:

$$(3) \qquad a_1 \to \{a_4, a_5, \cdots, a_m\}.$$

Assume that B contains at least two partite sets of T. By (2), there is a vertex $b' \in B$ with $b' \to b$. So, we see from (3) and (2) that $a_1 a_4 a_5 \cdots a_m b' b a_2$ is a ν-path of $a_1 a_0$, a contradiction. Therefore, $B = V_{n-1}$. Clearly, $m = n - 2$ and we may assume without loss of generality that $a_i \in V_i$ for $i = 2, 3, \cdots, n - 2$.

Let $H = N^-(a_0)\backslash(P_0 \cup V_{m-1})$. If $a_{m-1} \to x$ for some $x \in H \cup (V_0 - a_0)$, then $a_1 a_m b a_3 a_4 \cdots a_{m-1} x$ is of length m, a contradiction. Hence, we have that $H \cup V_0 \to a_{m-1}$. But now, the following two inequalities

$$\begin{aligned} |N^-(a_{m-1})| &\geq |V_0| + |H| + |\{a_1, a_2, a_{m-2}, b\}| = k + |H| + 4, \\ |N^-(a_0)| &\leq |V_{m-1}\backslash\{a_{m-1}\}| + |H| + |\{a_1, a_2, a_m\}| \\ &= k + |H| + 2 \end{aligned}$$

imply a contradiction to the regularity of T.

Suppose now that $m = 4$. Since $n \geq 7$, B contains at least two partite sets of T and there is a vertex $b' \in B$ which is adjacent with the vertex b.

If $b \to b'$, then $a_1 \to \{a_2, a_3, a_4\}$ holds by (1). It follows that $a_0 \to \{a_2, a_3\}$. Let $F = N^-(a_0)\backslash P_0$. Clearly, $|F| \geq |N^-(a_0)| - 2$. If there is a vertex $x \in F$ with $b' \to x$, then $a_1 a_4 b b' x$ is a ν-path of $a_1 a_0$, a contradiction. Hence, $F \to b'$. Now we see that $|N^-(b')| \geq |F| + |\{a_0, a_m, b\}| \geq |N^-(a_0)| + 1$ contradicts to the regularity of T.

Assume now that $b' \to b$. From (1) and (d), it is easy to check that $a_0 \to a_2$. Since $|N^-(a_0)| = |N^-(b)|$, we see by the same arguments as above and (1) that $|N^-(a_0) \cap P_0| \geq 3$, i.e. $\{a_3, a_4\} \to a_0$. From $V_{n-1} \to a_2$ and the regularity of T, we conclude that there is a vertex y with $a_2 \to y \to b$. Obviously, $y \notin \{a_0, a_1, a_3\}$ and $a_1 a_2 y b a_3$ is a ν-path of $a_1 a_0$, a contradiction.

(f): $a_3 \to B$ if $m \geq 5$.

Note by (e) that $N^+(B) \cap (V_0\backslash P_0) = \emptyset$. Suppose that $b \to a_3$ for some $b \in B$. It is obvious that $(N^+(a_1)\backslash P_0) \cap N^-(b) = \emptyset$. Hence, if $N^+(a_1)\backslash P_0 \neq \emptyset$, we have $b \to N^+(a_1)\backslash P_0$, and furthermore, $a_0 \to N^+(a_1)\backslash P_0$.

If there is a vertex a_j $(3 \leq j \leq m)$ such that $a_1 \to a_j$, but $a_0 \not\to a_{j-1}$, then $a_1 a_j \cdots a_m b a_2 \cdots a_{j-1}$ is a ν-path of $a_1 a_0$, a contradiction. This means that $|N^+(a_0) \cap P_0| \geq |N^+(a_1) \cap P_0| - 2$. It follows by the regularity of T that $|B| \leq 2$. So, by the assumption that $k \geq 2$, we have $|B| = |V_{n-1}| = 2$. Note that $m = n - 2$ and $T[P_0]$ is a tournament. Let a_0' be the vertex in $V_0 - a_0$. Then, it is easy to see that $a' \to V(P) \cup B$, a contradiction to the regularity of T. This completes the proof of (e).

According to (a)–(f), we have that $\{a_3, a_4, \cdots, a_m\} \to B \to \{a_1, a_2\}$. Since $k \geq 2$ and $m \leq n - 2$, we have $N^-(a_0)\backslash P_0 \neq \emptyset$. By (c) and (e),

$N^-(a_0)\backslash P_0 \to B$ holds. Since $|N^-(a_0)| = |N^-(b)|$, we have

$$|N^-(a_0) \cap P_0| \geq |N^-(b) \cap (P_0 \cup B)| \geq m - 1 \tag{4}$$

for any vertex $b \in B$.

If $T[B]$ contains an arc, say $b' \to b$, then, by (4), we have $|N^-(a_0) \cap P_0| = m$, this means that $\{a_1, a_2, \cdots, a_m\} \to a_0$. It is easy to show that $N^+(a_1) \cap V(P) = \{a_2\}$. So, $N^+(a_1)\backslash P_0 \neq \emptyset$. Clearly, $B \to N^+(a_1)\backslash P_0$. But now, $|N^+(b')| > |N^+(a_1)|$ yields a contradiction.

Suppose now that $B = V_{n-1}$ and let b be a vertex of B. Note that $m = n-2$ and $|V_i \cap V(P)| = 1$ for $i = 0, 1, 2, \cdots, n-2$. By (e), it is easy to see that $a_0 \to N^+(b)\backslash P_0$. Since $|N^+(a_0)| = |N^+(b)|$ and b has exactly two out-neighbors in P_0, $|B| = 2$, i.e., $k = 2$. Let a'_0 be the other vertex in V_0. Then we see that $a'_0 \to V(P) \cup B$, a contradiction to the regularity of T.

The proof of the theorem is complete.

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