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ESTIMATING CONVERGENCE FACTORS OF SCHWARZ ALGORITHMS FOR ORTHOGONAL SPLINE COLLOCATION METHOD

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ABSTRACT. A new and simple estimate using a convex function for convergence factors of Schwarz algorithms for orthogonal spline collocation method is presented. The estimated convergence factors in this context depend only on a way to split a given domain.

1. Introduction

Consider the following homogeneous Dirichlet Poisson equation

$$-\Delta u = f \qquad \text{in} \quad \Omega := (0,1)\times(0,1). \tag{1.1}$$

For a positive integer $N \geq 3$, let $\{t_i\}_{i=0}^{N}$ be a uniform partition of the interval $[0,1]$, that is, $t_i := ih, \quad i = 0,1,\cdots,N$, where $h := 1/N$ and let $t_i^* := (t_i + t_{i+1})/2$. Let $S_{h,3}(0,1)$ be the space of piecewise Hermite cubics on $[0,1]$ defined by

$$S_{h,3}(0,1) := \{u \in C^1[0,1] \;:\; u|_{[t_i,t_{i+1}]} \in P_3,\ i = 0,1\cdots,N-1\},$$

where P_3 denotes the set of all polynomials of degree ≤ 3. Define $S^0_{h,3}(0,1)$ as the set of all functions in $S_{h,3}(0,1)$ vanishing at 0 and 1 and

$$S_{h,3}(I) := \{u|_I : u \in S_{h,3}(0,1)\}, \quad \text{where} \quad I \subset (0,1).$$

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Let $\mathcal{G} := \{\xi_i\}_{i=1}^{2N}$ be the set of the Legendre-Gauss[:=LG] points on $[0,1]$, that is

$$\xi_{2i+1} = t_i^* - \frac{h}{2\sqrt{3}}, \quad \xi_{2i+2} = t_i^* + \frac{h}{2\sqrt{3}}, \quad i = 0, \cdots, N-1$$

and let

$$\tilde{\mathcal{G}} := \{(\xi, \eta) : \xi, \eta \in \mathcal{G}\}.$$

The cubic orthogonal collocation problem corresponding to (1.1) is;
Find $u_h \in \mathcal{V}_h := S_{h,3}^0(0,1) \otimes S_{h,3}^0(0,1)$ *such that*

$$-\Delta u_h(\xi, \eta) = f(\xi, \eta), \quad (\xi, \eta) \in \tilde{\mathcal{G}}. \tag{1.2}$$

The general convergence proof for Schwarz algorithms for orthogonal collocation method was appeared in [1], [2] and [3]. Therefore the interesting reader should refer to [1], [2] and [3] for more details. The aim of this paper is to provide a new way to estimate convergence factors on Schwarz algorithm considered in [1] for a two or three overlapping subrectangles case. Due to a convex function, the convergence factors in this context depend only on a way to decompose a given domain. An outline of this paper is as follows. In the following section we construct an increasing convex interpolating function and give a bound for the approximation solution $\phi_h \in S_{h,3}(0,1)$ at each knot t_k to $u''(\xi) = \mu u(\xi)$, $\mu > 0$ with particular boundary conditions. Finally we provide a new way estimating convergence factors.

2. Estimating convergence factor

It is well known (see Theorem 2.1 in [4]) that the following eigenvalue problem in the space $S_{h,3}^0(0,1)$

$$-u''(\xi) = \lambda u(\xi), \quad \xi \in \mathcal{G}, \qquad u(0) = u(1) = 0 \tag{2.1a}$$

has $2N$ distinct eigenvalues

$$\{\lambda_j = c_j h^{-2}, \quad 0 < c_j \leq 36\}_{j=1}^{2N} \qquad \left(h = \frac{1}{N}\right) \tag{2.1b}$$

and corresponding orthogonal eigenvectors $\{v_j\}_{j=1}^{2N}$ satisfying

$$\langle v_i, v_j \rangle_N = \delta_{i,j}, \quad i, j = 1, 2, \cdots, 2N \tag{2.1c}$$

where

$$\langle u, v\rangle_N := \frac{h}{2}\sum_{i=1}^{2N} u(\xi_i)v(\xi_i), \quad \text{for} \quad u, v \in S_{h,3}(0,1).$$

The existence and uniqueness for the solution to the following problem with $\mu = \lambda_j$, where λ_j is as in (2.1b), were known in [1].

For a given $\mu > 0$, *find* $\phi_h \in S_{h,3}(\alpha, \beta)$ $(\alpha, \beta \in \{t_i\}_{i=1}^{N})$ *satisfying*

$$\phi''(\xi) = \mu\phi(\xi), \quad \xi \in \mathcal{G} \tag{2.2a}$$

with

$$\phi(\alpha) = 0, \quad \phi(\beta) = 1, \quad (0 \le \alpha < \beta \le 1). \tag{2.2b}$$

For simplicity we let $(\alpha, \beta) = (0, 1)$. The solution ϕ_h of (2.2a,b) restricted on $[t_k, t_{k+1}]$ can be written as follows; for $k = 0, 1, 2, \cdots, N-1$,

$$\phi_h^k(t) := \alpha_{k+1} + \beta_{k+1}(t - t_k^*) + \gamma_{k+1}\frac{(t - t_k^*)^2}{2} + \delta_{k+1}\frac{(t - t_k^*)^3}{6}. \tag{2.3}$$

As in [4], the equation (2.2a,b) applied to (2.3) leads to

$$\alpha_{k+1} = (1/\mu - h^2/24)\gamma_{k+1}, \quad \beta_{k+1} = (1/\mu - h^2/72)\delta_{k+1} \tag{2.4}$$

and the C^1- condition at t_j, $j = 1, 2, \cdots, N-1$, yields

$$D\begin{pmatrix} \gamma_j \\ \delta_j \end{pmatrix} = \begin{pmatrix} \gamma_{j+1} \\ \delta_{j+1} \end{pmatrix}$$

where

$$D = \frac{1}{rt - s}\begin{pmatrix} rt + s & 2st \\ 2r & rt + s \end{pmatrix}$$

with

$$r = \frac{h^2}{12} + \frac{1}{\mu}, \quad s = \frac{h}{2}\left(\frac{h^2}{36} + \frac{1}{\mu}\right), \quad t = \frac{2}{h}\left(\frac{h^2}{9} + \frac{1}{\mu}\right). \tag{2.5}$$

The boundary conditions (2.2b) and (2.4) yield

$$r\gamma_1 = s\delta_1, \quad r\gamma_N + s\delta_N = 1.$$

From (2.5) we can see that $rt - s > 0$ and D is a positive matrix with $det(D) = 1$. The eigenvalues w_1 and w_2 of D are given by

$$w_1 := \frac{rt + s + 2\sqrt{rst}}{rt - s} > 0, \quad w_2 := \frac{rt + s - 2\sqrt{rst}}{rt - s} > 0,$$
$$\text{and} \qquad w_1 \cdot w_2 = 1.$$

From Lemma 2.2 in [1], the solution ϕ_h of (2.2) satisfies

$$\phi_h(t_k) = \sigma^{\frac{N-k}{2}} \frac{1 - \sigma^k}{1 - \sigma^N} \quad \text{where} \quad \sigma = \frac{w_2}{w_1}. \tag{2.6}$$

Using the eigenvalue w_1 of the matrix D, we construct an increasing convex function which interpolates the solution function ϕ_h to $(2.2a, b)$ in the following proposition.

PROPOSITION 1. *For $\phi_h \in S_{h,3}(0,1)$ satisfying $(2.2a, b)$, the sequence $\{\phi_h(t_k)\}$ is strictly increasing and*

$$0 \le \phi_h(t_k) \le g_h(t_k), \quad \text{where} \quad g_h(t) := \frac{t - \alpha}{\beta - \alpha}, \tag{2.7}$$

Proof. Since $det(D) = w_2 \cdot w_1 = 1$, $\sigma = w_2/w_1 = 1/w_1^2$. (2.6) yields

$$\phi_h(t_k) = w_1^{-N+k} \frac{1 - w_1^{-2k}}{1 - w_1^{-2N}} = \frac{w_1^N}{w_1^{2N} - 1}\left(w_1^k - \frac{1}{w_1^k}\right).$$

Define

$$f_h(t) := \frac{w_1^N}{w_1^{2N} - 1}\left(w_1^{Nt} - \frac{1}{w_1^{Nt}}\right), \quad 0 \le t \le 1.$$

An easy calculation shows that $f_h'(t) > 0$ and $f_h''(t) > 0$ for all $t \in [0, 1]$ and

$$f_h(t_k) = \phi_h(t_k) \quad k = 0, 1, \cdots, N.$$

The positivity of f_h' and $f_h(0) = 0$ and $f_h(1) = 1$ lead

$$0 < \phi_h(t_k) < \phi_h(t_{k+1}), \quad k = 0, 1, \cdots, N-1$$

and this implies (2.7) with the convexity of f_h. □

Because of this proposition, we can estimate the convergence factor of the Schwarz Algorithm for the cubic orthogonal collocation method. First consider three overlapping subrectangles. Let us decompose Ω as

$\Omega_1 = I_1 \times (0,1)$ and $\Omega_2 = I_2 \times (0,1)$ with $I_1 = (0, l_1) \cup (l_3, 1)$ and $I_2 = (l_2, l_4)$ where

$$0 < l_2 < l_1 < l_3 < l_4 < 1, \quad l_i \in \{t_i\}_{i=0}^N \tag{2.8a}$$

and

$$l_1 + l_3 = l_2 + l_4 = 1, \quad l_4 - l_2 = l_1. \tag{2.8b}$$

Let

$$\mathcal{G}_1 := \tilde{\mathcal{G}} \cup \Omega_1, \quad \text{and} \quad \mathcal{G}_2 := \tilde{\mathcal{G}} \cup \Omega_2.$$

Define two function spaces as

$$V_1 := S_{h,3}(I_1) \otimes S_{h,3}^0(0,1), \quad V_2 := S_{h,3}(I_2) \otimes S_{h,3}^0(0,1).$$

Let $\Gamma_i = \partial\Omega_i \cap \Omega, i = 1,2$ with $\Gamma_1^- = \{l_1\} \times (0,1)$ and $\Gamma_1^+ = \{l_3\} \times (0,1)$, so that we have $\Gamma_1 = \Gamma_1^- \cup \Gamma_1^+$.

The Schwarz algorithm applied to the cubic orthogonal collocation problem (1.2) can be written as follows:

For a given initial function $u_h^0 \in \mathcal{V}_h$, find a sequence $\{u_h^n\}$ such that $u_h^{2n+1} \in V_1, u_h^{2n+2} \in V_2$ for $n \geq 0$ such that

$$\begin{cases} -\Delta u_h^{2n+1} = f \quad in \quad \mathcal{G}_1 \\ u_h^{2n+1} = u_h^{2n} \quad on \quad \Gamma_1, \quad u_h^{2n+1} = 0 \quad on \quad \partial\Omega_1 \backslash \Gamma_1 \end{cases} \tag{2.9a}$$

and

$$\begin{cases} -\Delta u_h^{2n+2} = f \quad in \quad \mathcal{G}_2 \\ u_h^{2n+2} = u_h^{2n+1} \quad on \quad \Gamma_2, \quad u_h^{2n+2} = 0 \quad on \quad \partial\Omega_2 \backslash \Gamma_2. \end{cases} \tag{2.9b}$$

THEOREM 1. *Let u_h be the solution of (1.2) and let u_h^{2n+1} and $u_h^{2n+2} (n \geq 0)$ be defined by (2.9a, b). Under the assumptions (2.8a, b), there is a constant C, independent of h, satisfying, for $n \geq 1$ and $i = 1, 2$,*

$$\|u_h - u_h^{2n+i}\|_{H^1(\Omega_i)} \leq C \left(\frac{1 - l_1}{2l_1} \right)^n \|u_h - u_h^1\|_{H^1(\Omega_1)}.$$

Proof. Let $\{\lambda_j\}_{j=1}^{2N}$ and $\{v_j\}_{j=1}^{2N}$ be as in (2.1a,b,c) and let $\phi_h^{1,j}$ and $\phi_h^{2,j}$ $(j = 1, \cdots, 2N)$ be the solution of the problem (2.2a,b) with $\mu = \lambda_j, \alpha = 0, \beta = l_1$ and $\mu = \lambda_j, \alpha = l_2, \beta = l_4$, respectively.

Applying (2.7) to $\phi_h^{1,j}$ and $\phi_h^{2,j}$ for $j = 1, 2, \cdots, 2N$, we have

$$(2.10a) \qquad 0 \le \phi_h^{2,j}(l_1) \le \frac{l_1 - l_2}{l_4 - l_2}, \quad 0 \le \phi_h^{2,j}(l_3) \le \frac{l_3 - l_2}{l_4 - l_2}$$

and

$$(2.10b) \qquad 0 \le \phi_h^{1,j}(l_2) \le \frac{l_2}{l_1}.$$

The Theorem 3.2 and Corollary 3.2 in [1] imply

$$(2.11) \qquad ||u_h - u_h^{2n+i}||_{H^1(\Omega_1)} \le C\,(\kappa_{l_1})^n ||\, u_h - u_h^1\,||_{H^1(\Omega_1)}, \quad n \ge 1,$$

where C is an absolute constant and

$$\kappa_{l_1} := \max_{1 \le j \le 2N} \phi_h^{1,j}(l_2)\left[\phi_h^{2,j}(l_1) + \phi_h^{2,j}(l_3)\right] > 0.$$

From the estimations $(2.10a, b)$ and the assumptions $(2.8a, b)$ on domain decomposition, we have

$$\kappa_{l_1} \le \frac{l_2}{l_1} = \frac{1 - l_1}{2l_1} < 1.$$

Therefore the conclusion comes from (2.11). □

Note that the fact $l_2/l_1 < 1$ with $(2.8a, b)$ shows that we can choose l_1 and l_3 satisfying

$$(2.12) \qquad 0 < l_2 < 1/3 < l_1 < l_3 < 2/3 < l_4 < 1.$$

The domain decomposition satisfying above assumption is not the only condition to guarantee the convergence of Schwarz algorithm $(2.9a, b)$. It is remarkable that, under the conditions (2.12), the convergence factor $l_2/l_1 = (1 - l_1)/(2l_1)$ does not depend on the given mesh size h.

Now consider two overlapping subrectangles. Let $\Omega_1 = I_1 \times (0,1)$ and $\Omega_2 = I_2 \times (0,1)$ with $I_1 = (0, l_1)$ and $I_2 = (l_2, 1)$ where $0 < l_2 < l_1$, and $l_i \in \{t_k\}_{k=0}^N$. Let $\Gamma_i = \partial\Omega_i \cap \Omega$, $i = 1, 2$. Modifying the function spaces V_1 and V_2 defined for three overlapping subrectangle cases, the Schwarz alternating method for two overlapping subrectangles is defined similarly to the method defined in (2.9).

THEOREM 2. *Let u_h be the solution of (1.2) and let u_h^{2n+1} and $u_h^{2n+2}(n \ge 0)$ be defined by Schwarz algorithms $(2.9a, b)$ for two ovelapping subrectangles. There is a constant C, independent of h, satisfying,* *for $n \geq 1$ and $i = 1, 2$,*

$$\|u_h - u_h^{2n+i}\|_{H^1(\Omega_i)} \leq C \left(\frac{l_2(l_1 - l_2)}{l_1(1 - l_2)} \right)^n \|u_h - u_h^1\|_{H^1(\Omega_1)}.$$

Proof. As in the proof in theorem 1, let $\{\lambda_j\}_{j=1}^{2N}$ and $\{v_j\}_{j=1}^{2N}$ be as in (2.1a,b,c) and let $\phi_h^{1,j}$ and $\phi_h^{2,j}$ ($j = 1, \cdots, 2N$) be the solution of the problem (2.2a,b) with $\mu = \lambda_j, \alpha = 0, \beta = l_1$ and $\mu = \lambda_j, \alpha = l_2, \beta = 1$, respectively. Applying (2.7) to $\phi_h^{1,j}$ and $\phi_h^{2,j}$ for $j = 1, 2, \cdots, 2N$, we have

$$0 \leq \phi_h^{2,j}(l_1) \leq \frac{l_1 - l_2}{1 - l_2}, \quad \text{and} \quad 0 \leq \phi_h^{1,j}(l_2) \leq \frac{l_2}{l_1}.$$

Hence, from Theorem 3.1 and Corollary 3.1 in [1], for $n \geq 1$ and $i = 1, 2$,

$$\|u_h - u_h^{2n+i}\|_{H^1(\Omega_i)} \leq C \left(\max_{1 \leq j \leq 2N} \{\phi_h^{1,j}(l_2)\phi_h^{2,j}(l_1)\} \right)^n \|u_h - u_h^1\|_{H^1(\Omega_1)}.$$

Then the estimate (2.10) yields the conclusion. □

As pointed in three overlapping subrectangles, the convergence factor again depends only on the way to split domains. In this case there are no restrictions on l_1 and l_2 except $0 < l_2 < l_1$. Actual numerical results can be found in [1], in which they considered two overlapping subrectangles with $\Omega_1 = (0, 2/3) \times (0, 1)$ and $\Omega_2 = (1/3, 1) \times (0, 1)$.

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