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A METRIC ON NORMED ALMOST LINEAR SPACES

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ABSTRACT. In this paper, we introduce a semi-metric on a normed almost linear space X via functional. And we prove that a normed almost linear space X is complete if and only if V_X and W_X are complete when X splits as $X = W_X + V_X$. Also, we prove that the dual space X^* of a normed almost linear space X is complete.

Let $(X, \|\cdot\|)$ be a normed almost linear space. In general, the function $d(x,y) = \|x - y\|$ is not a metric on X whereas it is true for a normed linear space. G. Godini ([3]) proved that for a normed almost linear space X there exists a semi-metric which satisfies some properties. In this paper, we show that there exists a semi-metric μ induced by a norm on a normed almost linear space X via functional. Moreover, if X^* is total over X then the semi-metric μ is a metric. As an application, a normed almost linear space X is complete if and only if V_X and W_X are complete when X splits as $X = W_X + V_X$, which improves the result in [5]. Also, we prove that the dual space X^* of a normed almost linear space X is complete. All spaces involved in this paper are over the real field $\mathbb{R}$. Let us denote by $\mathbb{R}_+$ the set $\{\lambda \in \mathbb{R} : \lambda \geq 0\}$. We recall some definitions and results used in this paper.

An *almost linear space* (als) is a set X together with two mappings $s : X \times X \to X$ and $m : \mathbb{R} \times X \to X$ satisfying the conditions $(L_1)-(L_8)$ given below. For $x, y \in X$ and $\lambda \in \mathbb{R}$ we denote $s(x,y)$ by $x + y$ and $m(\lambda, x)$ by λx, when these will not lead to misunderstandings. Let $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$. (L_1) $x+(y+z) = (x+y)+z$; (L_2) $x+y = y+x$; (L_3) There exists an element $0 \in X$ such that $x+0 = x$ for each $x \in X$;

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(L_4) $1x = x$; (L_5) $\lambda(x+y) = \lambda x+\lambda y$; (L_6) $0x = 0$; (L_7) $\lambda(\mu x) = (\lambda\mu)x$; (L_8) $(\lambda + \mu)x = \lambda x + \mu x$ for $\lambda \geq 0,\ \mu \geq 0$.

We denote $-1x$ by $-x$, and $x - y$ means $x + (-y)$. Note that $x - x$ need not be equal to zero in an *als* since an element in an *als* does not have an inverse element. For an *als* X we introduce the following two sets:

$$V_X = \{x \in X : x - x = 0\}$$

$$W_X = \{x \in X : x = -x\}.$$

V_X and W_X are almost linear subspaces of X(i.e., closed under addition and multiplication by scalars) and, in fact, V_X is a linear space. Clearly an *als* X is a linear space iff $V_X = X$ iff $W_X = \{0\}$. Note that $V_X \cap W_X = \{0\}$ and $W_X = \{x - x : x \in X\}$.

A *norm* on an *als* X is a functional $||\cdot|| : X \to \mathbb{R}$ satisfying the conditions $(N_1)-(N_3)$ below. Let $x, y, z \in X$ and $\lambda \in \mathbb{R}$. (N_1) $||x-z|| \leq ||x-y||+||y-z||$; (N_2) $||\lambda x|| = |\lambda|\ ||x||$; (N_3) $||x|| = 0$ iff $x = 0$. An *als* X together with $||\cdot|| : X \to \mathbb{R}$ satisfying $(N_1)-(N_3)$ is called a *normed almost linear space* (*nals*). Using (N_1) we get $||x + y|| \leq ||x|| + ||y||$ and $||x - y|| \geq |\ ||x|| - ||y||\ |$ for $x, y \in X$. By the above axioms it follows that $||x|| \geq 0$ for each $x \in X$. We denote by B_X and S_X the sets $\{x \in X : ||x|| \leq 1\}$ and $\{x \in X : ||x|| = 1\}$, respectively.

The following proposition is needed in the sequel.

PROPOSITION 1 ([1]). *Let $(X, ||\cdot||)$ be a nals. Then,*

(a) *For $x \in X,\ w \in W_X$, we have $\max\{||x||,\ ||w||\} \leq ||x + w||$.*
(b) *The relations $w_1 + v_1 = w_2 + v_2,\ w_i \in W_X,\ v_i \in V_X,\ i = 1, 2$ imply that $w_1 = w_2$ and $v_1 = v_2$.*

Let X be an *als*. A functional $f : X \to \mathbb{R}$ is called an *almost linear functional* if f is additive, positively homogeneous and $f(w) \geq 0$ for each $w \in W_X$. Let $X^\#$ be the set of all almost linear functionals on X. Define addition in $X^\#$ by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ for $x \in X$ and the multiplication by scalars $(\lambda \circ f)(x) = f(\lambda x)$ for $x \in X,\ \lambda \in \mathbb{R}$. The element $0 \in X^\#$ is the functional which is 0 for each $x \in X$. Then $X^\#$ is an *als*. An almost linear subspace Γ of $X^\#$ is said to be *total* over

X if the relations $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ for each $f \in \Gamma$ imply that $x_1 = x_2$.

When X is a *nals*, for $f \in X^{\#}$ define $\|f\| = \sup\{|f(x)| : x \in X,\ \|x\| \le 1\}$, and let $X^* = \{f \in X^{\#} : \|f\| < \infty\}$. Then X^* is a *nals* (cf. [2]). We shall call such a space X^* the *dual space* of X. For a *nals* X and $f \in X^*$, an equivalent formula for the norm of f is

$$\|f\| = \sup\{|f(x)| : x \in S_X\} = \sup\left\{\frac{|f(x)|}{\|x\|} : x \in X,\ x \ne 0\right\},$$

hence

$$|f(x)| \le \|f\|\|x\|.$$

In the theory of a normed linear space an important tool is the Hahn-Banach theorem. An analogous theorem is no longer true in a *nals* ([1, 4.5 Example]). But we have the following proposition.

PROPOSITION 2 ([4]). *Let $(X, \|\cdot\|)$ be a nals. Then,*

(a) *For each $x \in X$, there exists $f \in B_{X^*}$ such that $f(x) = \|x\|$.*
(b) *If a nals X splits as $X = W_X + V_X$ and $f \in (V_X)^*$, then there exists $\bar{f} \in V_{X^*}$ such that $\|\bar{f}\| = \|f\|$ and $\bar{f}(v + w) = f(v)$ for each $v \in V_X$, $w \in W_X$.*
(c) *For each $x \in X$, $\|x\| = \sup\left\{\frac{|f(x)|}{\|f\|} : f \in X^*, f \ne 0\right\}$.*

G. Godini introduced ([3, Corollary 3.3]) a semi-metric ρ on a *nals* $(X, \|\cdot\|)$ which satisfies the following properties:

(1) $$\rho(x, v) = \|x - v\| \quad (x \in X,\ v \in V_X),$$

(2) $$\rho(x + z, y + z) = \rho(x, y) \quad (x, y, z \in X),$$

(3) $$\rho(\lambda x, \lambda y) = |\lambda|\rho(x, y) \quad (x, y \in X,\ \lambda \in \mathbb{R}),$$

(4) $$|\,\|x\| - \|y\|\,| \le \rho(x, y) \le \|x - y\| \quad (x, y \in X),$$

(5) $$\lim_{\lambda \to \lambda_0} \rho(\lambda x, x) = \rho(\lambda_0 x, x) \quad (x \in X,\ \lambda_0 > 0).$$

G. Godini's semi-metric ρ is a metric when X has a basis (cf. [5, Theorem 6]). Now, we construct a new semi-metric on a *nals* X via functional.

THEOREM 3. *Let* $(X, ||\cdot||)$ *be a nals with dual* X^*. *Define* $\mu : X \times X \to \mathbb{R}$ *by*

$$\mu(x,y) = \sup\{|f(x) - f(y)| : f \in B_{X^*}\} \quad (x, y \in X). \tag{6}$$

Then μ *is a semi-metric on* X *satisfying the properties of* ρ *in* (1) – (5). *Moreover, if* X^* *is total over* X *then* μ *is a metric on* X.

Proof. Clearly, μ is a semi-metric on X. For $x \in X$, $v \in V_X$, there exists $g \in B_{X^*}$ such that $g(x-v) = ||x-v||$ by Proposition 2(a). Hence

$$||x - v|| = g(x - v) = |g(x) - g(v)| \leq \mu(x, v).$$

Also, for each $f \in B_{X^*}$

$$|f(x) - f(v)| = |f(x - v)| \leq ||f|| \, ||x - v|| \leq ||x - v||,$$

whence $\mu(x,v) \leq ||x - v||$. Thus $\mu(x,v) = ||x - v||$. For $x, y, z \in X$ and $f \in B_{X^*}$, we have $|f(x+z) - f(y+z)| = |f(x) - f(y)|$. Hence, $\mu(x+z, y+z) = \mu(x,y)$. The property (3) is obvious for $\lambda \geq 0$. Let $\lambda < 0$. Then

$$\begin{aligned}
\mu(\lambda x, \lambda y) &= \sup\{|f(\lambda x) - f(\lambda y)| : f \in B_{X^*}\} \\
&= \sup\{|\lambda| \, |f(-x) - f(-y)| : f \in B_{X^*}\} \\
&= |\lambda| \sup\{|(-1 \circ f)(x) - (-1 \circ f)(y)| : f \in B_{X^*}\} \\
&= |\lambda| \sup\{|g(x) - g(y)| : g \in B_{X^*}\}, \quad \text{put } g = -1 \circ f \\
&= |\lambda| \mu(x, y).
\end{aligned}$$

Hence μ satisfies the property of ρ in (3). For $x, y \in X$ and $f \in B_{X^*}$,

$$|f(x)| \leq |f(x) - f(y)| + |f(y)| \leq \mu(x,y) + ||y||.$$

Hence $||x|| \leq \mu(x,y) + ||y||$. Similarly $||y|| \leq \mu(x,y) + ||x||$, whence the first inequality in (4) follows. Since $-f(-x) \leq f(x)$ for each $f \in B_{X^*}$, we have

$$f(x) - f(y) \leq f(x) + f(-y) = f(x - y) \leq ||x - y||.$$

Similarly, $f(y) - f(x) \leq ||y - x|| = ||x - y||$ for each $f \in B_{X^*}$. Hence $|f(x) - f(y)| \leq ||x - y||$ for each $f \in B_{X^*}$, whence the right-hand side inequality in (4) follows. For $\lambda > 0$,

$$\begin{aligned}\mu(\lambda x, x) &= \sup\{|f(\lambda x) - f(x)| : f \in B_{X^*}\}\\ &= \sup\{|\lambda - 1|\ |f(x)| : f \in B_{X^*}\}\\ &= |\lambda - 1|\ \sup\{|f(x)| : f \in B_{X^*}\}\\ &= |\lambda - 1|\ ||x||.\end{aligned}$$

Hence, for $x \in X$ and $\lambda_0 > 0$ we have

$$\lim_{\lambda \to \lambda_0} \mu(\lambda x, x) = \lim_{\lambda \to \lambda_0} |\lambda - 1|\ ||x|| = |\lambda_0 - 1|\ ||x|| = \mu(\lambda_0 x, x),$$

whence μ satisfies the property of ρ in (5). By definition of total, the second statement of the theorem is clear. The proof of the theorem is complete. □

From Theorem 6 in [5] and Theorem 3, we get the following corollary.

COROLLARY 4. *Let $(X, ||\cdot||)$ be a nals with dual X^*. If X has a basis or X^* is total over X, then there exists a metric on X satisfying the properties of ρ in (1)-(5).*

EXAMPLE 5. Let $\mathbb{R}^2$ be endowed with the Euclidean norm $||\cdot||$ and let $e_1 = (1, 0)$, $e_2 = (0, 1)$. Let $A_i = \{\lambda e_i : \lambda \geq 0\}$, $i = 1, 2$ and let $X = A_1 \cup A_2$. Define $s(x, y) = x + y$ if both $x, y \in A_i$, $i = 1, 2$, and $s(x, y) = s(y, x) = (||x|| + ||y||)e_2$ if $x \in A_i \setminus \{0\}$, $y \in A_j \setminus \{0\}$, $i \neq j$. And define $m(\lambda, x) = |\lambda| x$. Let $0 \in X$ be the element $0 \in \mathbb{R}^2$. Then X together with $||\cdot||$ is a *nals*. There is no metric on X with (1)-(5). Indeed, suppose μ is a metric on X. Then $\mu(e_1 + e_2, e_2 + e_2) = \mu(2e_2, 2e_2) = 0 \neq \mu(e_1, e_2)$. Therefore μ has no property (2). Therefore X has no basis. Also, X^* is not total over X by Corollary 4.

In a *nals* X the semi-metric μ defined by (6) generates a topology on X (which is not a Hausdorff in general) and in the sequel any topological concept such as closeness, completion, continuity, will be understood for this topology. Moreover the topology on $(V_X, ||\cdot||)$ generated by μ is the same as the topology generated by norm.

THEOREM 6. *Let $(X, ||\cdot||)$ be a nals. Then V_X is closed in X.*

Proof. Let $\{v_n\}$ be a sequence in V_X such that

$$\lim_{n\to\infty} \mu(v_n, x) = 0$$

for some $x \in X$. Since

$$\begin{aligned}\mu(0, x-x) &= \mu(v_n - v_n, x - x)\\ &\le \mu(v_n - v_n, x - v_n) + \mu(x - v_n, x - x)\\ &= \mu(v_n, x) + \mu(-v_n, -x)\\ &= \mu(v_n, x) + |-1|\mu(v_n, x)\\ &= 2\mu(v_n, x)\end{aligned}$$

for each $n \in N$, we have $\mu(0, x-x) = 0$. Hence $x - x = 0$. Therefore $x \in V_X$. □

THEOREM 7. *If a nals X splits as $X = W_X + V_X$ or X^* is total over X, then W_X is closed in X.*

Proof. Suppose that $\{w_n\}$ is a sequence in W_X which converges to $x \in X$. (1) Let $X = W_X + V_X$ and $x = w_0 + v_0$ where $w_0 \in W_X$, $v_0 \in V_X$. Note that

$$\lim_{n\to\infty} |f(w_n) - f(x)| = 0$$

for all $f \in B_{X^*}$. If $v_0 \neq 0$, then there exists $g \in B_{(V_X)^*}$ such that $g(v_0) = ||v_0|| \neq 0$ by Proposition 2(a). By Proposition 2(b), there exists $\overline{g} \in B_{X^*}$ such that $\overline{g}(w + v) = g(v)$ for each $w \in W_X$, $v \in V_X$. For this $\overline{g} \in B_{X^*}$,

$$\lim_{n\to\infty} |\overline{g}(w_n) - \overline{g}(x)| = \lim_{n\to\infty} |g(v_0)| = ||v_0|| \neq 0,$$

a contradiction. Thus $v_0 = 0$, whence $x = w_0 \in W_X$. Therefore W_X is closed in X.

(2) Let X^* be total over X. Since

$$\begin{aligned}\mu(x, -x) &\le \mu(x, w_n) + \mu(w_n, -x)\\ &= \mu(x, w_n) + \mu(-w_n, -x)\\ &= 2\mu(x, w_n)\end{aligned}$$

for each $n \in N$, we have $\mu(x, -x) = 0$. Since X^* is total over X, μ is a metric on X. Hence $x = -x$. Therefore $x \in W_X$. □

Thus, if a split *nals* X is complete, then V_X and W_X are complete. However, if a *nals* X is not split, then the converse does not hold (cf. [5, Example 9]). When X splits as $X = W_X + V_X$, Theorem 6 and Theorem 7 yield the following theorem.

THEOREM 8. *Let $(X, \|\cdot\|)$ be a split nals. Then X is complete if and only if V_X and W_X are complete.*

Proof. Let V_X and W_X be complete and let $\{x_n = v_n + w_n\}$ be a Cauchy sequence in $X = W_X + V_X$. For $v_n - v_m \in V_X$, there exists $g \in B_{(V_X)^*}$ such that $g(v_n - v_m) = \|v_n - v_m\|$. By Proposition 2(b), there exists $\bar{g} \in V_{X^*}$ such that $\|\bar{g}\| = \|g\|$ and $\bar{g}(v + w) = g(v)$ for each $v \in V_X$, $w \in W_X$. We have

$$\begin{aligned}\mu(v_n, v_m) &= \|v_n - v_m\| = g(v_n - v_m) = g(v_n) - g(v_m)\\ &= \bar{g}(v_n + w_n) - \bar{g}(v_m + w_m) \leq \mu(x_n, x_m),\end{aligned}$$

whence $\{v_n\}$ is a Cauchy sequence in V_X. Since V_X is complete, there exists $v_0 \in V_X$ such that

$$\lim_{n\to\infty} \mu(v_n, v_0) = 0.$$

For each $f \in B_{X^*}$, define $g(v + w) = f(w)$ for each $v \in V_X$, $w \in W_X$. Since $\|w\| \leq \|v + w\|$, we have

$$|g(v + w)| = |f(w)| \leq \|f\| \, \|w\| \leq \|f\| \, \|v + w\|.$$

Thus $g \in B_{X^*}$. Also, we have

$$|f(w_n) - f(w_m)| = |g(v_n + w_n) - g(v_m + w_m)| \leq \mu(x_n, x_m),$$

whence $\mu(w_n, w_m) \leq \mu(x_n, x_m)$. Therefore $\{w_n\}$ is a Cauchy sequence in W_X. Since W_X is complete, there exists $w_0 \in W_X$ such that

$$\lim_{n\to\infty} \mu(w_n, w_0) = 0.$$

Put $x = v_0 + w_0$. Then

$$\begin{aligned}\mu(x_n, x) &= \mu(v_n + w_n, v_0 + w_0)\\ &\le \mu(v_n + w_n, w_n + v_0) + \mu(w_n + v_0, v_0 + w_0)\\ &= \mu(v_n, v_0) + \mu(w_n, w_0).\end{aligned}$$

Thus $\{x_n\}$ converges to $x \in X$. □

For the dual space X^* of a *nals* X, we can define (cf. [2, Theorem 5.4]) a metric $d : X^* \times X^* \to \mathbb{R}$ by

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in B_X\} \quad (f, g \in X^*).$$

Then d satisfies the properties of ρ in (1)-(5). In the sequel, any topological concept on the dual space of a *nals* will be understood for this topology.

THEOREM 9. *The dual space X^* of a nals X is complete.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in X^*. Then

$$d(f_n, f_m) = \sup\{|f_n(x) - f_m(x)| : x \in B_X\} \to 0$$

as $m, n \to \infty$. Thus $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in B_X$. Since

$$|f_n(x) - f_m(x)| = ||x|| \left| f_n\left(\frac{x}{||x||}\right) - f_m\left(\frac{x}{||x||}\right)\right|$$

for each nonzero $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $x \in X$. Hence, we can define

$$f(x) := \lim_{n\to\infty} f_n(x) \quad (x \in X).$$

Clearly, f is an almost linear functional on X. Since $|\, ||f_n|| - ||f_m||\, | \le d(f_n, f_m)$, $\{||f_n||\}$ is a Cauchy sequence in $\mathbb{R}$ whence $\lim_{n\to\infty} ||f_n|| < \infty$. Since

$$\begin{aligned}||f|| &= \sup\{|f(x)| : x \in B_X\}\\ &= \sup\{|\lim_{n\to\infty} f_n(x)| : x \in B_X\}\\ &\le \sup\{\lim_{n\to\infty} ||f_n||\, ||x|| : x \in B_X\}\\ &= \lim_{n\to\infty} ||f_n||,\end{aligned}$$

we have $||f|| < \infty$. Now, we prove that $\{f_n\}$ converges to f. Let $\epsilon > 0$ be given. Since $d(f_n, f_m) \to 0$ as $m, n \to \infty$, there exists $n_0 \in N$ such that

$$d(f_n, f_m) = \sup\{|f_n(x) - f_m(x)| : x \in B_X\} < \epsilon$$

for all $n, m \geq n_0$. Therefore

$$\begin{aligned} d(f_n, f) &= \sup\{|f_n(x) - f(x)| : x \in B_X\} \\ &= \sup\{|f_n(x) - \lim_{m\to\infty} f_m(x)| : x \in B_X\} \\ &< \epsilon \end{aligned}$$

for all $n > n_0$. This shows that $\{f_n\}$ converges to f. □

REMARK. If X is a normed linear space, then $d(f, g) = \sup\{|f(x) - g(x)| : x \in B_X\} = ||f - g||$. Thus, the above theorem is a generalization of a fact in a normed linear space.

As in the case of a normed linear space, we can define a reflexive *nals* (cf. [4]). Then we have a generalized result of fact in a normed linear space.

COROLLARY 10. *A reflexive nals X is complete.*

If a *nals* X is reflexive, then X splits as $X = W_X + V_X$ (cf. [4, Theorem 2.6]), whence we have the following corollary from Theorem 8.

COROLLARY 11. *If a nals X is reflexive, then V_X and W_X are complete.*

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