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ESSENTIAL SPECTRA OF TOEPLITZ OPERATORS

M. W. Wong

ABSTRACT. The essential spectra for a class of Toeplitz operators on l^p, $1 \le p \le \infty$, are computed.

1. Introduction

For $1 \le p < \infty$, we let l^p be the set of all sequences

$$a = (\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots)$$

of complex numbers such that

$$\|a\|_p = \left\{ \sum_{n=-\infty}^{\infty} |a_n|^p \right\}^{\frac{1}{p}} < \infty,$$

and we let l^∞ be the set of all sequences

$$a = (\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots)$$

of complex numbers such that

$$\|a\|_\infty = \sup\{|a_n| : n = 0, \pm 1, \pm 2, \cdots\} < \infty.$$

Then l^p is a complex Banach space in which the norm is given by $\| \ \|_p$, $1 \le p \le \infty$.

Let c be a sequence of real numbers in l^1 given by

$$c = (\cdots, c_{-2}, c_{-1}, c_0, c_1, c_2, \cdots).$$

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Then, for any sequence

$$a = (\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots)$$

in l^p, $1 \le p \le \infty$, we define $T_c a$ to be the sequence of complex numbers by

$$T_c a = c * a,$$

where the mth entry of $c * a$ is equal to $\sum_{n=-\infty}^{\infty} c_{m-n} a_n$, $m = 0, \pm 1, \pm 2, \cdots$. By Young's inequality, we can see that $T_c : l^p \to l^p$ is a bounded linear operator for $1 \le p \le \infty$. We denote $T_c : l^p \to l^p$, $1 \le p \le \infty$, by T_{cp} and call it the Toeplitz operator on l^p corresponding to the sequence c. The following result is Theorem 2.5 in the paper [7] by Wong.

THEOREM 1.1. *Let c be a sequence of real numbers in l^1 given by*

$$c = (\cdots, c_{-2}, c_{-1}, c_0, c_1, c_2, \cdots),$$

where

$$c_n = c_{-n}, \quad n = 0, \pm 1, \pm 2, \cdots,$$

and let φ be the continuous function on $[-\pi, \pi]$ such that

$$\varphi(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}, \quad -\pi \le \theta \le \pi,$$

where the convergence is uniform and absolute on $[-\pi, \pi]$. Let m and M respectively be the minimum and maximum values of φ on $[-\pi, \pi]$. Then the spectrum $\sum(T_{cp})$ of T_{cp}, $1 \le p \le \infty$, is equal to $[m, M]$.

The aim of this paper is to supplement Theorem 1.1 by computing the essential spectrum $\sum_e(T_{cp})$ of T_{cp}, $1 \le p \le \infty$. In Section 2, we recall the spectrum and the essential spectrum of a bounded linear operator A from a complex Banach space X into X. The main result on the essential spectrum $\sum_e(T_{cp})$ of T_{cp}, $1 \le p \le \infty$, is given and proved in Section 3.

2. The spectrum and the essential spectrum

Let A be a bounded linear operator from a complex Banach space X into X. Let $\rho(A)$ be the resolvent set of A defined by

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is one to one and onto}\},$$

where I is the identity operator on X. The spectrum $\sum(A)$ of A is defined to be the set complement of $\rho(A)$ in $\mathbb{C}$. We denote the range of A by $R(A)$, and the null spaces of A and A^t by $N(A)$ and $N(A^t)$ respectively, where the adjoint A^t of A is defined as on, say, page 100 of the book [6] by Wong. We call A a Fredholm operator if $R(A)$ is a closed subspace of X, and the dimensions $\alpha(A)$ and $\beta(A)$ of $N(A)$ and $N(A^t)$ respectively are finite. For any Fredholm operator A, we define the index $i(A)$ of A by

$$i(A) = \alpha(A) - \beta(A).$$

Let $\Phi(A)$ be the set defined by

$$\Phi(A) = \{\lambda \in \mathbb{C} \; : \; A - \lambda I \text{ is Fredholm and } i(A - \lambda I) = 0\}.$$

Then the essential spectrum $\sum_e(A)$, defined by Schechter in [4], is the set complement of $\Phi(A)$ in $\mathbb{C}$. A proof that $\sum_e(A)$ is a closed subset of $\sum(A)$ can be found in Section 4 of Chapter 1 of the book [5] by Schechter. The following result, noted in the paper [1] by Coburn, is a consequence of a result in the paper [2] by Joichi, which is given as Theorem 3.2 on page 171 of the book [5] by Schechter.

THEOREM 2.1. *Let A be a bounded linear operator from a complex Banach space X into X. Then*

$$\sum_e(A^t) = \{\bar{\lambda} : \; \lambda \in \sum_e(A)\}.$$

REMARK 2.2. Theorem 2.1 is the analogue for the essential spectrum of a result of Phillips in the paper [3] to the effect that

$$\sum(A^t) = \{\bar{\lambda} : \; \lambda \in \sum(A)\}.$$

3. The main result

We can now state and prove the main result in this paper.

THEOREM 3.1. *Let c and φ be as in* Theorem 1.1. *Then, for* $1 \le p \le \infty$,

$$\sum\nolimits_e(T_{cp}) = [m, M],$$

where m and M are the minimum and maximum values of φ on $[-\pi, \pi]$ respectively.

Proof. Let $\lambda \in [m, M]$. Let us first suppose that φ is not equal to λ on a set with positive measure. Suppose $\lambda \in \Phi(T_{cp})$, $1 \le p \le 2$. Let a be a sequence in l^p, $1 \le p \le 2$, such that

$$(T_c - \lambda I)a = (\varphi - \lambda)f = 0,$$

where I is the identity operator on l^p, $1 \le p \le 2$, and f is the function in $L^2[-\pi, \pi]$ such that the sequence of Fourier coefficients of f is equal to a. Then $f = 0$, i.e., $a = 0$, and hence $T_{cp} - \lambda I$, $1 \le p \le 2$, is one to one and onto. This is a contradiction to Theorem 1.1 and hence $\lambda \in \sum_e(T_{cp})$, $1 \le p \le 2$, and by Theorem 2.1, $\lambda \in \sum_e(T_{cp})$, $1 \le p \le \infty$. Now, suppose that φ is equal to λ on a set S with positive measure. Let us write $S = \cup_{j=1}^{\infty} S_j$, where $\{S_j : j = 1, 2, \cdots\}$ is a countably infinite collection of pairwise disjoint subsets of S with positive measures. For $j = 1, 2, \cdots$, let χ_j be the characteristic function of S_j, and let a^j be the sequence of Fourier coefficients of χ_j. Then, for $j = 1, 2, \cdots$, and $p \ge 2$,

$$(T_{cp} - \lambda I)a^j = (\varphi - \lambda)\chi_j = 0.$$

Since $\{a^j : j = 1, 2, \cdots\}$ is a sequence of linearly independent sequences in l^p, $p \ge 2$, it follows that λ is an eigenvalue with infinite multiplicity of T_{cp}, $p \ge 2$. Therefore $\lambda \in \sum_e(T_{cp})$, $p \ge 2$, and hence, by Theorem 2.1, $\lambda \in \sum_e(T_{cp})$, $1 \le p \le \infty$. Thus, $[m, M] \subseteq \sum_e(T_{cp})$, $1 \le p \le \infty$, and, by Theorem 1.1, the proof is complete. □

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DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, 4700 KEELE STREET, TORONTO, ONTARIO M3J 1P3, CANADA